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ON CERTAIN TRANSFORMATION FORMULAS INVOLVING q-HYPERGEOMETRIC SERIES

Bindu Prakash Mishra, Sunil Singh* and Mohammad Shahjade**

Department of Mathematics,
M.D. College, Parel, Mumbai-400012, Maharashtra, INDIA.
E-mail: bindu1962@gmail.com

*Department of Mathematics, The Institute of Science,
15, Madam Cama Rd, Mantralaya, Fort,
Mumbai-400032, Maharashtra, INDIA.
E-mail: drsunilsingh912@gmail.com

**Department of Mathematics,
MANUU (Central University), Poly. 8th Cross,
1st Stage, 3rd Block, Nagarbhavi, Bangalore -560072, INDIA.
E-mail: mohammadshahjade@gmail.com

Dedicated to Prof. K. Srinivasa Rao on his 75^{th} Birth Anniversary

Abstract: In this paper transformations formulas involving q-hypergeometric series have been established. Certain identities have been deduced as special cases.

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1. Introduction, Notations and Definitions

Throughout the paper, we use the customary notation,

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{r=0}^{n-1}(1-aq^r), \quad n \geq 1,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad |q| < 1$$

and

$$(a_1, a_2, a_3, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n...(a_r;q)_n,$$

$(a_1, a_2, ..., a_r; q)_\infty = \prod_{i=1}^{r} (a_i; q)_\infty.$

Generalized basic (q-) hypergeometric series is defined as,

$$ {}_r\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} q^{\lambda n(n-1)/2}, \tag{1.1} $$

where $|q| < 1$.

Series in (1.1) converges for $|z| < \infty$ if $\lambda > 0$ and for $|z| < 1$ if $\lambda = 0$, provided no denominator parameter is of the form q^{-m} where m is a positive integer.

We shall make use of basic binomial theorem in our analysis, viz.,

$$ {}_1\Phi_0(\alpha; -; z) = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n z^n}{(q; q)_n} = \frac{(\alpha z; q)_\infty}{(z; q)_\infty}, \quad |q| < 1, \quad |z| < 1. \tag{1.2} $$

[5; App IV (IV. II)]

2. Main Result

In this section we establish the following transformation formula;

$$ {}_r\Phi_{s+1} \left[\begin{array}{l} a_1, a_2, ..., a_r; q; -xz \\ b_1, b_2, ..., b_s, \alpha x; q^\lambda \end{array} \right] $$

$$ = \frac{(x; q)_\infty}{(\alpha x; q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha; q)_m x^m}{(q; q)_m} \; {}_{r+1}\Phi_{s+1} \left[\begin{array}{l} a_1, a_2, ..., a_r, q^{-m}; q; zq^m \\ b_1, b_2, ..., b_s, \alpha; q^{\lambda-1} \end{array} \right], \tag{2.1} $$

for convergence $|q| < 1, |z| < 1$ and $|x| < 1$.

Proof of (2.1)

$$ \frac{(\alpha x; q)_\infty}{(x; q)_\infty} \; {}_r\Phi_{s+1} \left[\begin{array}{l} a_1, a_2, ..., a_r; q; -xz \\ b_1, b_2, ..., b_s, \alpha x; q^\lambda \end{array} \right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n (-1)^n (xz)^n q^{\lambda n(n-1)/2}}{(q, b_1, b_2, ..., b_s; q)_n} \times \frac{(\alpha x q^n; q)_\infty}{(x; q)_\infty}. $$

Using (1.2) we have;

$$ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n (-1)^n (xz)^n q^{\lambda n(n-1)/2}}{(q, b_1, b_2, ..., b_s; q)_n} \sum_{n=0}^{\infty} \frac{(\alpha q^n; q)_m}{(q; q)_m} x^m, $$

Putting $m-n$ for m and changing the order of summations we obtain,

$$= \sum_{m=0}^{\infty} \frac{(\alpha;q)_m x^m}{(q;q)_m} \sum_{n=0}^{m} \frac{(a_1, a_2, ..., a_r; q)_n (q^{-m}; q)_n (zq^m)^n q^{(\lambda-1)n(n-1)/2}}{(q, b_1, b_2, ..., b_s; q)_n (\alpha; q)_n},$$

which gives (2.1).

