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ON ANALYSIS OF THE FRÉCHET DERIVATIVES

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Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: This work reviews various aspects of the Fréchet derivatives basically from the first principle through some numerous results bordering on the basic theories. On displaying thorough proofs to the numerous analytical results, we hence showed that the Fréchet derivatives is not a number at all but a linear operator and in the end we succeeded in establishing the existence of various generalized results on the Fréchet derivatives.

Keyword and Phrases: Banach Space, Operator, Derivatives, Chain Rule, Mean Value Theorem and Implicit Function Theorem.

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1. Introduction

Definition 1.1. *Let T be an operator mapping a Banach space X into a Banach space Y. If there is a bounded linear operator φ from S a bounded subset of X into Y such that*

$$T\left(x+\Delta x\right)-T\left(x\right)=T'\left(x\right),\Delta x+\varphi\left(x,\Delta x\right),$$

, and

$$\underset{\|x\|\to 0}{Lim}\frac{\|\varphi\left(x,\Delta x\right)\|}{\|\Delta x\|}=0 \tag{1.1A}$$

or

$$\underset{\|\Delta x\|\to 0}{Lim}\frac{\|T\left(x_0+\Delta x\right)-T\left(x_0\right)-\varphi\left(\Delta x\right)\|}{\|\Delta x\|}=0 \tag{1.1B}$$

or equivalently

$$\underset{\|t\|\to 0}{Lim}\frac{\|T(x+t)-T(x)-\varphi(t)\|}{\|t\|}=0 \tag{1.1C}$$

1.1A and 1.1B are due to [3]. For Banach space X and Y and an open subset, S with the map $T: S \to Y, x \in N$ be such that

$$T(x+\Delta x)-T(x)=T'(x)\Delta x+\varphi(x,\Delta x)$$

for every $h \in X$ with $x+h \in S$ where $T'(x): X \to Y$ is a linear operator and

$$\lim_{\Delta x\to 0}\frac{\varphi(x,\Delta x)}{\Delta x}=0$$

where $T'(x)\Delta x$ is the Fréchet differential of T at x with increment Δx. It is important to note that with (1.1) satisfied, T is said to be Fréchet differential at x_0 and the bounded linear operator

$$T'(x_0)=\varphi \tag{1.2}$$

is called the Fréchet derivative of T at x_0 and the limit in (1.1) is supposed to hold independently of the way that Δx is approaches 0 and the he Fréchet differential

$$\partial T(x_0,\Delta x)=T'(x_0)\Delta x \tag{1.3}$$

is an arbitrary closed approximation to the difference

$$T(x_0+\Delta x)-T(x_0)$$

relative to Δx for Δx small.

Definition 1.2. *A definition by induction due to a Mosaic states that if T is $(m-1)$ times Fréchet differentiable ($m \geq 2$, an integer) and an m linear operator L from X into Y exists such that*

$$\underset{\|\Delta x\|\to 0}{lim}\frac{\|T^{(m-1)}(x_0+\Delta x)-T^{(m-1)}(x_0)-\varphi(\Delta x)\|}{\|\Delta x\|}=0 \tag{1.4}$$

is due to [3] Then φ is called the $m-$ Fréchet derivative of T at x_0 and

$$L=T^m(x_0) \tag{1.5}$$

Note the Fréchet derivative of T at x is usually denoted by $dT(x)$ or $T'(x)$, is called Fréchet differentiable on its domain if $T'(x)$ exists at every point of the domain.

Remark 1.1.

a) *if* $X = R,\ Y = R$, *then the classical derivative* $T^{'}(x)$ *of real function* $T : R \to R$ *at* x *defined by*

$$T'(x) = \lim_{\Delta x \to 0} \frac{T(x + \Delta x) - T(x)}{\Delta x} \tag{1.6}$$

is a number representing the slope of the graph of the function T *at* x. *The Fréchet derivative of* T *is not a number, but a linear operator on* R *into* R. *The existence of the classical derivative* $T^{'}(x)$ *implies the existence of the Fréchet derivative at* x, *and by comparison of the right-hand side and left-hand side of equation (1.6) below*

$$T(x + \Delta x) - T(x) = T^{'}(x)\Delta x + \Delta F(\Delta x)$$

we find that S *is the operator which multiplies every* Δx *by the number* $T^{'}(x)$.

b) *In elementary calculus, the derivative at a point* x *is a local linear approximation of the given function in the neighborhood of* x. *Similarly, the Fréchet derivative can be interpreted as the best local linear approximation. We consider the change in* T *when its argument changes from* Δx *to* $x + \Delta x$ *and then approximates this change by a linear operator* φ *so that*

$$T(x + \Delta x) = T(x) + \varphi(\Delta x) + E,$$

where E *is the error in the linear approximation. Thus,* E *has the same order of magnitude as* Δx *except when* φ *is equal to the Fréchet derivative of*T. $E = 0(\Delta x)$, *so that*E *is much smaller than* Δx *as* $\Delta x \to 0$. *In this way, the Fréchet derivative gives the best linear approximation of* T *near* x.

c) *It is clear from the definition that if* T *is linear, then*

$$dT(x) = T(x)$$

that is, if T *is a linear operator, then the Fréchet derivative (linear approximation) of* T *is* T *itself.*

2. Basic Fréchet Results

Theorem 2.1. [2] *If an operator has the Fréchet derivative at a point, then it has the Gateaux derivative at that point and both derivatives have equal values.*
Proof. Let $T : X \to Y$, and let T have the Fréchet derivative at x, then

$$\lim_{(\|\Delta x\| \to 0)} \frac{\|T(x + \Delta x) - T(x) - \varphi(\Delta x)\|}{\|\Delta x\|} = 0$$

for some bounded linear operator $S : X \to Y$. In particular for any fixed nonzero $\Delta x \in X$, we have

$$\left\| \lim_{\eta \to 0} \frac{T(x+\eta\Delta x) - T(x)}{\eta} - S(\Delta x) \right\| = \lim_{\eta \to 0} \frac{\|T(x+\eta\Delta x) - T(x) - S(\Delta x)\|}{\|\Delta x\|} \|\Delta x\|$$

this implies that K is the Gateaux derivative of T at x. In view of the uniqueness Theorem of Gateaux derivative is unique and here Fréchet derivative is also unique. However example show that the converse theorem is not true, in general.

