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EULER-DARBOUX EQUATION ASSOCIATED WITH EXPONENTIAL FUNCTION OF CONVOLUTION TYPE-II

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Dedicated to Prof. A.M. Mathai on his 80^{th} birth anniversary

Abstract: In this paper, authors have established new and interesting results of Euler-Darboux equations associated with exponential functions of convolution type-II.

Key words and Phrases: Euler-Darboux equation, fractional integral operator, Gauss hypergeometric function, Holder continuity.

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1. Introduction

The paper is devoted to solve a boundary value problem for the Euler-Darboux equation

$$u_{xy} - (\beta u_x - \alpha u_y)/(x-y) = 0 \quad (\alpha > 0, \beta > 0, \alpha + \beta < 1)$$

in the domain $[(x,y)|0 < x < y < 1]$ by reducing it to a dominant singular integral with Cauchy kernel. Boundary conditions are $u(x,x) = \varphi_1(x)$ and $AI_{ox}^{a,b,-\alpha+\beta-1}u(o,x) + BJ_{x1}^{a+\alpha-\beta,c,-\alpha+\beta-1} = \varphi_2(x)$, where I and J stand for generalized fractional integral operators. In the recent paper Gaur, V.K. and Bajpai, U.K. (2008), there has been discussed a generalized Goursat problem for the Euler-Darboux equation.

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{\exp x - \exp y}\frac{\partial u}{\partial y} + \frac{\alpha}{\exp x - \exp y}\frac{\partial u}{\partial y} = 0 \tag{1.1}$$

By using the generalized fractional calculus Saigo, M. (1978, 80). Similar work on fractional operators did time to time by eminent mathematicians like Saxena and Sethi (1975), Sethi and Banerji (1975), Bajpai and Gaur (2008) and Gakhov (1966).

The present section is intended to solve a problem for the equation (1.1) in the domain $[(x,y)|0 < x < y < 1]$ assuming the value of the solution of the noncharacteristic boundary $y = x$ and the value of a linear combination of the generalized fractional integrals or derivatives of the solution on two characteristic segments $x = 0$ and $y = 1$. Such a problem has been discussed by A.M. Nahusev, H.G. Bzihatlov, A.V. Bicalze, T.I. Lanina, S.K. Kumykova, V.A. Eleev, M.M. Smirnov, R.K. Saxena, P.L. Sethi etc. As in their problems the calculus is of the sense of Riemann-Liouville of fixed orders, in our problems the calculus is of the sense of Riemann-Liouville of fixed orders, in our problem the calculus is of the generalized sense and orders may be chosen between some positive and negative number depending on α and β.

In section (2) the problem is formulated and solved, where some of calculations will be left to sections (3) and (4). The investigation in section (2) requires various formulas of the Gauss hypergeometric function, whose definition and properties will be summarized in section (5), where those of our fractional calculus will also be started. The formulas in section (5) will be cited as (I.1), (II.2) etc. hereafter.

2. Formulation of Problem and its Solution

In this section, we set up our main problem and its conclusion leaving the details in the later sections. Consider the Euler-Darboux equation (1.1) in the triangular domain $\Omega = OAB$, where $O = (0,0), A = (0,1)$ and $B = (1,1)$. A solution $u(x,y)$ of (1.1) under the conditions

$$u\Big|_{y=x} = \tau(x), (\exp y - \exp x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)\Big|_{y=x} = v(x)$$

is given in the form Darboux, (1972).

$$u(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \tau[\exp x(\exp y - \exp x)\exp t]$$

$$(\exp t)^{\beta-1}(1-\exp t)^{\alpha-1}dt + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)}(\exp y - \exp x)^{1-\alpha-\beta}$$

$$\int_0^1 v[\exp x + (\exp y - \exp x)\exp t](\exp t)^{-\alpha}(1-\exp t)^{-\beta}dt$$

Hence, it is easily seen that the values of u on the characteristics OA and AB are written as follows

$$u^{(1)}(y) \equiv u(0,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{oy}^{\alpha,0,\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)} I_{oy}^{1-\beta,\alpha+\beta-1,\beta-1}\upsilon, \quad 0<y<1 \tag{2.1}$$

and

$$u^{(2)}(x) \equiv u(x,1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x,1}^{\beta,0,\alpha-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{1-\alpha,\alpha+\beta-1,\alpha-1}\upsilon, \quad 0<x<1 \tag{2.2}$$

Respectively, by means of the generalized fractional integrals (I.1) and (I.2).

Let $H^k(T)$ be a class of Holder continuous functions on a real interval T with the Holder index k. Denote the open interval (0,1) by U and its closure by $\bar{U}$. Now let us state our main problem.

Problem. Find a solution $u(x,y)$ of (1.1) in Ω satisfying the boundary conditions.

$$u(x,x) = \varphi_1(x), \quad x \in \bar{U} \tag{2.3}$$

and

$$A I_{ox}^{a,b,-\alpha+\beta-1} u^{(1)} + B J_{x1}^{a+\alpha-\beta,c,-a+\beta-1} u^{(2)} = \varphi_2(x), \quad x \in U, \tag{2.4}$$

and having the properties that $\upsilon(x) \in H^k(U)$ for some k $(0<k<1)$ and $\upsilon(x)$ may have infinities of integrable order of end points of U, where A and B are non zero constants, a,b and c are constants such that $-\alpha < a < \beta$, $-\alpha < a+b < 1-a$ and $-\alpha < a+c < \beta$, and $\varphi_1 \in H^{k_1}(\bar{U})$ and $\varphi_2 \in H^{k_2}(\bar{U})$ are given functions with $1-a-\alpha > k_1 > max(1-\alpha-\beta, c)$, $1 > k_2 > a-\beta+1$ and $\varphi(1)=0$.

