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A VARIANT OF THE TYPE-1 BETA AND DIRICHLET INTEGRALS

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} ***Birth Anniversary***

Abstract: A double integral, which can be taken as an extension of a type-1 beta integral, is introduced and its properties are studied. Than a k-variate multiple integral on a unit cube in Euclidean k-space is introduced, which will produce the total integral equivalent to the total integral in a $(k-1)$-variate type-1 Dirichlet integral, or the integral over a simplex in a $(k-1)$-flat. Several properties of this multiple integral are studied. Marginal functions and Mellin transform are examined. Several transformations are given, finally leading to a type-1 Dirichlet integral. Statistical densities connected with the integrand of the extended Dirichlet integral are also discussed.

Keywords: Extended Dirichlet integral, Mellin transform, power transformations, extended Dirichlet average.

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1. Introduction

Type-1 beta integral and beta function are very important in many areas. Type-1 beta model is a popular prior probability measure in Bayesian analysis. Variation in the chance of occurrence of any event is usually modeled by a type-1 beta model. In population studies, the probability of conception is usually modeled by a type-1 beta model. In random cuts or random division of an interval, a type-1 beta model and its generalization of type-1 Dirichlet model come in naturally. A very popular reliability model in reliability analysis or system reliability is based on a type-1 beta model, often a power transformed beta model. When inter-arrival times are independently exponentially or gamma distributed then proportional arrival times are often type-1 beta or Dirichlet distributed, the arrivals may be vehicles arriving into a service station, patients arriving into a hospital, phone calls coming into a telephone, floods occurring in a river, occurrence of lightning in a certain place etc

A variant of type-1 beta integral and the corresponding type-1 Dirichlet integral are dealt with in this article. Averaging any function in the extended Dirichlet model will give an extension of Dirichlet averages in applied analysis. An interesting bivariate integral is seen to be equivalent to a univariate type-1 beta integral. Similarly a certain k-variate integral over a unit cube is seen to be equivalent to a $(k-1)$-variate Dirichlet integral. Some properties of this integral and the corresponding integrand are examined in this article.

Type-1 and Type-2 Dirichlet functions and Dirichlet integrals are given various types of extensions in the scalar variable case and to matrix-variate cases, see for example Mathai (2003) for one type of extension in the scalar variable case, Mathai (1997), Mathai and Provost (1995), Thomas and Mathai (2008) for some extensions to matrix-variate cases. But in the present article, an entirely different type of extension is given. This type of extension is intriguing and there is novelty in this respect. This type of extension does not seem to be available in the literature.

This article is organized as follows: In section 2 an extended type-1 beta integral is introduced and its properties, such as marginal functions, Mellin transforms, power transformations etc, are studied. In section 3, the corresponding extended Dirichlet integral is examined and several of its properties are studied. Then the corresponding statistical densities are also introduced. The extended type-1 beta and type-1 Dirichlet integrals are intriguing and interesting from a the point of view of mathematical manipulations but this author does see any direct applications to any problem at present. Hopefully, someone will find interesting applications of the results in this article in the near future.

2. Extended Type-1 Beta Integral

For $\delta > 0, \delta_j > 0, \alpha_j > 0, j = 1, 2$, consider the following integral:

$$A = \int_{x_1=0}^{1}\int_{x_2=0}^{1} x_1^{\alpha_1\delta_1-1}x_2^{\alpha_2\delta_2-1}(1-x_2^{\delta_2})^{\alpha_1} \times [1-(a-x_1^{\delta_1})(1-x_2^{\delta_2})]^{-(\alpha_1+\alpha_2-\delta)}\mathrm{d}x_1\wedge \mathrm{d}x_2 \tag{2.1}$$

It will be shown that this double integral $\delta\delta_1\delta_2 A$ is equivalent to a type-1 beta integral. We may write (2.1) as the following:

$$A = \int_{x_1=0}^{1}\int_{x_2=0}^{1} x_1^{\alpha_1\delta_1-1}x_2^{\alpha_2\delta_2-1}(1-x_2^{\delta_2})^{\alpha_1} \\ \times \sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2-\delta)_m}{m!}(1-x_1^{\delta_1})^m(1-x_2^{\delta_2})^m \mathrm{d}x_1\wedge \mathrm{dx}_2$$

by using a binomial expansion, where, for example, $(a)_m = a(a+1)...(a+m-1), a\neq 0, (a)_0 = 1$ is the Pochhammer symbol, and $\Re(\cdot)$ means the real part of $(\cdot)$. Integration over x_1 gives

$$\int_{x_1=0}^{1} x_1^{\alpha_1\delta_1-1}(1-x_1^{\delta_1})^m \mathrm{d}x_1 = \frac{1}{\delta_1}\int_{y_1=0}^{1} y_1^{\alpha_1-1}(1-y_1)^m \mathrm{d}y_1 \\ = \frac{1}{\delta_1}\frac{\Gamma(\alpha_1)\Gamma(m+1)}{\Gamma(\alpha_1+m+1)}, \Re(\alpha_1)>0.$$

Integral over x_2 yields

$$\int_{x_2=0}^{1} x_2^{\alpha_2\delta_2-1}(1-x_2^{\delta_2})^{\alpha_1+m}\mathrm{d}x_2 = \frac{1}{\delta_2}\int_{y_2=0}^{1} y_2^{\alpha_2-1}(1-y_2)^{\alpha_1+m}\mathrm{d}y_2 \\ = \frac{1}{\delta_2}\frac{\Gamma(\alpha_2)\Gamma(\alpha_1+m+1)}{\Gamma(\alpha_1+\alpha_2+m+1)}, \Re(\alpha_2)>0.$$

Hence the product gives

$$\frac{1}{\delta_1\delta_2}\Gamma(\alpha_1)\Gamma(\alpha_2)\frac{\Gamma(m+1)}{\Gamma(\alpha_1+\alpha_2+1`)} = \frac{1}{\delta_1\delta_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2+1)}\frac{(1)_m}{(\alpha_1+\alpha_2+1)_m}.$$