Theorem 2.2. [8] *Let Ω be an open subset of X and $T : \Omega \to Y$ be Fréchet derivative at an arbitrary point a of Ω. Then T is continuous. This means that every Fréchet differentiable operator defined on a subset of a Banach space is continuous.*
Proof. For $a \in \Omega$, let $\varepsilon > 0$ be such that $a + t \in \Omega$ whenever $\|t\| < \varepsilon$. Then

$$\|T(a + \Delta x) - T(a)\| = \|K(\Delta x) + \varphi(a, \Delta x)\| \to 0$$

This means that T is continuous at a.

Most of the results of classical calculus can be easily generalized Fréchet derivatives. For example, the usual rules of sum and product produce cases of functions of two or more that variable apply to Fréchet derivative.

We now present extensions of the chain theorem, the mean value theorem, the implicit function theorem and Taylor's formula to Fréchet differentiable operators.

Theorem 2.3. (CHAIN RULE) [8] *Let X, Y, Z be real Banach spaces, $S : X \to Y$ and $T : Y \to Z$ are Fréchet differentiable at x and $y = S, Y$, respectively, then $U = T \circ S$ is Fréchet differentiable at x and*

$$U'(x) = T'(S(x))S'(x)$$

Proof. For $x, t \in X$, we have

$$U(x + \Delta x) - U(x) = T(S(x + \Delta x) - T(S(x))$$

$$= T(S(x + \Delta x) - S(x) + S(x) - T(y) = T(x + y) - T(y)$$

where $z = S(x + \Delta x) - S(x)$. Thus,

$$\|U(x + \Delta x) - U(x) - T'(y)z\| = \circ(\|z\|)$$

$$\|U(x + \Delta x) - U(x) - T'(y)S'(x)\Delta x\|$$
$$= \|U(x + \Delta x) - U(x) - T'(y)z + T'(y)S'(x)\Delta x\|$$

$$= \circ(\|\Delta x\|) + \circ(\|z\|)$$

In view of the fact that S is continuous at x, and by a known theorem, we obtain $\|z\| = \circ(\|\Delta x\|)$ and so

$$U^{'}(\Delta x)t = T^{'}(S(x))S^{'}(x)\Delta x$$

We require the following notation in the Mean Value Theorem and Taylor's formula. If a and b are two points of a vector space, the notation

$$[a, b] = x = \alpha a + (1-\alpha)b \in X/\alpha \in [0, 1]$$

$$[a, b] = x = \alpha a + (1-\alpha)b \in X/\alpha \in (0, 1)$$

are used to denote, respectively, the closed and open segments with end-point a and b.

Theorem 2.4. (MEAN VALUE THEOREM) [8] *Let $T : K \to Y$, where is an open convex set containing a and b,Y is a normed space and $T^{'}(x)$ exists for each $x \in [a, b]$ and $T(x)$ is continuous on $[a, b]$. Then*

$$\|T(b) - T(a)\| \leq \sup_{(y \in [a,b])} \|T^{'}(y)\| \|b - a\|$$

Proof. Let $F \in T^{*}$ and the real function h be defined by

$$h(\alpha) = F((T(1 - \alpha(b))) \; for \; \alpha \in [0, 1].$$

Applying the mean value theorem of the classical calculus to h, we have, for some $\bar{a} \in [0, 1]$ and $\bar{x} \; = (1 - \bar{\alpha})a + \bar{a}b)$,

$$F(T(b) - T(a) = F(T(b) - F(T(a)) = h(1) - h(0)$$

$$= h^{'}(\bar{a}) = F(T^{'}(\bar{x})(b - a)),$$

where we have used the chain rule and the fact that a bounded linear functional is its own derivative. Therefore, for each $F \in Y^{*}$,

$$\|F(T(b) - T(a))\| \leq \|F\| \|T^{'}(\bar{x})\| \|b - a\|$$

now, if we define the function $F_{\circ}$ on the subspace $[T(b) - T(a)]$ of Y

$$F_{\circ}(\lambda(F(B)) - F(a) = \lambda,$$

then $\|F_\circ\| = \|T(b) - T(a)\|^{-1}$. If F is a Hahn-Banach extension of $F_\circ$ the entire space Y, by making required substitution, then

$$1 = \|F(T(b) - T(a))\|T(b) - T(a)\|^{-1}\|T^{'}(\bar{x})\|\|b - a\|$$

Theorem 2.5. (IMPLICIT FUNCTION THEOREM) [2] *Suppose that X, Y, Z are Banach spaces, C is an open subset of $X \times Y$ and $T : C \to Z$ is continuous. Suppose further that for some $(x_1, y_1) \in C$*

i $T(x_1, y_1) = 0$,

ii *The Fréchet derivative of $T(x_1)$ when x is fixed is denoted by $T_y(x, y)$, called the partial Fréchet derivative with respect to y, exists at each point in a neighborhood of (x_1, y_1) and is continuous at (x, y)*

iii $[T_y(x_1, y_1)]^{-1} \in B[Z, Y]$.

Then there is an open subset D of X containing x, and a unique continuous mapping $y : D \to Y$ such that $T(x, y(x)) = 0$ and $y(x_1) = y_1$.