3. Special Cases

In this section we shall deduce certain special cases of (2.1).
Putting $\alpha = 0$ and $\lambda = 1$ in (2.1) we obtain the transformation,

$$ {}_r\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; -xz \\ b_1, b_2, ..., b_s; q \end{array} \right] $$

$$= (x;q)_\infty \sum_{m=0}^{\infty} \frac{x^m}{(q;q)_m} \; {}_{r+1}\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r, q^{-m}; q; zq^m \\ b_1, b_2, ..., b_s \end{array} \right]. \tag{3.1}$$

Taking $\lambda = 1$, $r = 1$, $s = 0$, $a_1 = a$ and $z = z/\alpha$ in (2.1) we get,

$$ {}_1\Phi_1 \left[\begin{array}{l} a; q; -\frac{\alpha x}{a} \\ \alpha x; q \end{array} \right] = \frac{(x;q)_\infty}{(\alpha x;q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha;q)_m x^m}{(q;q)_m} \; {}_2\Phi_1 \left[\begin{array}{l} a, q^{-m}; q; \frac{\alpha}{a} q^m \\ \alpha \end{array} \right]. \tag{3.2}$$

Summing the inner ${}_2\Phi_1$ by using [5; App. IV (IV.3)] and finally applying basic binomial theorem (1.2) we find the summation formula,

$$ {}_1\Phi_1 \left[\begin{array}{l} a; q; -\frac{\alpha x}{a} \\ \alpha x; q \end{array} \right] = \frac{(\alpha x/a;q)_\infty}{(\alpha x;q)_\infty}, \tag{3.3}$$

which is a known result [4; (13), p. 146]. As $a \to \infty$, (3.3) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n} (\alpha x)^n}{(q, \alpha x; q)_n} = \frac{1}{(\alpha x; q)_\infty}. \tag{3.4}$$

Putting $\alpha x = -\lambda q$ in (3.4) we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \lambda^n}{(q, -\lambda q; q)_n} = \frac{1}{(-\lambda q; q)_\infty}. \tag{3.5}$$

If we take $\lambda = -1$ in (3.5) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}, \tag{3.6}$$

which is known identity.
For $\lambda = 1$, (3.5) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(-q;q)_\infty} = (q;q^2)_\infty. \tag{3.7}$$

Taking $r = 1$, $a_1 = \alpha$, $s = 0$ and $\lambda = 1$ in (2.1) and applying binomial theorem (1.2) on the right hand side we get,

$$ {}_1\Phi_1 \left[\begin{array}{l} \alpha; q; -zx \\ \alpha x; q \end{array} \right] = \frac{(x;q)_\infty}{(\alpha x;q)_\infty} \, {}_2\Phi_0 \left[\begin{array}{l} \alpha, z; q; x \\ - \end{array} \right]. \tag{3.8}$$

Putting $\frac{x}{\alpha}$ for x in (3.8) and then taking $\alpha \to \infty$ we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(zx)^n}{(q,x;q)_n} = \frac{1}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(z;q)_n x^n}{(q;q)_n}. \tag{3.9}$$

Putting $z = q$ in (3.9) we find

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(x)^n}{(q,x;q)_n} = \frac{1}{(x;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} x^n. \tag{3.10}$$

Putting $x = -q$ in (3.10) and using [1; (1.1.7), p.11] we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n} = (q^2;q^2)_\infty. \tag{3.11}$$

Putting $x = aq$ in (3.10) and using a known result [1; (6.2.29), p. 152] we obtain,

$$(aq;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}a^n}{(q;q)_n(aq;q)_n} = \frac{1}{1+}\,\frac{aq}{1+}\,\frac{a(q^2-q)}{1+}\,\frac{aq^3}{1+}\,\frac{a(q^4-q^2)}{1+\ldots}. \tag{3.12}$$

Taking $\alpha = q$ in (3.8) we get

$$\sum_{n=0}^{\infty} \frac{(-zx)^n q^{n(n-1)/2}}{(x;q)_{n+1}} = \sum_{n=0}^{\infty} (z;q)_n x^n \tag{3.13}$$

Taking $r = 2$, $a_1 = \alpha$, $a_2 = a$, $s = 1$, $b_1 = b$ and $\lambda = 1$ in (2.1) we get

$$ {}_2\Phi_2 \left[\begin{array}{l} a, \alpha; q; -zx \\ b, \alpha x; q \end{array} \right] = \frac{(x;q)_\infty}{(\alpha x;q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha;q)_m x^m}{(q;q)_m} \, {}_2\Phi_1 \left[\begin{array}{l} a, q^{-m}; q; zq^m \\ b \end{array} \right]. \tag{3.14}$$