Therefore

$$A = \frac{1}{\delta_1\delta_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2+1)}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2-\delta)_m}{m!}\frac{(1)_m}{(\alpha_1+\alpha_2+1)_m} \\ = \frac{1}{\delta_1\delta_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}{}_2F_1(\alpha_1+\alpha_2-\delta,1;\alpha_1+\alpha_2+1;1)$$

where ${}_2F_1$ is a Gauss' hypergeometric function with argument 1. But a ${}_2F_1$ with argument 1 has the following simplification:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

whenever the gammas are defined. Then

$$\begin{aligned} A &= \frac{1}{\delta_1\delta_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2+1)}\frac{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\delta)}{\Gamma(\delta+1)\Gamma(\alpha_1+\alpha_2)} \\ &= \frac{1}{\delta\delta_1\delta_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}, \Re(\alpha_1)>0, \Re(\alpha_2)>0 \\ &= \frac{1}{\delta\delta_1\delta_2}\int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2-1}\mathrm{d}x, \Re(\alpha_1)>0, \Re(\alpha_2)>0 \end{aligned} \tag{2.2}$$

Hence

$$\delta\delta_1\delta_2 A = \int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2-1}\mathrm{d}x, \Re(\alpha_1)>0, \Re(\alpha_2)>0.$$

Thus, we can have the following theorem:

Theorem 2.1 *Consider the integral on the right side of (2.1), denoted by A. Then for* $\delta>0, \delta_1>0, \delta_2>0, \Re(\alpha_1)>0, \Re(\alpha_2)>0$

$$\begin{aligned} \delta\delta_1\delta_2 A &= \int_{x_1=0}^1\int_{x_2=0}^1 x_1^{\alpha_1\delta_1-1}x_2^{\alpha_2\delta_2-1}(1-x_2^{\delta_2})^{\alpha_1} \\ &\quad\times[1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})]^{-(\alpha_1+\alpha_2-\delta)}\mathrm{d}x_1\wedge\mathrm{d}x_2 \\ &= \int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2-1}\mathrm{d}x. \end{aligned} \tag{2.3}$$

2.1. Various representations of this pseudo type-1 beta integral

Make the transformation $y_j = x_j^{\delta_j}, j=1,2$. Then

Corollary 2.1.

$$\delta\delta_1\delta_2 A = \int_{y_1=0}^1\int_{y_2=0}^1 y_1^{\alpha_1-1}y_2^{\alpha_2-1}(1-y_2)^{\alpha_1}[1-(1-y_1)(1-y_2)]^{-(\alpha_1+\alpha_2-\delta)}\mathrm{d}y_1\wedge\mathrm{d}y_2 \tag{2.4}$$

Make the transformation $z_j = 1-y_j, j=1,2$.

Corollary2.2.

$$\delta\delta_1\delta_2 A = \int_{z_1=0}^1\int_{z_2=0}^1 (1-z_1)^{\alpha_1-1}(1-z_2)^{\alpha_2-1}z_2^{\alpha_1}[1-z_1z_2]^{-(\alpha_1+\alpha_2-\delta)}\mathrm{d}z_1\wedge\mathrm{d}z_2 \tag{2.5}$$

Consider the integrand in (2.5) with $\delta=1$. Let

$$\begin{aligned} f(x_1,z_2) &= (1-z_1)^{\alpha_1-1}(1-z_2)^{\alpha_2-1}z_2^{\alpha_1}[1-z_1z_1]^{-(\alpha_1+\alpha_2-1)} \\ &= \left[\frac{z_2(1-z_1)}{1-z_1z_2}\right]^{\alpha_1-1}\left[\frac{1-z_2}{1-z_1z_2}\right]^{\alpha_2-1}\frac{z_2}{1-z_1z_2}. \end{aligned} \tag{2.6}$$

Let $u_1 = z_1 z_2, u_2 = z_2$. Then $\mathrm{d}u_1 \wedge \mathrm{d}u_2 = z_2 \mathrm{d}z_1 \wedge \mathrm{d}z_2$. Therefore

$$\begin{aligned} f_1(u_1, u_2)\mathrm{d}u_1 \wedge \mathrm{d}u_2 &= \left[\frac{u_2(1-\frac{u_1}{u_2})}{1-u_1}\right]^{\alpha_1-1} \left[\frac{1-u_2}{1-u_1}\right]^{\alpha_2-1} \frac{1}{1-u_1}\mathrm{d}u_1 \wedge \mathrm{d}u_2 \\ &= \left[\frac{u_2-u_1}{1-u_1}\right]^{\alpha_1-1} \left[\frac{1-u_2}{1-u_1}\right]^{\alpha_2-1} \frac{1}{1-u_1}\mathrm{d}u_1 \wedge \mathrm{d}u_2 \end{aligned} \tag{2.7}$$

for $0 \le u_1 \le u_2 \le 1$. Let $v_1 = \frac{u_2-u_1}{1-u_1}, v_2 = \frac{1-u_2}{1-u_1}$. Then $v_1 + v_2 = 1$ or $v_1 = 1 - v_2$. From (2.7), $0 \le u_1 \le u_2 \le 1$. Therefore

$$\begin{aligned} \int_{u_2=u_1}^{1} (u_2-u_1)^{\alpha_1-1}(1-u_2)^{\alpha_2-1}\mathrm{d}u_2 &= \int_0^{1-u_1} z^{\alpha_1-1}(1-z-u_1)^{\alpha_2-1}\mathrm{d}z \\ &= (1-u_1)^{\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \Re(\alpha_1) > 0, \Re(\alpha_2) > 0. \end{aligned}$$

But $(1-u_1)^{\alpha_1+\alpha_2-1}$ is canceled for $\delta = 1$. If $\delta \neq 1$ then we have $(1-u_1)^{\delta-1}$ and the integral over u_1 gives $\frac{1}{\delta}$ and this δ is canceled. Then the final result is $\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$.

2.2. Statistical densities

From (2.2) it is clear that

$$\begin{aligned} g(x_1, x_2) = &\frac{\delta\delta_1\delta_2\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1\delta_1-1} x_2^{\alpha_2\delta_2-1}(1-x_2^{\delta_2})^{\alpha_1} \\ &\times [1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})]^{-(\alpha_1+\alpha_2-\delta)} \end{aligned} \tag{2.9}$$

for $\delta > 0, \delta_1 > 0, \delta_2 > 0, \alpha_1 > 0, \alpha_2 > 0, 0 \le x_1 \le 1, 0 \le x_2 \le 1$, and zero elsewhere, is a statistical density. In statistical densities the parameters are real but the integrals here will hold for complex parameters α_1 and α_2 with $\Re(\alpha_1) > 0, \Re(\alpha_2) > 0$. Let us compute the h-th moment of $u = \frac{x_2^{\delta_2}}{[1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})]}$. We will state it as a theorem.