Proof. For the sake of convenience. We may take $x_1 = 0, y_1 = 0$. Let $A = [T_y(0, 0)]^{-1} \in B[Z, Y]$. Since C is an open set containing $(0, 0)$, we find that $0 \in C_x = y \in Y/(x, y) \in C$ for all x sufficiently small, say $\|x\| \leq \delta$. For each x having this property, we define a function

$$S(x, .) : C_x \to Y \tag{2.2.1}$$

by

$$S(x, y) = y - AT(x, y) \tag{2.2.2}$$

in order to prove the theorem, we must prove (i) the existence of a fixed point for $S(x, .)$ under the condition that $\|x\|$ is sufficiently small, and (ii) continuity of the mapping $x \to y(x)$ and $y(x_1) = y_1$. Now

$$S_y(x, y)(u) = u - AT_y(x, y)(u)$$

and $AA^{-1} = AT_y(0, 0)$, therefore, assumption on T guarantee the existence of $S_y(x, y)$ for sufficiently small $\|x\|$ and$\|y\|$

$$S_y(x, y)(u) = A[T_y(0, 0) - T_y(x, y)](u).$$

Hence

$$\|S(x, y)\| \leq = \|A\|\|T_y(0, 0) - T_y(x, y)\|.$$

Since T_y is continuous at $(0,0)$. There exists a constant $L>0$ such that

$$\|S_y(x,y)\|\leq L \tag{2.2.3}$$

for sufficiently small $\|x\|$ and $\|y\|$, say, $\|x\|\leq\epsilon_1\leq\delta$ and $\|y\|\leq\epsilon_2$. Since T is continuous at$(0,0)$, there exists an $\varepsilon\leq\varepsilon_1$ such that

$$\|S\left(x,0\right)\|=\|AT\left(x,0\right)\|<\epsilon_2\left(1-L\right) \tag{2.2.4}$$

for all x with $\|x\|\leq\epsilon$. We now show that $S(x,\centerdot)$ maps the closed ball $(\bar{S_\epsilon})(0)=\{y{\in}Y/\|y\|{\leq}\epsilon_2\}$ into itself. For this, let $\|x\|\leq\epsilon$ and $\|y\|\leq\epsilon_2$. Then by (2.2.1), (2.2.3) and (2.2.4), we have

$$\|S(x,y)\|\leq\|S(x,y)-S(x,0)\|+\|S(x,0)\|$$

$$\leq\underset{0\leq\epsilon\leq1}{sup}\|S_y(x,y)\|\|y\|+\|S(x,0)\|$$

$$\leq L\epsilon_2+\epsilon_2(1-L)=\epsilon_2$$

Therefore, for $\|x\|<\epsilon, S\left(x,.\right):\bar{S}\epsilon_2(0)\to\bar{S}\epsilon_2(0)$. Also, for $y_1,y_2\in\bar{S}\epsilon_2(0)$, we obtain by (2.2.1) and (2.2.3)

$$\|S(x,y_1)-S(x,y_2)\|\leq\underset{\|y\|\leq\epsilon_2}{sup}\|S_y(x,y)\|\|y_1-y_2\|{\leq}L\|y_1-y_2\|$$

The Banach Contraction Mapping Theorem guarantees that for each x with $\|x\|<\epsilon$, there exists a unique $y(x)\in\bar{S}\epsilon_2(0)$ such that

$$y(x)=S(x,y(x))=y(x)-AT(x,y(x))$$

that is, $T(x,y(x))=0$. By uniqueness of y, we have $y(0)=0$ since $T(0,0)=0$. Finally, we show that $x\to y(x)$ is continuous. For if $\|x\|<\epsilon$ and $\|x_2\|<\epsilon$, then selecting$y_0=y(x_2)$ and $y_1=S(x_1,y_0)$, we have by the error bound for fixed point iteration on the mapping $S(x_1,\centerdot)$

$$\|y(x_2)-y(x_1)\|\leq 1/(1-L)\|y_0-y_1\|$$

We can write

$$y_0-y_1=y(x_2)-S(x_1,y(x_2)=S(x_2,y(x_2))-S(x_1,y(x_2))$$

$$=-A[T(x_2,y(x_2)-T(x_1,y(x_2))]$$

Therefore, by the continuity of T, $\|y(x_2) - y(x_1)\|$ can be made arbitrarily small for $\|x_2 - x_1\|$ sufficiently small.

Corollary 2.6. [2] *If, in addition to conditions of the above theorem $T_x(x, y)$, also exists on the open set, and is continuous at (x_1, y), then $F : x \to y(x)$ has a Fréchet derivative at x_1 given by*

$$F^{'}(x_1) = -[T_y(x_1, y_1)]^{-1}T_x(x_1, y)$$

Proof. We set $x = x_1 + h$ and $G(h) = F(x) - y_1$. Then $G(0) = 0$, and

$$\|G(h) + [T_y(x_1, y_1)]^{-1}h\| \leq \|[T_y(x_1, y_1)]^{-1}\|\|T_y(x_1, y_1)G(h) + T_x(x_1, y_1)h\|$$

and

$$T_y(x_1, y_1)G(h) - T_x(x_1, y_1)$$
$$= -T(x_1 + h, y_1 + G(h)) + T(x_1, y_1) + T_y(x_1, y_1)G(h) + T_x(x_1, y_1)h.$$

if θ_1, θ_2 are numbers in $(0, 1)$, then

$$\|T_y(x_1, y_1)G(h) + T_x(x_1, y_1)h\|$$

$$\leq \underset{\theta_1,\theta_2}{sup}\|T_x(x_1 + \theta_1 h, y_1 + \theta_2 G(h)) - T_x(x_1, y_1)\|\|h\|$$

$$+\underset{\theta_1,\theta_2}{sup}\|T_y(x_1 + \theta_1 h, y_1 + \theta_2 G(h)) - T_y(x_1, y_1)\|\|G(h)\|$$

Thus, applying continuity of T_x, T_y for $\epsilon > 0$, we find that $\delta = \delta(\epsilon)$ such that on $\|x - x_1\| \leq \delta$, we have

$$\|G(h)+[T_y(x_1, y_1)]^{-1}T_x(x_1, y_1)h\| \leq \frac{[T_y(x_1, y_1)]^{-1} \in [1 + [T_y(x_1, y_1)]^{-1}T_x(x_1, y_1)]\|h\|)}{1 - \|[T_y(x_1, y_1)]^{-1}\|}$$

The coefficient of $\|h\|$ can be made as small as needed as $\|h\| \to 0$. Thus,

$$\|F(x) - [F(x_1) - [T_y(x_1, y_1)]^{-1}T_x(x_1, y_1)(x - x_1)] = 0\,(\|x - x_1\|)\,\|$$

Definition 2.7. *If $T : X \to Y$ is Fréchet differentiable on an open set $\Omega \subset X$ and the first Fréchet derivative $T^{'}$ at$x \in \Omega$ is Fréchet differentiable at x, then the Fréchet derivative of $T^{'}$ at x is called the second derivative of T at x and is denoted by $T^{"}(x)$. It may be observed that if $T : X \to Y$ is Fréchet differentiable on an open set $\Omega \subset X$, then $T^{'}$ is a mapping on X into $B[X, Y]$. Consequently, if $T^{"}(x)$ exists, it is a bounded linear mapping from X into $B[X, Y]$. If $T^{"}$ exists at every* *point of* Ω*, then* $T^{"} : X \to B[X, [X, Y]]$.

