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NEW PROOFS OF TRIANGLE INEQUALITIES

Norihiro Someyama and Mark Lyndon Adamas Borongan*

Shin-yo-ji Temple, 5-44-4 Minamisenju,
Arakawa-ku, Tokyo 116-0003 JAPAN

E-mail: philomatics@outlook.jp

*Mathematics Department,
Silliman University, 1 Hibbard Avenue,
Dumaguete City, Negros Oriental, 6200 PHILIPPINES

E-mail: lyndonaborongan@su.edu.ph

Dedicated to Prof. A.K. Agarwal on his 70^{th} Birth Anniversary

Abstract: We give three new proofs of the triangle inequality in Euclidean Geometry. There seems to be only one known proof at the moment. It is due to properties of triangles, but our proofs are due to circles or ellipses. We aim to prove the triangle inequality as simple as possible without using properties of triangles. In other words, we suggest proofs without using paper and pen.

Keyword and Phrases: Triangle inequality, circle, ellipse, focci of ellipse, Euclidean geometry, algebraic geometry.

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1. Introduction

We consider triangle inequalities for triangles on Euclidean space. For convenience, let the dimension of the space be 2. We write $\triangle\mathrm{ABC}$ for the triangle with three vertices $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathbb{R}^2$. Hereafter, AB denotes the segment from a point A to a point B, and $\angle\mathrm{ABC}$ the angle made by segments AB, BC. Moreover, $\overline{\mathrm{PQ}}$ stands for the length of the segment PQ in $\mathbb{R}^2$.

The triangle inequality asserts that the sum of any two sides of a triangle is *strictly* bigger than the remaining third side. This geometric inequality is well known as one of the most fundamental and classical theorems in Euclidean geometry:

Theorem 1.1. (Triangle Inequalities). *For any triangle* $\triangle$ABC*, an inequality*

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} > \overline{\mathrm{BC}} \tag{1.1}$$

holds (regardless of the dimension of the space).

This was probably proved by the ancient Greeks for the first time, but the proof is still considered important. See Subsection 1.1 for details of 'the first proof'. However, we want to give new and more natural proofs in this article. Furthermore, it is also well known that we can prove the triangle inequality *in the broad sense*, i.e. the '$\geq$'-version of (1.1), by algebraic argument. It is not exactly the triangle inequality in the sense of Euclidean geometry, because the point A is on the segment BC in case $\overline{\mathrm{AB}} + \overline{\mathrm{AC}} = \overline{\mathrm{BC}}$. For details of them, Subsection 1.2 (in particular Remark 1.3) will mention.

Throughout this article, we always assume the following:

Assumption 1.1. BC is the longest side among three sides of $\triangle$ABC. That is,

$$\overline{\mathrm{BC}} > \max\{\overline{\mathrm{AB}}, \overline{\mathrm{AC}}\}. \tag{1.2}$$

1.1. The Best Known Proof by Euclidean Geometrical Method

We consider a triangle $\triangle$ABC with Assumption 1.1 and plot a point P, obeying

$$\overline{\mathrm{AC}} = \overline{\mathrm{AP}}, \tag{1.3}$$

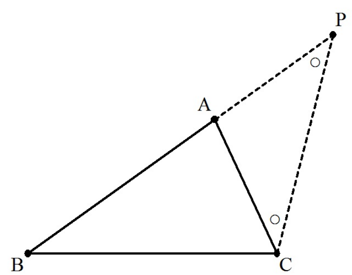

Figure 1:

on an extension of the segment BA. (See Figure 1.) Then,

$$\angle\mathrm{BCP} > \angle\mathrm{ACP} = \angle\mathrm{APC} = \angle\mathrm{BPC}. \tag{1.4}$$

Hence, $\triangle$PBC satisfies

$$\overline{\mathrm{BC}} < \overline{\mathrm{BP}} = \overline{\mathrm{BA}} + \overline{\mathrm{AP}} = \overline{\mathrm{AB}} + \overline{\mathrm{AC}},$$

from (1.3) and (1.4). This completes the proof.

Remark 1.1. The above proof also applies to obtuse angled triangles, but there is another short proof for obtuse angled triangles. In fact, if $\angle$BCA is the maximum angle,

$$\overline{\mathrm{AB}} > \max\{\overline{\mathrm{BC}}, \overline{\mathrm{AC}}\}$$

holds, so we obviously gain (1.1).

1.2. Proofs of Algebraic Triangle Inequalities

The names 'triangle inequalities' often appear in some fields such as linear algebra and functional analysis. These 'triangle inequalities' are able to argue by algebraic method.