Theorem 2.2. *For the u defined above, the h-th moment of u in the density (2.9) is equivalent to the h-th moment of a real type-1 beta random variable with the parameters (α_2, α_1).*

Proof:

$$\begin{aligned} E\left[\frac{x_2^{\delta_2 h}}{[1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})]^h}\right] &= c\int_{x_1=0}^{1}\int_{x_2=0}^{1} x_1^{\alpha_1\delta_1-1} x_2^{\alpha_2\delta_2+\delta_2 h-1} \\ &\times (1-x_2^{\delta_2})^{\alpha_1}[1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})]^{-(\alpha_1+\alpha_2+h-\delta)}\mathrm{d}x_1 \wedge \mathrm{d}x_2. \end{aligned} \tag{2.10}$$

where

$$c=\frac{\delta\delta_1\delta_2\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Make the transformation $x_1^{\delta_1}=y_1, x_2^{\delta_2}=y_2$. Then

$$E\left[\frac{y_2}{1-(1-y_1)(1-y_2)}\right]^h=\frac{c}{\delta_1\delta_2}\int_{y_1=0}^{1}\int_{y_2=0}^{1}y_1^{\alpha_1-1}y_2^{(\alpha_2+h)-1}(1-y_2)^{\alpha_1}$$
$$\times[1-(1-y_1)(1-y_2)]^{-(\alpha_1+\alpha_2+h-\delta)}\mathrm{d}y_1\wedge\mathrm{d}y_2. \quad (2.11)$$

Let $z_1=1-y_1, z_2=1-y_2$. Then

$$E\left[\frac{1-z_2}{1-z_1z_2}\right]^h=\frac{c}{\delta_1\delta_2}\int_{z_1=0}^{1}\int_{z_2=0}^{1}(1-z_1)^{\alpha_1-1}(1-z_2)^{\alpha_2+h-1}z_2^{\alpha_1}$$
$$\times[1-z_1z_2]^{-(\alpha_1+\alpha_2+h-\delta)}\mathrm{d}z_1\wedge\mathrm{d}z_2 \quad (2.12)$$
$$=\frac{c}{\delta_1\delta_2}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2+h-\delta)_m}{m!}\int_{z_1=0}^{1}z_1^m(1-z_1)^{\alpha_1-1}\mathrm{d}z_1$$
$$\times\int_{z_2=0}^{1}z_2^{\alpha_1+m}(1-z_2)^{\alpha_2+h-1}\mathrm{d}z_2$$

Evaluating the various integrals with the help of beta integrals one has the following:

$$E\left[\frac{1-z_2}{1-z_1z_2}\right]^h=\frac{c}{\delta_1\delta_2}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2+h-\delta)_m}{m!}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2+h+1)}{\Gamma(\alpha_1+\alpha_2+h+1)}\frac{(1)_m}{(\alpha_1+\alpha_2+h+1)_m}$$
$$=\delta\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+h)}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2+h+1)}{}_2F_1(\alpha_1+\alpha_2+h-\delta,1;\alpha_1+\alpha_2+h+1;1)$$
$$=\delta\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+h)}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2+h+1)}\frac{\Gamma(\alpha_1+\alpha_2+h+1)\Gamma(\delta)}{\Gamma(\delta+1)\Gamma(\alpha_1+\alpha_2)}$$
$$=\frac{\Gamma(\alpha_2+h)}{\Gamma(\alpha_2)}\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2+h)}=E(x^h),\Re(\alpha_2+h)>0 \quad (2.13)$$

where x has a type-1 beta density with the parameters (α_2,α_1), or $1-x$ has a type-1 beta density with the parameters (α_1,α_2). The corresponding Mellin transform is equivalent to all the integrals in (2.10),(2.11),(2.12),(2.13) with $h=s-1$. Observe that the right side of (2.12) can also be written as follows for $\delta = 1$:

$$E\left[\frac{1-z_2}{1-z_1z_2}\right]^h = \frac{\delta\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{z_1=0}^{1}\int_{z_2=0}^{1}\left[\frac{z_2(1-z_1)}{1-z_1z_2}\right]^{\alpha_1-1} \times \left[\frac{1-z_2}{1-z_1z_2}\right]^{\alpha_2+h-1}\frac{z_2}{1-z_1z_2}\mathrm{d}z_1\wedge\mathrm{d}z_2 \tag{2.14}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{u_2=0}^{1}(1-u_2)^{\alpha_1-1}u_2^{\alpha_2+h-1}\mathrm{d}u_2, u_2 = \frac{1-z_2}{1-z_1z_2} \tag{2.15}$$

$$= \frac{\Gamma(\alpha_2+h)}{\Gamma(\alpha_2)}\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2+h)}, \Re(\alpha_2+h)>0, \Re(\alpha_1)>0.$$

2.3. Marginal functions

Let

$$f(x_1,x_2) = (1-x_1)^{\alpha_1-1}(1-x_2)^{\alpha_2-1}x_2^{\alpha_1}[1-x_1x_2]^{-(\alpha_1+\alpha_2-\delta)}.$$

Integral over x_1 gives

$$\int_{x_1=0}^{1} f(x_1,x_2)\mathrm{d}x_1 = x_2^{\alpha_1}(1-x_2)^{\alpha_2-1}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2-\delta)_m}{m!}\int_0^1 x_1^m(1-x_1)^{\alpha_1-1}\mathrm{d}x_1 x_2^m$$

$$= x_2^{\alpha_1}(1-x_2)^{\alpha_2-1}\sum_{m=0}^{\infty}\frac{(\alpha_1+\alpha_2-\delta)_m}{m!}\frac{\Gamma(\alpha_1)}{\Gamma(\alpha+1)}\frac{(1)_m}{(\alpha_1+1)_m}x_2^m$$

$$= x_2^{\alpha_1}(1-x_2)^{\alpha_2-1}\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+1)}{}_2F_1(\alpha_1+\alpha_2-\delta,1;\alpha_1+1;x_2). \tag{2.16}$$