Theorem 2.8. (TAYLOR'S FORMULA FOR DIFFERENTIABLE FUNCTIONS) [3] *Let* $T : \Omega \subset\to Y$ *and let* $[a, a+h]$ *be any closed segment in* Ω*. If* T *is Fréchet differentiable at* a *, then*

$$T(a+h) = T(a) + T^{'}(a)h + \|h\|\epsilon(h),\ \underset{(h\to 0)}{lim}\,\epsilon(h) = 0$$

Theorem 2.9. (TAYLOR'S FORMULA FOR TWICE FRÉCHET DIFFERENTIABLE FUNCTIONS) [3] *Let* $T : \Omega \subset X \to Y$ *and let* $[a, a+h]$ *be any closed segment lying in* Ω*. If* T *is differentiable in* Ω *and twice differentiable at* a*, then*

$$T(a+h) = T(a) + T^{'}(a)\,h + \frac{1}{2}\left(T^{"}(a)\,h\right)h + \|h\|^2\epsilon(h)\,,\underset{h\to 0}{lim}\epsilon(h) = 0$$

for proofs of these two theorems and other related results, we refer to Cartan[3] and Nashed[6].

Theorem 2.10. [15] *Let* $f : R^3 \to R$ *possess continuous second partial derivatives with respect to all three variable, and let* $F : C^1[a, b] \to R$ *be defined by*

$$F(x) = \int_a^b f(x(t), x^{'}(t), t)dt$$

Then the Fréchet derivative of $F, dF(x)h$*, is given by*

$$dF(x)h = \int_b^a \left[\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial x'}\right)\right] hdt + \left[\frac{\partial f}{\partial x'}\right]_b^a$$

Proof.

$$F(x+h) - F(x) = \int_b^a f(x(t)+h(t), x^{'}(t)+h(t), t)dt$$

$$-f(x(t), x^{'}(t), t)dt = \int_b^a \left(\frac{\partial f}{dx}(x(t), x^{'}(t), t)h(t)\right)$$

$$+\frac{\partial f}{\partial x'}(x(t), x^{'}(t), t)h(t)dt + r(h, h)$$

Where $r(h, h) = 0\left(\|h\|_{C[a,b]}\right)$, i.e.,

$$\frac{|r(h, h)|}{\|h\|_{C^1[a,b]}}) \to 0as\|h\|C[a, b] \to 0$$

Hence,

$$dF(x)h = \int\limits_b^a \left[\frac{\partial f}{\partial x}(x(t), x^{'}(t), t)h(t) + \frac{\partial f}{\partial x'}(x(t), x^{'}(t), t)h(t)\right] dt$$

$$= \int_a^b \left[\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial x'}\right)\right] hdt + \left[\frac{\partial f}{\partial x'}h\right]_b^a$$

after integration by part.

Theorem 2.11. [6] *Let* $a(.,.) : X \times X \to R$ *be a bounded symmetric bilinear form on a Hilbert spaces* X *and* J *a functional on* X*. often called "energy" functional", defined by*

$$J(u) = \frac{1}{2}a(u, u) - F(u),$$

where $F \in X^*$ *then the Fréchet derivative of* $J = dJ(u)\varnothing = a(u, \varnothing)$
Proof. For an arbitrary

$$\varnothing \in X J(u + \varnothing) = \frac{1}{2}a(u + \varnothing, u + \varnothing) - F(u + \varnothing)$$

$$\frac{1}{2}a(u, u) + \frac{1}{2}a(\varnothing, u) + \frac{1}{2}a(u, \varnothing) + \frac{1}{2}a(\varnothing, \varnothing) - F(u) - F(\varnothing)$$

$$J(u + \varnothing) = \left\{\frac{1}{2}a(u, u) - F(u)\right\} + a(\varnothing, u) - F(\varnothing) + \frac{1}{2}a(\varnothing, \varnothing)$$

$$= J(u) + a(u, \varnothing) - F(\varnothing) + \frac{1}{2}a(\varnothing, \varnothing)$$

$$\frac{|[J(u - \varnothing) - J(u) - a(u, \varnothing) - F(\varnothing)]|}{\|\varnothing\|X} = \frac{1}{2}\frac{|a(\varnothing, \varnothing)|}{\|\varnothing\|X}$$

$$\leq \frac{1}{2}\frac{M\|\varnothing\|X\|\varnothing\|X}{\|\varnothing\|X}$$

as a(.,.)is bounded. This implies that

$$\lim_{\|\varnothing\|X\to 0} \frac{|J(u+\varnothing)-J(u)-a(u,\varnothing)-F(\varnothing)|}{\|\varnothing\|X} = 0dJ(u)\varnothing = a(u,\varnothing)-F(\varnothing)$$

Theorem 2.12. [6] *A linear operator T from a Banach space X into a Banach space Y is Fréchet differentiable if and only if T is bounded.*

Proof Let T be a linear operator and Fréchet differentiable at a point. Then T is continuous (and hence bounded) by Theorem 5.2.4. conversely, if T is bounded linear operator, then $\|T(x+t)-Tx-Tt\| = 0$, proving that T is Fréchet differentiable and $T^{'} = T$

Example 2.1. Prove that for $f \in C_0^\infty(\Omega), \Omega \subset R^2$, there is a constant K depending on Ω such that

$$K\int_\Omega f^2 dx \le \int_W \left[\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2\right] dx$$

Solution. Let $f \in C_0^\infty(\Omega)$. Consider a rectangle $Q = [a,b]\times[c,d]$ with $\bar{\Omega} \subset IntQ$. Note that f vanishes outside Ω. Then

$$f(x,y) = \int_c^y \frac{\partial f}{\partial y} f(x,t) \quad forall(x,y) \in Q.$$