Putting $z = b/a$ and summing the inner ${}_2\Phi_1$ series on the right hand side of (3.14) we obtain

$$\sum_{n=0}^{\infty} \frac{(a, \alpha; q)_n q^{n(n-1)/2} (-bx/a)^n}{(q, b, \alpha x; q)_n} = \frac{(x; q)_\infty}{(\alpha x; q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha; q)_m (b/a; q)_m x^m}{(q; q)_m (b; q)_m}. \tag{3.15}$$

As $a \to \infty$, (3.15) yields

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n q^{n(n-1)} (bx)^n}{(q, b, \alpha x; q)_n} = \frac{(x; q)_\infty}{(\alpha x; q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha; q)_m x^m}{(q, b; q)_m}. \tag{3.16}$$

For $b = q$, (3.16) yields;

$$\sum_{n=0}^{\infty} \frac{(\alpha; q)_n q^{n^2} x^n}{(q; q)_n^2 (\alpha x; q)_n} = \frac{(x; q)_\infty}{(\alpha x; q)_\infty} \sum_{m=0}^{\infty} \frac{(\alpha; q)_m x^m}{(q; q)_m^2}. \tag{3.17}$$

Putting x/α for x and then taking $\alpha \to \infty$ in (3.17) we have,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{3}{2}n^2 - \frac{1}{2}n} x^n}{(q; q)_n^2 (x; q)_n} = \frac{1}{(x; q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m-1)/2} x^m}{(q; q)_m^2}. \tag{3.18}$$

Putting $x = q$ in (3.18) we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(q; q)_n^3} = \frac{1}{(q; q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{(q; q)_m^2}. \tag{3.19}$$

For $x = -q$, (3.18) yields;

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}}{(q; q)_n (q^2; q^2)_n} = (q; q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q; q)_m^2}. \tag{3.20}$$

Taking $r = s = 0$ in (3.1) and then using (1.2) on its right hand side we get,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} (-zx)^n}{(q; q)_n} = (xz; q)_\infty, \tag{3.21}$$

which is a known identity.
Taking $r = 1$, $s = 1$, $a_1 = a$, $b_1 = b$ in (3.1) we get

$${}_1\Phi_1 \left[\begin{array}{l} a; q; -zx \\ b; q \end{array} \right] = (x; q)_\infty \sum_{m=0}^{\infty} \frac{x^m}{(q; q)_m} \, {}_2\Phi_1 \left[\begin{array}{l} a, q^{-m}; q; zq^m \\ b \end{array} \right]. \tag{3.22}$$

Taking $z = b/a$ in (3.22) and summing the inner ${}_2\Phi_1$ series on its right hand side by using [5; App. IV (IV.3)] we find,

$$ {}_1\Phi_1\left[\begin{array}{l} a;q;-bx/a \\ b;q \end{array}\right] = (x;q)_\infty \, {}_1\Phi_1\left[\begin{array}{l} b/a;q;x \\ b \end{array}\right], \tag{3.23}$$

which is basic analogue of Kummer's transformation.
As $a \to \infty$, (3.23) yields

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(bx)^n}{(q,b;q)_n} = (x;q)_\infty \sum_{n=0}^{\infty} \frac{x^n}{(q,b;q)_n} \tag{3.24}$$

for $b = q$, (3.24) yields

$$\sum_{n=0}^{\infty} \frac{q^{n^2}x^n}{(q;q)_n^2} = (x;q)_\infty \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n^2}, \tag{3.25}$$

which is a known result [1; (6.7.4), p. 170].
Putting $x = aq$ in (3.25) we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}a^n}{(q;q)_n^2} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{q^n a^n}{(q;q)_n^2}, \tag{3.26}$$

which is known result [1; (6.7.2), p. 169].
Putting $x = q$ in (3.25) and then comparing with (3.12) we find,

$$(q;q)_\infty^2 \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n^2} = \frac{1}{1+}\frac{q}{1+}\frac{q^2-q}{1+}\frac{q^3}{1+}\frac{q^4-q^2}{1+\ldots}. \tag{3.27}$$

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