Similarly, the marginal function of x_1 is available by integrating out x_2. By using the above steps we have the following:

$$\int_{x_2=0}^{1} f(x_1,x_2)\mathrm{d}x_2 = (1-x_1)^{\alpha_1-1}\frac{\Gamma(\alpha_2)\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+\alpha_2+1)}{}_2F_1(\alpha_1+\alpha_2-\delta,\alpha_1+1;\alpha_1+\alpha_2+1;x_1) \tag{2.17}$$

for $0 \le x_1 \le 1, \alpha_1 > -1, \alpha_2 > 0$ and zero elsewhere. Corresponding marginal densities are available by multiplying (2.16) and (2.17) by $\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$. If the marginal densities are denoted by $f_1(x_1)$ and $f_2(x_2)$ respectively, then

$$f_1(x_1) = \frac{\alpha_1}{(\alpha_1+\alpha_2)}(1-x_1)^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2-\delta,\alpha_1+1;\alpha_1+\alpha_2+1;x_1) \tag{2.18}$$

for $0 \leq x_1 \leq 1, \alpha_1 > 0, \alpha_2 > 0$ and zero elsewhere, and

$$f_2(x_2) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_2)\Gamma(\alpha_1+1)} {}_2F_1(\alpha_1+\alpha_2-\delta, 1; \alpha_1+1; x_2) \tag{2.19}$$

for $0 \leq x_2 \leq 1, \alpha_1 > -1, \alpha_2 > 0$ and zero elsewhere.

Let us compute $E[x_1^{h_1} x_2^{h_2}]$ in the pseudo type-1 beta density, which is the Mellin transform of the extended type-1 beta density when $h_1 = s_1 - 1, h_2 = s_2 - 1$ where s_1 and s_2 are the Mellin parameters. That is,

$$E[x_1^{h_1} x_2^{h_2}] = \frac{\delta\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \int_0^1 x_1^{h_1} x_2^{h_2} (1-x_1)^{\alpha_1-1} \\ \times (1-x_2)^{\alpha_2-1} x_2^{\alpha_1} [1-x_1x_2]^{-(\alpha_1+\alpha_2-\delta)} \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

Expanding $[1-x_1x_2]^{-(\alpha_1+\alpha_2-\delta)}$ by using a binomial expansion and then integrating out x_1 and x_2 and writing the resulting series in terms of a hypergeometric series, one has the following:

$$E[x_1^{h_1} x_2^{h_2}] = \delta\Gamma(\alpha_1+\alpha_2) \frac{\Gamma(h_1+1)\Gamma(\alpha_1+h_2+1)}{\Gamma(\alpha_1+h_1+1)\Gamma(\alpha_1+\alpha_2+h_2+1)} \\ \times {}_3F_2(\alpha_1+\alpha_2-\delta, h_1+1, \alpha_1+h_2+1; \alpha_1+h_1+1, \alpha_1+\alpha_2+h_2+1; 1) \tag{2.20}$$

for $\delta > 0, \Re(h_1) > -1, \Re(h_2) > -1, \alpha_1 > 0, \alpha_2 > 0$.

Theorem 2.3. *For the density in (2.16) the (h_1, h_2)-th product moment is given in (2.20).*

When $h_1 = h_2 = h$ we have

$$E[(x_1x_2)^h] = \frac{\delta\Gamma(h+1)\Gamma(\delta)}{\Gamma(\delta+h+1)} = \delta E(u^h) \tag{2.21}$$

whrere u is a real type-1 beta variable with the parameters $(1, \delta)$ or with the density

$$g(u) = \delta(1-u)^{\delta-1}, 0 \leq u \leq 1, \delta > 0.$$

When $\delta = 1$ then $u = x_1x_2$ has a uniform density over $[0, 1]$.

Corollary 2.4. *For $h_1 = h_2 = h$ in Theorem 2.3, then the h-th moment of x_1x_2 is given in (2.21).*

3. Multivariate Case

Consider the following multiple integral

$$A_k = \int_0^1 ... \int_0^1 x_1^{\alpha_1\delta_1 - 1} ... x_k^{\alpha_k\delta_k - 1}(1-x_2^{\delta_2})^{\alpha_1}(1-x_3^{\delta_3})^{\alpha_1+\alpha_2}...$$
$$\times (1-x_k^{\delta_k})^{\alpha_1+...+\alpha_{k-1}}[1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})]^{-(\alpha_1+...+\alpha_k-\delta)}\mathrm{d}x_1\wedge ...\wedge \mathrm{d}x_k. \tag{3.1}$$

Theorem 3.1. *For the A_k defined in (3.1)*

$$A_k = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_k)}{\delta\delta_1...\delta_k\Gamma(\alpha_1+...+\alpha_k)}, \Re(\alpha_j)>0, \delta>0, \delta_j>0, j=1,...,k.$$

Proof: Make the transformations $y_j = x_j^{\delta_j}$, then $z_j = 1 - y_j$ for $j = 1, ..., k$. Then, following through the steps in the case of $k = 2$, we have the following result:

$$A_k = \frac{1}{\delta_1...\delta_k}\int_0^1 ... \int_0^1 y_1^{\alpha_1-1}...y_k^{\alpha_k-1}(1-y_2)^{\alpha_1}(1-y_3)^{\alpha_1+\alpha_2}$$
$$\times ...(1-y_k)^{\alpha_1+...+\alpha_{k-1}}[1-(1-y_1)...(1-y_k)]^{-(\alpha_1+...+\alpha_k-\delta)}\mathrm{d}y_1\wedge ...\wedge \mathrm{d}y_k \tag{3.2}$$
$$= \frac{1}{\delta_1...\delta_k}\int_{z_1=0}^1 ... \int_{z_k=0}^1 (1-z_1)^{\alpha_1-1}...(1-z_k)^{\alpha_k-1}$$
$$\times z_2^{\alpha_1}...z_k^{\alpha_1+...+\alpha_{k-1}}[1-z_1...z_k]^{-(\alpha_1+...+\alpha_k-\delta)}\mathrm{d}z_1\wedge ...\wedge \mathrm{d}z_k. \tag{3.3}$$