By Hlder's inequality, we get

$$|f(x,y)|^2 \le \left(\int_c^y dt\right)\left(\int_c^y \left(\frac{\partial f}{\partial y}\right)^2 dt\right) \le (d-c)\int_c^d \left(\frac{\partial f}{\partial y}(x,t)\right)^2$$

Integrating over Q, we get

$$\int_Q f^2 dx \le (d-c)^2 \int_Q \left(\frac{\partial f}{\partial y}\right)^2 dx$$

3. Main Results

Theorem 3.1. **[ON ITERATIVE FRÉCHET DERIVATIVES]** *Let x be a fixed point in a Banach space X and Y another space and also let the continuous linear operator $S : X \to Y$ be the Fréchet derivative of the operator $T : X \to Y$ such that*

$$\lim_{\|\Delta x\|\to 0} \frac{\|T(x+\Delta x)-T(x)-\varphi(\Delta x)\|}{\|\Delta x\|}$$

Then the higher order Fréchet derivatives successively can be generated in iterative manner such that

$$\lim_{\sum_{i=1}^{n}\|\Delta x_i\|\to 0}\frac{\left\|T\left[x+\sum_{i=1}^{n}(\Delta x_i)\right]-T\left[x+\sum_{i=1}^{n-1}(\Delta x_i)\right]-\sum_{i=1}^{n}\varphi_i(\Delta x_i)\right\|}{\left\|\sum_{i=1}^{n-1}\Delta x_i\right\|}=0$$

$n \geq 2$ and an integer.

Proof. Given that

$$T^{'}(x)=\|\Delta x\|\overset{lim}{\to} 0\frac{||T[x+\Delta x]-T(x)]-S(\Delta x)\|}{\|\Delta x\|}=0$$

For $n = 1$ but if $n = 2$, where

$$T''(x)=\lim_{\sum_{i=1}^{2}\|\Delta x_i\|\to 0}\frac{\left\|T\left[x+\sum_{i=1}^{2}(\Delta x_i)\right]-T[x+\Delta x_i]-\sum_{i=1}^{2}\varphi_i(\Delta x_i)\right\|}{\sum_{i=1}^{2}\|\Delta x_i\|}=0$$

Continuing, we can assume that the above Fréchet result holds for every $n = k$, we now show it holds for $n = k + 1$. So

$$T^{(k+1)}(x)=\lim_{\sum_{i=1}^{k+1}\|\Delta x\|\to 0}\frac{\left\|T\left[x+\sum_{i=1}^{k+1}(\Delta x_i)\right]-T\left[x+\sum_{i=1}^{k}(x_i)\right]-\varphi\sum_{i=1}^{k+1}(\Delta x_i)\right\|}{\sum_{i=1}^{k+1}n\|\Delta x\|}=0$$

Since this holds for $n = k + 1$ then it holds for all $n = k$ the earlier assumption that the proof holds for $n = k$ is hereby justified, hence

$$T^{(n)}(x)=\lim_{\|\Delta x\|\to 0}\frac{\left\|T\left[x+\sum_{i=1}^{n}(\Delta x_i)\right]-T\left[x+\sum_{i=1}^{n-1}(\Delta x_i)\right]-\varphi\sum_{i=1}^{n}(\Delta x_i)\right\|}{\sum_{i=1}^{n}n\|\Delta x\|}=0$$

Theorem 3.2. [ON THE CHAIN RULE FOR ITERATED FRÉCHET DERIVATIVE *Let A, B and C be a unitary spaces, if $S : A \to B$ and $T : B \to C$ are Fréchet differentiable at z and $u'(x) = u(s(x)) \circ s'(x)$. Then the higher* *order Fréchet derivative for $U^{(n)}(x)$ can be generated with $U = S \circ T$ generating $\varphi^{(n)}(x) = U^{(n)}(x)$ if and only if*

$$\lim_{\left\|\sum_{i=1}^{n}\Delta x_i\right\|\to 0} \frac{\left\|U\left[x+\left(\sum_{i=1}^{n}\Delta x_i\right)\right]-U\left[x-\left(\sum_{i=1}^{n-1}\Delta x_i\right)\right]-\varphi\left(\sum_{i=1}^{n}\Delta x_i\right)\right\|}{\left\|\sum_{i=1}^{n}\Delta x_i\right\|}$$
$$= \lim_{\left\|\sum_{i=1}^{n}\Delta x_i\right\|\to 0} \frac{\left\|S\circ T\left[x+\left(\sum_{i=1}^{n}\Delta x_i\right)\right]-S\circ T\left[x-\left(\sum_{i=1}^{n-1}\Delta x_i\right)\right]-\varphi\left(\sum_{i=1}^{n}\Delta x_i\right)\right\|}{\left\|\sum_{i=1}^{n}\Delta x_i\right\|}$$

Proof. Let and suppose $U^n(x)$ can be generated with $U = S \circ T$ such that the generalized Fréchet derivative

$$\varphi^n(x) = U^n(x) = U\left[x+\left(\sum_{i=1}^{n}\Delta x_i\right)\right]-U\left(\sum_{i=1}^{n}\Delta x_i\right)$$
$$= T\left[S\left(x_0+\sum_{i=1}^{n}\Delta x_i\right)\right]-T\left[S\left(\sum_{i=1}^{n}\Delta x_i\right)\right] = T\left[x+\sum_{i=1}^{n}\Delta y_i\right]-T\left(\sum_{i=1}^{n}\Delta y_i\right)$$

where

$$\sum_{i=1}^{n}\Delta x_i = S\left(x+\sum_{i=1}^{n}\Delta x_i\right)-S\left(\sum_{i=1}^{n}\Delta x_i\right)$$

Thus

$$\left\|U\left(x+\Delta x_i\right)-U\left(\sum_{i=1}^{n}\Delta x_i\right)-T^n\left(x_0\right)x\right\| = \sigma\left\|(z)\right\|$$

Since

$$\left\|\sum_{i=1}^{n}\Delta x_i - S^n(x)\sum_{i=1}^{n}\Delta x_i\right\| = \sigma'\left(\|x\|\right)$$