1.2.1. Triangle Inequalities for Absolute Values of Vectors

We consider three vectors

$$\boldsymbol{a} := \overrightarrow{\mathrm{BA}}, \quad \boldsymbol{b} := \overrightarrow{\mathrm{AC}}, \quad \boldsymbol{c} := \overrightarrow{\mathrm{BC}}.$$

Then, since $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, it is sufficient to prove

$$|\boldsymbol{a} + \boldsymbol{b}| < |\boldsymbol{a}| + |\boldsymbol{b}| \tag{1.5}$$

so as to obtain (1.1), where $| \quad |$ is an Euclidean norm:

$$|\boldsymbol{x}| := \sqrt{x_1^2 + x_2^2} \qquad \text{for} \quad \boldsymbol{x} = (x_1, x_2).$$

Note that, of course, $|\boldsymbol{a}|, |\boldsymbol{b}| > 0$. To see (1.5), we consider the following equation:

$$(|\boldsymbol{a}| + |\boldsymbol{b}|)^2 - |\boldsymbol{a} + \boldsymbol{b}|^2 = 2(|\boldsymbol{a}||\boldsymbol{b}| - \boldsymbol{a} \cdot \boldsymbol{b}) \tag{1.6}$$

where '$\cdot$' is the inner product. The inner product defines by

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos(180^\circ - \angle\mathrm{BAC}) = -|\boldsymbol{a}||\boldsymbol{b}| \cos\angle\mathrm{BAC}$$

where $0^\circ < \angle\mathrm{BAC} < 180^\circ$. Thus,

$$|\boldsymbol{a}||\boldsymbol{b}| - \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|(1 + \cos\theta) > 0, \tag{1.7}$$

since $0 \leq \cos\theta < 1$. Hence, (1.5) is proved from (1.6) and (1.7).

Remark 1.2. It follows in more general that

$$|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|. \tag{1.8}$$

Let us call this the *algebraic triangle inequality.* To see (1.8), we consider (1.6) and the Cauchy-Schwarz-Bunyakovsky's inequality

$$|\boldsymbol{a} \cdot \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|.$$

They immediately derive (1.8).

1.2.2. Algebraic Triangle Inequalities for Complex Numbers

(1.5) can be shown by using complex values. We write

$$\boldsymbol{a} := (a_1, a_2), \qquad \boldsymbol{b} := (b_1, b_2)$$

for two real vectors in Subsection 1.2.1. Since the mapping $\mathbb{R}^2 \ni (x, y) \mapsto x+iy \in \mathbb{C}$ is isomorphic: $\mathbb{R}^2 \approx \mathbb{C}$, let $\boldsymbol{a}, \boldsymbol{b}$ correspond to two complex numbers

$$z_1 := a_1 + ia_2 = r_1 e^{i\theta_1}, \qquad z_2 := b_1 + ib_2 = r_2 e^{i\theta_2}$$

respectively. Here $i := \sqrt{-1}$ and $0^\circ < \theta_j < 180^\circ$ $(j = 1, 2)$. Note that $\theta_1 \neq \theta_2$. Then, we should prove

$$|z_1 + z_2| < |z_1| + |z_2| \tag{1.9}$$

to obtain (1.5). We can see (1.9) as follows:

$$\begin{aligned}
|z_1 + z_2| &= |(r_1 \cos\theta_1 + r_2 \cos\theta_2) + i(r_1 \sin\theta_1 + r_2 \sin\theta_2)| \\
&= \sqrt{(r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2} \\
&= \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)} \\
&< \sqrt{r_1^2 + 2r_1 r_2 + r_2^2} \\
&= r_1 + r_2 \\
&= |z_1| + |z_2|.
\end{aligned}$$

We here use that $|\cos(\theta_1 - \theta_2)| < 1$. Hence, this completes the proof.

Remark 1.3. More generally, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.10}$$

This corresponds to (1.8). The condition for (1.8) (resp. (1.10)) to become an equation is what $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel (resp. that $\theta := \theta_1 - \theta_2 = n\pi$ for any $n \in \mathbb{Z}$). Geometrically, triangles cannot of course be made if these conditions are satisfied, so the algebraic triangle inequalities such as (1.5) and (1.9) are triangle inequalities in the broad sense.

2. New Proofs by Circles

Let us present our proofs of triangle inequalities from this section. This section gives us two proofs of (1.1) by circles.

2.1. New Proof 1 by Circles

We consider two circles, one is a circle C_1 whose center is the point B and whose radius is the segment AB, and the other is a circle C_2 whose center is the point C and whose radius is the segment AC. (See Figure 2.) If

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} \leq \overline{\mathrm{BC}},$$

then C_1 and C_2 come in contact with each other or become separated from each other. In either case, segments AB, BC and AC never make a triangle. Hence, (1.1) must hold.