Now, expand $[1-z_1...z_k]^{-(\alpha_1+...+\alpha_k-\delta)}$ by using a binomial expansion and integrate out $z_1, ..., x_k$. Then

$$A_k = \frac{1}{\delta_1...\delta_k}\sum_{m=0}^{\infty}\frac{(\alpha_1+...+\alpha_k-\delta)_m}{m!}\int_0^1 z_1^m(1-z_1)^{\alpha_1-1}\mathrm{d}z_1$$
$$\times \int_0^1 z_2^{\alpha_1+m}(1-z_2)^{\alpha_2-1}\mathrm{d}z_2 ... \int_0^1 z_k^{\alpha_1+...+\alpha_{k-1}+m}(1-z_k)^{\alpha_k-1}\mathrm{d}z_k$$
$$= \frac{\Gamma(\alpha_1)...\Gamma(\alpha_k)}{\delta_1...\delta_k}\frac{1}{\Gamma(\alpha_1+...+\alpha_k+1)}\sum_{m=0}^{\infty}\frac{(1)_m(\alpha_1+...+\alpha_k-\delta)_m}{(\alpha_1+...+\alpha_k+1)_m m!}$$
$$= \frac{\Gamma(\alpha_1)...\Gamma(\alpha_k)}{\delta_1...\delta_k}\frac{1}{\Gamma(\alpha_1+...+\alpha_k+1)}{}_2F_1(\alpha_1+...+\alpha_k-\delta, 1; \alpha_1+...+\alpha_k+1; 1)$$
$$= \frac{\Gamma(\alpha_1)...\Gamma(\alpha_k)}{\delta_1...\delta_k\Gamma(\alpha_1+...+\alpha_k+1)}\frac{\Gamma(\alpha_1+...+\alpha_k+1)\Gamma(\delta)}{\Gamma(\delta+1)\Gamma(\alpha_1+...+\alpha_k)}$$
$$= \frac{\Gamma(\alpha_1)...\Gamma(\alpha_k)}{\delta\delta_1...\delta_k\Gamma(\alpha_1+...+\alpha_k)}, \Re(\alpha_j)>0, \delta_j>0, \delta_j>0, j=1,...,k, \delta>0. \tag{3.4}$$

Hence

$$f(x_1,...,x_k) = \frac{\delta\delta_1...\delta_k\Gamma(\alpha_1+...+\alpha+k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)} x_1^{\alpha_1\delta_1-1}...x_k^{\alpha_k\delta_k-1}$$
$$\times (1-x_2^{\delta_2})^{\alpha_1}...(1-x_k^{\delta_k})^{\alpha_1+...+\alpha_{k-1}}$$
$$\times [1-(1-x_1^{\delta_1})(1-x_2^{\delta_2})...(1-x_k^{\delta_k})]^{-(\alpha_1+...+\alpha_k-\delta)} \quad (3.5)$$

for $\delta > 0, \delta_j > 0, \alpha_j > 0, 0 \le x_j \le 1, j = 1,...,k$, and zero elsewhere, is a statistical density.

$$f_1(y_1,...,y_k) = \frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)} y_1^{\alpha_1-1}...y_k^{\alpha_k-1}$$
$$\times (1-y_2)^{\alpha_1}(1-y_3)^{\alpha_1+\alpha_2}...(1-y_k)^{\alpha_1+...+\alpha_{k-1}}$$
$$\times [1-(1-y_1)...(1-y_k)]^{-(\alpha_1+...+\alpha_k-\delta)} \quad (3.6)$$

for $\delta > 0, \alpha_j > 0, 0 \le y_j \le 1, j = 1,...,k$, and zero elsewhere, is a statistical density.

$$f_2(z_1,...,z_k) = \frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}(1-z_1)^{\alpha_1-1},,,(1-z_k)^{\alpha_k-1} z_2^{\alpha_1}...z_k^{\alpha_1+...+\alpha_{k-1}}$$
$$\times [1-z_1...z_k]^{-(\alpha_1+...+\alpha_k-\delta)}, \delta > 0, \alpha_j > 0, 0 \le z_j \le 1, j = 1,...,k \quad (3.7)$$

and zero elsewhere, is a statistical density.

The product moments coming from (3.7), $E[z_1^{h_1}...z_k^{h_k}]$, which for $h_j = s_j - 1, j = 1,...,k$ is also the Mellin transform of the function $f_2(z_1,...,z_k)$, is the following:

$$E[z_1^{h_1}...z_k^{h_k}] = \frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)} \int_{z_1=0}^{1} ... \int_{z_k=0}^{1} z_1^{h_1}...z_k^{h_k}(1-z_1)^{\alpha_1-1}...(1-z_k)^{\alpha_k-1}$$
$$\times z_2^{\alpha_1}...z_k^{\alpha_1+...+\alpha_{k-1}}[1-z_1...z_k]^{-(\alpha_1+...+\alpha_k-\delta)}\mathrm{d}z_1 \wedge ... \wedge \mathrm{d}z_k.$$

Expanding $[1-z_1...z_k]^{-(\alpha_1+...+\alpha_m-\delta)}$, integrating out $z_1,...,z_k$ and following the steps in the previous computations, we have the following:

$$E[z_1^{h_1}...z_k^{h_k}] = \delta\Gamma(\alpha_1+...+\alpha_k)$$
$$\times \frac{\Gamma(h_1+1)\Gamma(h_2+\alpha_1+1)...\Gamma(h_k+\alpha_1+...+\alpha_{k-1}+1)}{\Gamma(\alpha_1+h_1+1)\Gamma(\alpha_1+\alpha_2+h_2+1)...\Gamma(\alpha_1+...+\alpha_k+h_k+1)}$$
$$\times {}_{k+1}F_k(\alpha_1+...+\alpha_k-\delta, h_1+1, h_2+\alpha_1+1, ..., h_k+\alpha_1+...+\alpha_{k-1}+1;$$
$$\alpha_1+h_1+1, \alpha_1+\alpha_2+h_2+1, ..., \alpha_1+...+\alpha_k+h_k+1; 1) \quad (3.8)$$

Then, when $h_j = s_j - 1, j = 1, ..., l$, (3.8) gives the Mellin transform of $f_2(z_1, ..., z_k)$, with Mellin parameters $s_1, ..., s_k$.