We get

$$\left\|U\left(x+\sum_{i=1}^{n}\Delta x_i\right)-U\left(\sum_{i=1}^{n}\Delta x_i\right)-T^n\left(x_0\right)S^n(x)\,\Delta x\right\|$$
$$= \left\|\left[U\left(x+\sum_{i=1}^{n}\Delta x_i\right)-U\left(\sum_{i=1}^{n}\Delta x_i\right)\right]-T^n(y)\,x-T^n(y)\,S^n(x)\,\Delta x\right\|$$

In view of the fact that S is continuous at x, we obtain

$$\left\|\sum_{i=1}^{n}\Delta x_i\right\| = \sigma\left(\|\Delta x\|\right)$$

Therefore

$$\varphi^n(x) = \lim_{\sum\limits_{i=1}^{n}\|\Delta x_i\|\to 0}\frac{\left\|S\circ T\left[x+\left(\sum\limits_{i=1}^{n}\Delta x_i\right)\right]-S\circ T\left[x-\left(\sum\limits_{i=1}^{n-1}\Delta x_i\right)-\varphi\left(\sum\limits_{i=1}^{n}\Delta x_i\right)\right]\right\|}{\sum\limits_{i=1}^{n}\|\Delta x_i\|}$$

Conversely

$$\lim_{\sum\limits_{i=1}^{n}\|\Delta x_i\|\to 0}\frac{\left\|S\circ T\left[x+\left(\sum\limits_{i=1}^{n}\Delta x_i\right)\right]-S\circ T\left[x-\left(\sum\limits_{i=1}^{n-1}\Delta x_i\right)-\varphi\left(\sum\limits_{i=1}^{n}\Delta x_i\right)\right]\right\|}{\sum\limits_{i=1}^{n}\|\Delta x_i\|}$$

implies

$$\begin{aligned}&T\circ S\left[x+\sum_{i=1}^{n}\Delta x_i\right]-T\circ S(x)\\&=T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right]+\varphi\left[x,\sum_{i=1}^{n}\Delta x_i\right]=T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right]+\varphi\left[\sum_{i=1}^{n}\Delta x_i\right]\end{aligned}$$

and

$$\begin{aligned}&T\circ S\left[x+\sum_{i=1}^{n-1}\Delta x_i\right]+S\circ T(x)\\&=T\circ S\left[\sum_{i=1}^{n-1}\Delta x_i\right]+\varphi\left[x-\sum_{i=1}^{n}\Delta x_i\right]=T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right]+\varphi\left[\sum_{i=1}^{n}\Delta x_i\right]\end{aligned}$$

$$\lim_{\sum\|\Delta x_i\|\to 0}\frac{\left\|T\circ S\left[x+\sum\Delta x_i\right]-T\circ S\left[x-\sum\limits_{i=1}^{n-1}\Delta x_i\right]-\varphi\left[\sum\limits_{i=1}^{n}\Delta x_i\right]\right\|}{\sum\limits_{i=1}^{n}\|\Delta x_i\|}$$

$$=\lim_{\sum\|\Delta x_i\|\to 0}\frac{\left\|\sum\limits_{i=1}^{n}\Delta x_i\right\|}{\sum\|\Delta x_i\|}=\lim_{\Delta x_n\to 0}\frac{0}{\Delta x_i}=0 \quad \text{(By L'Hospital rule)}$$

Hence $U^n(x) = \varphi^n(x)$ is Fréchet differentiable and the proof is complete.

Theorem 3.3. [**GENERALIZED FRÉCHET MEAN VALUE THEOREM**] *Let $T : A \to B$ where A is an open convex set containing x_1 and x_n and C is a normed space. $T^{(n)}(x)$ exists for each $x_1 \in [x_1, x_n]$, then*

$$\|T^{(n-1)}(x_n) - T^{(n-1)}(x_1)\| \leq \underset{x \in [x_1, x_n]}{Sup} \|T^{(n)}(x_1)\, T^{(n-2)} x_n - T^{(n-2)}(x_1)\|$$

Proof. Let $T : K \to B$ where K is an open convex set containing x_1 and x_n. B is a normal space and $T^{(n)}(x)$ exists for each $x \in (x_1, x_n)$ and $T'(x)$ is continuous in (x_1, x_n) such that

$$\|T(x_n) - T(x_1)\| \leq \underset{x \in [x_1, x_n]}{\text{Sup}} \|T'(x)\, x_n - x_1\|$$

Then by induction for the nth complex iterative Fréchet derivative of T, the mean value theorem becomes

$$\|T^{(n-1)}(x_n) - T^{(n-1)}(x_1)\| \leq \underset{x \in [x_1, x_n]}{\text{Sup}} \|T^{(n)}(x_1)\| \|T^{(n-2)} x_n - T^{(n-2)}(x_1)\|$$

Theorem 3.4. (Generalized Implicit Function Theorem) *Suppose that $A, B.C$ are complex Banach Spaces. D is an open subset of $A \times B$ and $T : D \to C$ is continuous. Suppose further that for some $({x_1}^*, {x_n}^*) \in D$, then*

$$T^{(n)}({x_1}^*, {x_n}^*) = 0$$

The nth Fréchet derivative of $T(.,.)$ where x is fixed is denoted by ${T^{(n)}}_{{x_n}^*}({x_1}^*, {x_n}^*)$ called the nth partial derivative with respect to x_n exists at each point in a neighborhood of $({x_1}^*, {x_n}^*)$ and is continuous $({x_1}^*, {x_n}^*)$.

$${T_x}^{(n)}\left({x_1}^*, {x_n}^*\right)^{-1} \in B(C, B)$$

Then there is a subset E of A containing ${x_1}^*$ and a unique continuous mapping $S : E \to C$ such that $T^{(n)}({x_1}^*, {x_n}^*({x_1}^*)) = 0$ and $S^{(n)}({x_1}^*) = {x_n}^*$.

Proof.