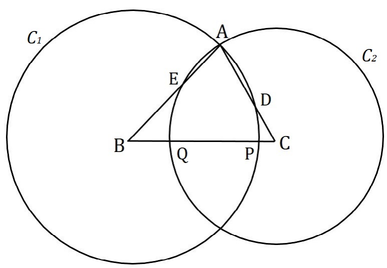

Figure 2:

2.2. New Proof 2 by Circles

If C_1 and C_2 come in contact with each other, the point A is on the segment BC. So, it is sufficient to argue only the case that C_1 intersects C_2 at two points. Then, A is one of two points of intersection of C_1 and C_2. (See Figure 2.) Remark that the point C (resp. B) is outside C_1 (resp. C_2) from (1.2). We write P for a point of intersection of BC and C_1. Similarly, we write Q for a point of intersection of BC and C_2. Then,

$$\overline{\mathrm{BC}} = \overline{\mathrm{BP}} + \overline{\mathrm{PC}} = \overline{\mathrm{AB}} + \overline{\mathrm{PC}}$$

because $\overline{\mathrm{AB}} = \overline{\mathrm{BP}}$. So it is sufficient to prove

$$\overline{\mathrm{AC}} > \overline{\mathrm{PC}}, \tag{2.1}$$

but, since C_1 intersects C_2, we obviously obtain (2.1) as follows:

$$\overline{\mathrm{PC}} < \overline{\mathrm{QC}} = \overline{\mathrm{AC}}.$$

Hence, this completes the proof.

Remark 2.1. A proof for obtuse angled triangles is known and is similar to the above proof. (See [2] for example.) That is, if ∠BAC is the maximum angle, we consider a circle whose center is C and whose radius is BC. We leave the details to the reader.

3. New Proof by Ellipses and Properties of $\mathbb{R}$

We first give a *partial* proof of the triangle inequality by ellipses in Section 3.1. We next complement the proof by certain property of real numbers in Section 3.2. That complementary proof is however only an appendix.

3.1. Proof for Special Triangles

We consider an ellipse

$$\mathcal{E}: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

and set that vertices B and C of △ABC are focci of $\mathcal{E}$ given by

$$\mathrm{B}(-\sqrt{a^2 - b^2}, 0), \quad \mathrm{C}(\sqrt{a^2 - b^2}, 0).$$

Then, if the vertex A of △ABC is any point on $\mathcal{E}$, we gain

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} = 2a \tag{3.1}$$

by definition of an ellipse. If $\mathrm{P}(a, 0)$, it satisfies that

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} = \overline{\mathrm{BP}} + \overline{\mathrm{PC}} \ (= 2a) \tag{3.2}$$

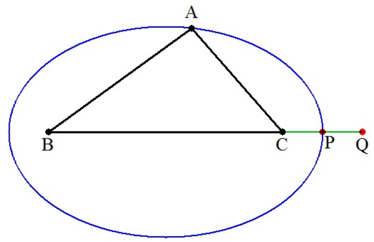

Figure 3:

from (3.1). We plot a point Q, on an extension of the segment BP, obeying

$$\overline{\mathrm{PC}} = \overline{\mathrm{PQ}}. \tag{3.3}$$

(See Figure 3.) Hence, by (3.2) and (3.3),

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} = \overline{\mathrm{BP}} + \overline{\mathrm{PQ}} = \overline{\mathrm{BQ}} > \overline{\mathrm{BC}},$$

so (1.1) holds if (3.1).

From the above, we can also see that (1.1) holds if

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} > 2a. \tag{3.4}$$

Thus, we obtain the following result:

Proposition 3.1. *Let $a > 0$ be the semi-major axis of $\mathcal{E}$. If*

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} \geq 2a,$$

then (1.1) holds.

Remark 3.1.

1) The above proof of Proposition 3.1 is the argument by analytic geometry, but we can also prove by the algebraic argument as follows:

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} \geq 2a > 2\sqrt{a^2 - b^2} = \overline{\mathrm{BC}}.$$

2) The above proof is acceptable wheather $\triangle$ABC is acute or not.

3.2. Appendix: Completion of the Proof

As we saw in Section 3.1, it was easy to see (1.1) if

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} \geq 2a.$$

This is a partial proof of (1.1). In order to complete the proof of (1.1), it is necessary to prove the following statement.

Proposition 3.2. *Let $a > 0$ be the semi-major axis of $\mathcal{E}$. If*

$$\overline{\mathrm{AB}} + \overline{\mathrm{AC}} < 2a,$$

then (1.1) holds.

To see this, we use the following property for real numbers.

Lemma 3.3. *(e.g. [1], p.42) Let $\alpha, \beta, \gamma \in \mathbb{R}$. If $\alpha < \gamma$ for any γ such that $\beta < \gamma$, then $\alpha \leq \beta$.*

Remark 3.2. Lemma 3.3 is a statement which paraphrases the following well-known proposition used frequently in measure theory:

'*Let $a, b \in \mathbb{R}$. If $a < b + \varepsilon$ for any $\varepsilon > 0$, then $a \leq b$.*'

To see Lemma 3.3 directly, we use reductio ad absurdum. In fact, we should assume $\alpha > \beta$ and set $\gamma = (\alpha + \beta)/2$.

Actually, we can immediately prove Proposition 3.2 by putting

$$\alpha = \overline{\mathrm{BC}}, \quad \beta = \overline{\mathrm{AB}} + \overline{\mathrm{AC}}, \quad \gamma = 2a$$

in Lemma 3.3.

Hence, by virtue of Proposition 3.1 and Proposition 3.2, this completes the proof of (1.1).

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