Theorem 3.2 *For the density given in (3.5) the* $(h_1, ..., h_k)$*-th product moment is given by (3.8)*

When $h_1 = ... = h_k = h$ then the h-th moment of the product $(z_1...z_k)$ is available from (3.8) by putting $h_1 = ... = h_k = h$. Observe that a lot of gammas are canceled in this case, and the final result is the following:

$$E[(z_1...z_k)^h] = \delta\Gamma(\alpha_1 + ... + \alpha_k)\frac{\Gamma(h+1)}{\Gamma(\alpha_1 + ... + \alpha_k + h + 1)} \times {}_2F_1(\alpha_1 + ... + \alpha_k - \delta, h+1; \alpha_1 + ... + \alpha_k + h + 1; 1).$$

But

$${}_2F_1(\alpha_1 + ... + \alpha_k - \delta, h+1; \alpha_1 + ... + \alpha_k + h + 1; 1) = \frac{\Gamma(\alpha_1 + ... + \alpha_k + h + 1)\Gamma(\delta)}{\Gamma(h+\delta+1)\Gamma(\alpha_1 + ... + \alpha_k)}.$$

Therefore

$$E[(z_1...z_k)^h] = \frac{\delta\Gamma(h+1)\Gamma(\delta)}{\Gamma(h+\delta+1)} = \begin{cases} 1 \text{ for } h = 0 \\ \frac{1}{h+1} \text{ for } \delta = 1 \\ \delta E(u^h) \end{cases} \quad (3.9)$$

where u is a real type-1 beta random variable with the parameters $(1, \delta)$, or with the density $g(u) = \delta(1-u)^{\delta-1}, 0 \le u \le 1$ and zero elsewhere. Thus, when $\delta = 1$, $z_1...z_k$ is uniform over $[0, 1]$, or $z_1...z_k$ is uniformly distributed when $\delta = 1$.

Corollary 3.1. *The* h*-th moment of the product* $z_1...z_k$ *is given in (3.9), and when* $h = 1$*, the product* $z_1...z_k$ *is uniformly distributed over* $[0, 1]$.

4. Marginal Functions

Consider $f(x_1, ..., x_k)$ of (3.5). If x_1 is integrated out then one gets the marginal function of $x_2, ..., x_k$. Let us denote the marginal function of $x_2, ..., x_k$ as $f_{2...k}$ $(x_2, ..., x_k)$. Expand $[1 - (1 - x_1^{\delta_1})...(1 - x_k^{\delta_k})]^{-(\alpha_1 + ... + \alpha_k - \delta)}$ by using a binomial expansion. Then the integral over x_1 gives the following:

$$\int_0^1 x_1^{\alpha_1\delta_1 - 1}(1 - x_1^{\delta_1})^m \mathrm{d}x_1 = \frac{1}{\delta_1}\frac{\Gamma(\alpha_1)\Gamma(m+1)}{\Gamma(\alpha_1 + m + 1)} = \frac{1}{\delta_1}\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+1)}\frac{(1)_m}{(\alpha_1+1)_m}.$$

Hence the sum over m gives

$${}_2F_1(\alpha_1 + ... + \alpha_k - \delta, 1; \alpha_1 + 1; (1 - x_2^{\delta_2})...(1 - x_k^{\delta_k})).$$

Then

$$f_{2...k}(x_2,...,x_k) = \frac{\delta\delta_2...\delta_k\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2)...\Gamma(\alpha_k)} x_2^{\alpha_2\delta_2-1}...x_k^{\alpha_k\delta_k-1}$$
$$\times (1-x_2^{\delta_2})^{\alpha_1}(1-x_3^{\delta_3})^{\alpha_1+\alpha_2}...(1-x_k^{\delta_k})^{\alpha_1+...+\alpha_{k-1}}$$
$$\times {}_2F_1(\alpha_1+...+\alpha_k-\delta,1;\alpha_1+1;(1-x_2^{\delta_2})...(1-x_k^{\delta_k})) \quad (4.1)$$

for $\alpha_1 > -1, \alpha_j > 0, \delta_j > 0, \delta > 0, j = 2,...,k$. If x_1 and x_2 are integrated out then the marginal function of $x_3,...,x_k$, denoted by $f_{3...k}(x_3,...,x_k)$, is the following:

$$f_{3...k}(x_3,...,x_k) = \frac{\delta\delta_3...\delta_k\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_3)...\Gamma(\alpha_k)} x_3^{\alpha_3\delta_3-1}...x_k^{\alpha_k\delta_k-1}$$
$$\times (1-x_3^{\delta_3})^{\alpha_1+\alpha_2}...(1-x_k^{\delta_k})^{\alpha_1+...+\alpha_{k-1}}$$
$$\times {}_2F_1(\alpha_1+...+\alpha_k-\delta,1;\alpha_1+\alpha_2+1;(1-x_3^{\delta_3})...(1-x_k^{\delta_k})). \quad (4.2)$$

In a similar manner one can integrate out the first r variables $x_1,...,x_r$ to get the marginal function of the remaining variables in terms of a ${}_2F_1$. One can also integrate out any of the r variables, need not be the first r. The steps are similar but the final expression will not be in terms of a Gauss' hypergeometric function ${}_2F_1$.

5. Some Transformations

In (3.7) let us consider the transformation $u_1 = z_1...z_k, u_2 = z_2...\, z_k, ..., u_k = z_k$. Then $\mathrm{d}u_1 \wedge ... \wedge \mathrm{d}u_k = z_2z_3^2...z_k^{k-1}\mathrm{d}z_1 \wedge ... \wedge \mathrm{d}z_k$. Note that we may write

$$z_2^{\alpha_1}z_3^{\alpha_1+\alpha_2}...z_k^{\alpha_1+...+\alpha_{k-1}} = z_2^{\alpha_1-1}z_3^{(\alpha_1-1)+(\alpha_2-1)}...z_k^{(\alpha_1-1)+...+(\alpha_{k-1}-1)}z_2z_3^2...z_k^{k-1}.$$

If the joint measure of $u_1,...,u_k$ is written as $g(u_1,...,u_k)$ then, observing that $z_j = \frac{u_j}{u_{j+1}}, j = 1,...,k-1$, we have

$$g(u_1,...,u_k) = \frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}[1-\frac{u_1}{u_2}]^{\alpha_1-1}...[1-\frac{u_{k-1}}{u_k}]^{\alpha_{k-1}-1}$$
$$\times (1-u_k)^{\alpha_k-1}u_2^{\alpha_1-1}u_3^{(\alpha_1-1)+(\alpha_2-1)}...u_k^{(\alpha_1-1)+...+(\alpha_{k-1}-1)}$$
$$\times [1-u_1]^{-(\alpha_1+...+\alpha_k-\delta)}. \quad (5.1)$$