For the sake of convenience we may take $x_1 = 0$, $x_2 = 0$. Let A=${T^{(n-1)}}_{x_2}(0,0)]^{-1}$ $\in B(C, B)$. Since D is an open set containing (0,0), we find that

$$0 \in D_{x_1} = \{x_2 \in C / (x_1, x_2) \in D\}$$

For all x sufficiently small, say $x_1 \leq \delta$ for each x_1 having this property, we define a function

$$S^{(n-1)}(x_1, .) : D_{x_1} \to C$$

by

$$S^{(n-1)}(x_1, x_2) = x_2 - FT^{(n-1)}(x_1, x_2)$$

In order to prove the theorem, we must prove the existence of a fixed point for $S^{(n-1)}(x_1, .)$ under the condition that x_1 is sufficiently small and Continuity of the mapping $x_1 \to x_2(x_1)$ and $x_2(x_1) = x_2{}^*$. Now,

$$S_{x_2}{}^{(n-1)}(x_1, x_2)(U) = U - FT_{x_2}(x_1, x_2)(U),$$

and

$$FF^{-1} = FT_{x_2}{}^{(n-1)}(0, 0);$$

therefore, assumptions on $T^{(n-1)}$ guarantees the existence of $S^{(n-1)}{}_{x_2}(x_1, x_2)$ for sufficiently small $\|x_1\|$ and $\|x_2\|$ and

$$S_{x_2}{}^{(n-1)}(x_1, x_2)(U) = F[T_{x_2}{}^{(n-1)}(0, 0) - T_{x_2}{}^{(n-1)}((x_1, x_2)](U),]$$

Hence

$$\|S_{x_2}{}^{(n-1)}(x_1, x_2)\| \leq \|F\|\|T_{x_2}{}^{(n)}(0, 0) - T_{x_2}{}^{(n-1)}(x_1, x_2)\|$$

Since $T_{x_2}{}^{(n-1)}$ is continuous at (0,0) there exists a constant $L > 0$ such that

$$S_{x_2}{}^{(n-1)}(x_1, x_2) \leqslant L$$

For sufficiently small $\|x_1\|$ and $\|x_2\|$, we say that $\|x_1\| \leq \varepsilon_1 \leq \delta$ and $\|x_2\| \leq \varepsilon_2$. Since $T^{(n-1)}$ is continuous at (0,0), there exists an $\varepsilon \leq \varepsilon_1$ such that

$$\|S^{(n-1)}(x_1, 0)\| = \|FT^{(n)}(x_1, 0)\| < \varepsilon_2(1 - L)$$

For all x_1 with $\|x_1\| < \varepsilon$ We now show that $S^{(n-1)}(x_1, .)$ maps the closed ball

$$S_\varepsilon{}^{(n-1)}(O) = \{x_2 \in B / x_2 \leq \varepsilon_2\}$$

into itself. For this, let $\|x_1\| \leq \varepsilon$ and $\|x_2\| \leq \varepsilon_2$. Then by the mean value theorem and, we have

$$\left\|S^{(n-1)}(x_1, x_2)\right\| \leqslant \left\|S^{(n-1)}(x_1, x_2) - S^{(n-1)}(x_1, 0)\right\| + \left\|S^{(n-1)}(x_1, 0)\right\|$$

$$\leqslant \operatorname*{Sup}_{0 \leqslant \alpha \leqslant 1} \left\|S_{x_2}{}^{(n)}(x_1, x_2)\right\| \|x_2\| + \left\|S^{(n-1)}(x_1, 0)\right\| \leqslant L_{\varepsilon_2} + \varepsilon_2(1 - L) = \varepsilon_2$$

Therefore for $\|x_1\| \le \varepsilon$,

$$S^{(n-1)}(x_1, .): \ \overline{S_{\varepsilon_2}}^{(n-1)}(O) \to \overline{S_{\varepsilon_2}}^{(n-1)}(O)$$

Also for $x_2{}^*, x_2{}^{**} \in \overline{S_{\varepsilon_2}}(O)$. We obtain by mean value theorem 3.3

$$\left\|S^{(n-1)}(x_1, x_2{}^*) - S^{(n-1)}(x_1, x_2{}^{**})\right\| \le \underset{\|x_2\| \le \varepsilon_2}{Sup} \left\|S^{(n-1)}(x_1, x_2)\right\| \|x_2{}^* - x_2{}^{**}\| \|x_2\|$$

$$\le \varepsilon_2 \le L\|x_2{}^* - x_2{}^{**}\|$$

The Banach contraction mapping theorem guarantees that for each x_1 with $x_1 \le \varepsilon$, there exists a unique $x_2(x_1) \in S_{\varepsilon_2}{}^{(n-1)}(O)$ such that

$$x_2(x_1) = S^{(n-1)}(x_1, x_2(x_1)) = x_2(x_1) - FT^{(n-1)}(x_1, x_2(z_1))$$

That is $T^{(n-1)}(x_1, x_2(x_1)) = 0$. By uniqueness of x_2, we have $x_2(O) = 0$ since $T^{(n-1)}(0,0) = 0$. Finally, we show that $x_1 \to x_2(x_1)$ is continuous. For if $\|x_1{}^*\| < \varepsilon$ and $\|x_1{}^{**}\| < \varepsilon$, . then selecting $x_2{}^{\circ} = x_2(x^{**})$ and $x_2{}^* = S^{(n-1)}\left(x_1{}^*,\ x_2{}^{\circ}\right)$, we have by the error bound for fixed point iteration on the mapping $S^{(n-1)}(x_1{}^*,\ 0)$

$$\|x_2(x_1{}^{**}) - x_2(x_1)\| \le \frac{1}{1-L}\|x_2{}^{\circ} - x_2{}^*\|$$

can write

$$x_2{}^{\circ} - x_2{}^* = x_2(x_1{}^{**}) - S^{(n-1)}(x_1{}^*,\ x_2(x_1{}^{**}))$$

$$= S^{(n-1)}(x_1{}^{**},\ x_2(x_1{}^{**})) - T^{(n-1)}(x_1{}^*,\ x_2(x_1{}^{**}))$$

$$= -F\left[T^{(n-1)}(x_1{}^*,\ x_2(x_1{}^{**})) - T^{(n-1)}(x_1,\ x_2(x_1{}^{**}))\right]$$

Therefore by the continuity of $T^{(n-1)}\|x_2(x_1{}^{**} - \ x_2(x_1{}^*))\|$ can be made arbitrarily small and hence the proof.