But

$$[1-\frac{u_1}{u_2}]^{\alpha_1-1} = \frac{(u_2-u_1)^{\alpha_1-1}}{u_2^{\alpha_1-1}}, ..., [1-\frac{u_{k-1}}{u_k}]^{\alpha_{k-1}-1} = \frac{(u_k-u_{k-1})^{\alpha_{k-1}-1}}{u_k^{\alpha_{k-1}-1}}. \quad (5.2)$$

Observe that all the denominator factors are canceled and then the final result is the following:

$$g(u_1,...,u_k) = \frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}(u_2-u_1)^{\alpha_1-1}...(u_k-u_{k-1})^{\alpha_{k-1}-1}(1-u_k)^{\alpha_k-1} \times (1-u_1)^{-(\alpha_1+...+\alpha_k-\delta)}, u_1 \le u_2 \le ... \le u_k \le 1 \tag{5.3}$$

for $\delta > 0, \alpha_j > 0, j = 1,...,k$, and zero elsewhere.

Theorem 5.1. *The function $g(u_1,...,u_k)$ of (5.3) is a statistical density.*

Proof: It is easy to show that (5.3) is a probability measure or a statistical density. Integrate out $u_1,...,u_k$, starting with u_k. For example, integral over u_k gives the following:

$$\int_{u_k=u_{k-1}}^{1}(u_k-u_{k-1})^{\alpha_{k-1}-1}(1-u_k)^{\alpha_k-1}\mathrm{d}u_k = \int_{v_k=0}^{1-u_{k-1}} v_k^{\alpha_{k-1}-1}[1-u_{k-1}-v_k]^{\alpha_k-1}\mathrm{d}v_k$$
$$= (1-u_{k-1})^{\alpha_{k-1}+\alpha_k-1}\frac{\Gamma(\alpha_{k-1})\Gamma(\alpha_k)}{\Gamma(\alpha_k+\alpha_{k-1})},$$

for $\alpha_k > 0, \alpha_{k-1} > 0$. Integration over u_{k-1} yields

$$\frac{\Gamma(\alpha_{k-2})\Gamma(\alpha_k+\alpha_{k-1})}{\Gamma(\alpha_k+\alpha_{k-1}+\alpha_{k-2})}(1-u_{k-2})^{\alpha_k+\alpha_{k-1}+\alpha_{k-2}-1}.$$

Continueing like this, the integral over u_2 gives

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_k+...+\alpha_2)}{\Gamma(\alpha_k+...+\alpha_1)}(1-u_1)^{\alpha_k+...+\alpha_1-1}, \alpha_1 > 0.$$

Integral over u_1 gives

$$\int_{u_1=0}^{1}(1-u_1)^{\alpha_1+...+\alpha_k-1}(1-u_1)^{-(\alpha_1+...+\alpha_k-\delta)}\mathrm{d}u_1 = \int_0^1(1-u_1)^{\delta-1}\mathrm{d}u_1 = \frac{1}{\delta}.$$

Taking the product we get unity, and hence the result.

Consider the transformation

$$v_1 = \frac{u_2-u_1}{1-u_1}, v_2 = \frac{u_3-u_2}{1-u_1}, ..., v_{k-1} = \frac{u_k-u_{k-1}}{1-u_1}, v_k = \frac{1-u_k}{1-u_1}. \tag{5.4}$$

Theorem 5.2. *The transformation in (5.4) yields a type-1 Dirichlet measure for $v_2,...,v_k$.*

Proof: Under the transformation (5.4)

$$\frac{1}{(1-u_1)^{k-1}}\mathrm{d}u_2\wedge...\wedge\mathrm{d}u_k=\mathrm{d}v_2\wedge...\wedge\mathrm{d}v_k.$$

Note that $v_1+...+v_k=1$ or $v_1=1-v_2-...-v_k$. Then the joint measure of $v_2,...,v_k$ and u_1, denoted by $g_1(v_2,...,v_k,u_1)$, is the following:

$$\begin{aligned}g_1(v_2,...,v_k,u_1)&=\frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}(1-v_2-...-v_k)^{\alpha_1-1}v_2^{\alpha_2-1}...v_k^{\alpha_k-1}\\&\times[\frac{1}{(1-u_1)^{1-\delta}}].\end{aligned}$$

Let the joint measure of $v_2,...,v_k$ be denoted by $g_2(v_2,...,v_k)$. Then

$$\begin{aligned}g_2(v_2,...,v_k)&=\frac{\delta\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}(1-v_2-...-v_k)^{\alpha_1-1}v_2^{\alpha_2-1}...v_k^{\alpha_k-1}\\&\times\int_0^1\frac{1}{(1-u_1)^{1-\delta}}\mathrm{d}u_1\\&=\frac{\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}(1-v_2-...-v_k)^{\alpha_1-1}v_2^{\alpha_2-1}...v_k^{\alpha_k-1}.\end{aligned}\tag{5.5}$$

which is a type-1 Dirichlet measure with the parameters $(\alpha_2,...,\alpha_k;\alpha_1)$. Note that the integral over u_1 gives δ and this δ is canceled. Here (5.5) is a $(k-1)$-variate Dirichlet measure, $0\le v_j\le 1, j=2,...,k, v_2+...+v_k\le 1$. In terms of the variables $z_1,...,z_k$ of (3.7), this result can be stated as follows:

Corollary 5.1. *Let $z_1,...,z_k$ have the joint measure or joint density as in (3.7). Consider the transformation*

$$v_1=\frac{z_2...z_k(1-z_1)}{1-z_1...z_k},v_2=\frac{z_3...z_k(1-z_2)}{1-z_1...z_k},...,v_{k-1}=\frac{z_k(1-z_{k-1})}{1-z_1...z_k},v_k=\frac{1-z_k}{1-z_1...z_k}.\tag{5.6}$$

Hence $v_2,...,v_k$ have a real type-1 Dirichlet density with the parameters $(\alpha_2,\ ...,\ \alpha_k;$ $\alpha_1)$.

In terms of $y_1,...,y_k$ of (3.6) the result in (5.5) can be stated as follows:

Corollary 5.2. *Let $y_1,...,y_k$ have the joint density as in (3.6). Consider the transformation*

$$\begin{aligned}v_1&=\frac{y_1(1-y_2)...(1-y_k)}{1-(1-y_1)...(1-y_k)},v_2=\frac{y_2(1-y_3)...(1-y_k)}{1-(1-y_1)...(1-y_k)},...\\v_{k-1}&=\frac{y_{k-1}(1-y_k)}{1-(1-y_1)...(1-y_k)},v_k=\frac{y_k}{1-(1-y_1)...(1-y_k)}.\end{aligned}\tag{5.7}$$

. *Then the joint density of* $v_2, ..., v_k$ *is a real type-1 Dirichlet density with the parameters* $(\alpha_2, ..., \alpha_k; \alpha_1)$ *for* $\alpha_j > 0, j = 1, ..., k$.

In terms of the original variables $x_1, ..., x_k$ of (3.5) the result in (5.5) can be described as follows:

Corollary 5.3. *Let* $x_1, ..., x_k$ *have the joint density as in (3.5). Consider the transformation*

$$v_1 = \frac{x_1^{\delta_1}(1-x_2^{\delta_2})...(1-x_k^{\delta_k})}{1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})}, v_2 = \frac{x_2^{\delta_2}(1-x_3^{\delta_3})...(1-x_k^{\delta_k})}{1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})}, ...$$
$$v_{k-1} = \frac{x_{k-1}^{\delta_{k-1}}(1-x_k^{\delta_k})}{1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})}, v_k = \frac{x_k^{\delta_k}}{1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})}. \quad (5.9)$$

Then the joint density of $v_2, ..., v_k$ *is a real type-1 Dirichlet density with the parameters* $(\alpha_2, ..., \alpha_k; \alpha_1)$ *for* $\alpha_j > 0, j = 1, ..., k$.

Theorem 5.3. *Let* $x_1, ..., x_k$ *have the density as in (3.5). Let*

$$u = \sum_{j=2}^{k} \frac{x_j^{\delta_j}(1-x_{j+1}^{\delta_{j+1}})...(1-x_k^{\delta_k})}{1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})}.$$

Then the Mellin transform of u *or the Mellin transform of the density of* u *is equivalent to that of a real type-1 beta density with the parameters* $(\alpha_2 + ... + \alpha_k, \alpha_1)$.

Proof:

$$E[u^h] = c_k \int_0^1 ... \int_0^1 u^h x_1^{\alpha_1\delta_1 - 1} ... x_k^{\alpha_k\delta_k - 1}(1-x_2^{\delta_2})^{\alpha_1}(1-x_3^{\delta_3})^{\alpha_1+\alpha_2}$$
$$\times ...(1-x_k^{\delta_k})^{\alpha_1+...+\alpha_{k-1}}[1-(1-x_1^{\delta_1})...(1-x_k^{\delta_k})]^{-(\alpha_1+...+\alpha_k)} \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_k$$

where

$$c_k = \frac{\delta\delta_1...\delta_k\Gamma(\alpha_1+...+\alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)}, \alpha_j > 0, \delta_j > 0, j = 1, ..., k, \delta > 0.$$

Make the transformations $y_j = x_j^{\delta_j}, z_j = 1 - y_j, j = 1, ..., k$, then make the transformations $u_1 = z_1...z_k, u_2 = z_2...z_k, ..., u_{k-1} = z_{k-1}z_k, u_k = z_k$, then $v_1 = \frac{u_2 - u_1}{1-u_1}, v_2 = \frac{u_3-u_2}{1-u_1}, ..., v_{k-1} = \frac{u_k - u_{k-1}}{1-u_1}, v_k = \frac{1-u_k}{1-u_1}$. Then we note that $u = v_2 + ... + v_k$. But the joint density of $v_2, ..., v_k$ is $g_2(v_2, ..., v_k)$ of (5.5). Therefore $E[1-u]^h$ is the following, where the integration is done over the simplex $\Omega = \{(v_2, .., v_k)|0 \leq v_j \leq 1, j = 2, ..., k, v_2 + ... + v_k \leq 1\}$:

$$E[(1-u)^h] = \frac{\Gamma(\alpha_1 + ... + \alpha_k)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)} \int_\Omega (1 - v_2 - .. - v_k)^{h+\alpha_1-1} v_2^{\alpha_2-1} ... v_k^{\alpha_k-1} \mathrm{d}v_2 \wedge ... \wedge \mathrm{d}v_k$$
$$= \frac{\Gamma(\alpha_1 + h)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_1 + ... + \alpha_k)}{\Gamma(\alpha_1 + ... + \alpha_k + h)}, \Re(\alpha_1 + h) > 0. \quad (5.10)$$

Since the h-th moment, for arbitrary h, uniquely determines the distribution and since (5.10) is the h-th moment of a type-1 beta random variable with the parameters $(\alpha_1, \alpha_2 + ... + \alpha_k)$, $1 - u$ has this beta density. Therefore u has a type-1 beta density with the parameters $(\alpha_2 + ... + \alpha_k, \alpha_1)$, and hence the result.

Corollary 5.4. *For the u defined in Theorem 5.3, u and $1-u$ are both type-1 beta distributed.*

Corollary 5.5 *For the density in (3.6) let $u = \sum_{j=2}^{k} \frac{y_j(1-y_{j+1})...(1-y_k)}{1-(1-y_1)...(1-y_k)}$ then u has a type-1 beta density with the parameters $(\alpha_2 + ... + \alpha_k, \alpha_1)$ and $1 - u$ has a type-1 beta density with the parameters $(\alpha_1, \alpha_2 + ... + \alpha_k)$.*

Corollary 5.6. *Let $z_1, ..., z_k$ be as defined in (3.7). Let $u = \sum_{j=2}^{k} \frac{z_{j+1}...z_k(1-z_j)}{1-z_1...z_k}$. Then u and $1 - u$ are type-1 beta distributed as in Theorem 5.3.*

Corollary 5.7. *Let $u_1, ..., u_k$ have the density in (5.3). Let $u = \sum_{j=2}^{k} \frac{(u_{j+1}-u_j)}{1-u_1}$ with $u_{k+1} = 1$. Then u and $1 - u$ have the same densities as in Theorem 5.3, Corollaries 5.4,5.5 and 5.6.*

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