Corollary 2.4. *If in addition to conditions of theorem 3.4* $T_{x_1}{}^{(n-1)}(x_1,\ x_2)$ *also exists on the open set, and is continuous at* $(x_1{}^*, x_2{}^*)$, *then* $F : x_1 \to x_2(x_1)$ *has the nth Fréchet derivative at* $x_1{}^*$ *given by*

$$T^{(n)}(x_1{}^*) = -\left[T_{x_2}{}^{(n-1)}(x_1,\ x_2)\right]^{-1} T_{x_1}{}^{(n-1)}(x_1{}^*, x_2{}^*)$$

Proof. We set $x_1 = x_1{}^* + \Delta x_1{}^*$, and $G^{(n)}(\Delta x_1{}^*) = F^{(n)}(x_1) - x_2$.
Then $G^{(n)}(O) = 0$ and

$$\|G^{(n-1)}(\Delta x^*) + \left[T_{x_2}{}^{(n-1)}(x_1,\ x_2)\right]^{-1} \Delta x_1{}^*\|$$

$\leq \|[T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*)]^{-1}\| \|T_{x_2}{}^{(n-1)}(x_1,\ x_2)\, G^{(n-1)}(\Delta x_1{}^*) + T_{x_1}{}^{(n-1)}(x_1,\ x_2)\, \Delta x_1{}^*\|$

and

$$T_{x_2}{}^{(n-1)}(x_1,\ x_2)\, G^{(n-1)}(\Delta x_1{}^*) - T_{x_1}{}^{(n-1)}(x_1,\ x_2)\, \Delta x_1{}^*$$
$$= -T^{(n-1)}\left(x_1 + \Delta x_1{}^*, x_2 + G^{(n-1)}(\Delta x_1{}^*)\right) + T^{(n-1)}(x_1{}^*, x_2{}^*)$$
$$+ T_{x_2}{}^{(n-1)}(x_1,\ x_2)\, G^{(n-1)}(\Delta x_1{}^*) + T_{x_1}{}^{(n-1)}(x_1 +\ x_2)\, \Delta x_1{}^*$$

If O_1, O_2 are numbers in (0,1), then

$$\|T_{x_2}{}^{(n-1)}(x_1,\ x_2)\, G^{(n-1)}(\Delta x_1{}^*) + T_{x_1}{}^{(n-1)}(x_1,\ x_2)\, \Delta x_1{}^*\|$$

$$\leqslant \operatorname*{Sup}_{O_1,O_2} \left\| T_{x_1}{}^{(n-1)}(x_1{}^* + O_1 \Delta x_1{}^*, x_2 + O_2 G^{(n-1)}(\Delta x_1{}^*) - T_{x_1}{}^{(n-1)}(x_1{}^*, x_2{}^*) \right\| \|\Delta x_1{}^*\|$$
$$+ \operatorname*{Sup}_{O_1,O_2} \left\| T_{x_2}{}^{(n-1)}\left(x_1{}^* + O_1 \Delta x_1{}^*, x_2 + O_2 G^{(n-1)}(\Delta x_1{}^*) - T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*)\right) \right\|$$
$$\left\| G^{(n-1)}(\Delta x_1{}^*) \right\|$$

Thus applying continuity of $T_{x_1}{}^{(n-1)}, T_{x_2}{}^{(n-1)}$ for $\varepsilon > 0$, we find that $\delta = \delta(\varepsilon)$ such that on $x_1 - x_1{}^* \leq \delta$, we have

$$\left\| G^{(n-1)}(\Delta x_1{}^*) + \left[T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*)\right]^{-1} T_{x_1}{}^{(n-1)}(x_1,\ x_2)\, \Delta x_1{}^* \right\|$$
$$\leqslant \frac{\left[T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*)\right]^{-1} \varepsilon \left[1 + \left\| T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*) \right\|\right]^{-1} T_{x_1}{}^{(n-1)}(x_1{}^*, x_2{}^*)\, \|\Delta x_1{}^*\|}{1 - \left\| \left[T_{x_2}{}^{(n-1)}(x_1,\ x_2)\right]^{-1} \right\|}$$

The coefficient of $\Delta x_1{}^*$ can be made as small as required as $\Delta x_1{}^* \to 0$. Thus

$$\left\| F^{(n)} x_1 - \left(F^{(n)}(x_1{}^*) - \left[T_{x_2}{}^{(n-1)}(x_1,\ x_2)\right]^{-1} T_{x_1}{}^{(n-1)}(x_1, x_1{}^*)\,(x_1 - x_1{}^*)\right) \right\|$$
$$= \sigma(x_1 - x_1{}^*)$$

Hence

$$F^{(n)}(x_1{}^*) = -[T_{x_2}{}^{(n-1)}(x_1,\ x_2)]^{-1} T_{x_2}{}^{(n-1)}(x_1{}^*, x_2{}^*)$$

Theorem 3.5. [Taylor's Formula for the nth Fréchet differentiable functions] *Let $T : \Omega \subset A \to B$ and let $[(a, a + n\Delta x)$ be any closed segment lying in Ω. If T is differentiable in Ω and nth differentiable at a, then*

$$T(a + n\Delta x) = T(a) + T'(a)\, \Delta x + \frac{1}{2}(T''(a)\, \Delta x)\, \Delta x + \ldots + \frac{1}{n}\left(T^{(n)}(a)\, (\Delta x)^{(n)}\right)$$

$$+\Delta x^n \varepsilon(\Delta x)$$

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

Theorem 3.6. [On the Generalized Inverse Function Theorem] *Let X and Y be Banach Space and let S be an open subset of X. Define a continuously differentiable function $f : S \to Y$ on S and assume that f is one to one and onto Fréchet differentiable function at the point $(S_0, f(s_0))$ with*

$$S_{i-1} = f^{-1}(y_{i-1}), \quad \forall S_{i-1} \in S, \quad y_{i-1} \in Y$$

then

$$(S, Y) = \bigcup_{i=1}^{n} S_i, f(s_i)$$

with

$$S = \bigcup_{i=1}^{n} S_i = \bigcup_{i=1}^{n} S_i f^{-1}(y_i)(y_i) = f^{-1}(Y), \quad s \in S, \ y \in Y$$

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