Remark. Nahusev's result (1969) in the case that A and B are given function of x, $\alpha=\beta$ and $a=-b=-c=\alpha-1$.

Substituting (2.1) and (2.2) into (2.4) and using product rules (I.4) and (I.5) we have the relation,

$$\frac{A\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{ox}^{a+\alpha,b,-a+\beta+1}\tau + \frac{A\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)} J_{ox}^{a-\beta+1,b+\alpha+\beta-1,a+\beta-1}\upsilon$$

$$\frac{B\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x1}^{a+\alpha,c,-a+\beta-1}\tau + \frac{B\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta+1}\upsilon = \varphi_2(x) \tag{2.5}$$

Then by operating $(J_{ox}^{a-\beta+1,b+\alpha+\beta-1,-a+\beta-1})^{-1} = I_{ox}^{-a+\beta-1,-b-\alpha-\beta+1,0}$ (see (I.7)) on both sides of (2.5) and by noting (I.3), (I.4) and (I.6), conditions (2.3) and (2.4) can be unified in the form,

$$\frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)}\upsilon(x) + I_{ox}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1}\upsilon$$

$$= -\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1}\varphi_1 - \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}$$

$$J_{x1}^{a+\alpha,c,-a+\beta-1}\varphi_1 + \frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}\varphi_2 \tag{2.6}$$

Thus our problem is reduced to find $v(x)$ from (2.6) so let us investigate the equation (2.6).

The second term on the L.H.S. of (2.6) may be written as,

$$I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1} v$$

$$= \frac{\sin\pi(a-\beta)}{\pi}(\exp x)^{b+\alpha+\beta-1}\int_0^1 (\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u - \exp x)^{-1}$$

$$v(u)du - \cos\pi(a-\beta)(\exp x)^{a+b+\alpha}(1-\exp x)^{-a-c-\alpha}v(x) \tag{2.7}$$

which will be derived in the next section. If we set

$$\mu(x) = (\exp x)^{a-\beta+1}(1-\exp x)^{-a-c-\alpha}v(x) \tag{2.8}$$

And use the relation (2.7), we find that (2.6) is reduced to the dominate singular integral equation for $\mu(x)$ having the Cauchy kernel.

$$P(x)\mu(x) + \int_0^1 \frac{\mu(u)}{\exp u - \exp x}du = Q(x) \quad 0 < x < 1 \tag{2.9}$$

where

$$P(x) = \pi\cos\pi(a-\beta) - \frac{\pi}{\sin\pi(a-\beta)}\frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)}(\exp x)^{-a-b-\alpha}(1-\exp x)^{a+c+\alpha},$$

$$Q(x) = \frac{\pi}{\sin\pi(a-\beta)}(\exp x)^{b-\alpha-\beta+1}\left[\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)}R_{0x}^{\alpha+\beta-1}\varphi_1 + \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)}\right.$$

$$\left. I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a+\alpha,c,-a+\beta-1}\varphi_1 - \frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}\varphi_2\right]$$

Let us solve the integral equation (2.9) by applying the theory in Gakhov (1996), whose results have been summarized in Saxena and Sethi (1973).

In order to guarantee the Holder continuity of coefficients, we multiply the both sides of the equation (2.9) by $(\exp x)^{a+b+\alpha}$ and set

$$a(x) = (\exp x)^{a+b+\alpha}P(x), \quad b(x) = \pi i(\exp x)^{a+b+\alpha} \tag{2.10}$$

and $f(x) = (\exp x)^{a+b+\alpha}Q(x)$

The continuity of $a(x)$ and $b(x)$ is evident and that of $f(x)$ will be shown in section (2), consider the function $G(x) = \dfrac{a(x) - b(x)}{a(x) + b(x)}$ then we obtain the values

$$G(+0) = 1, \quad G(1-0) = \exp[2\pi(-a+\beta)i]$$

Thus, $\theta = \arg G(+0) = 0$. If we assume Δ, the change of $\arg G(x)$ on $\bar{U}$, to be equal to $2\pi(-a+\beta-1)$ then the index k of the equation (2.9) is equal to zero in a class of solutions $h(1)$, whose function are Holder continuous in U, bounded at $x = 0$ and unbounded but having integrable singularities at $x = 1$.
Therefore we obtain a solution of the equation (2.9) in the class $h(1)$ represented by,

$$\mu(x) = \frac{P(x)Q(x)}{P^2(x)+\pi^2} - \frac{Z(x)}{[P^2(x)+\pi^2]^{1/2}} \int_0^1 \frac{Q(u)}{Z(u)[P^2(u)+\pi^2]^{1/2}} \frac{1}{\exp u - \exp x} du \tag{2.11}$$

where

$$Z(x) = \exp\left[\frac{1}{2\pi i}\int_0^1 \frac{1}{\exp u - \exp x} \log\frac{P(u)-\pi i}{P(u)+\pi i} du\right] \tag{2.12}$$

Here the branch of the logarithm in (2.12) should be selected such that the value at $u = 1$ is equal to $2\pi(-a+\beta-1)i$. Thus by (2.8), $v(x)$ in the relation (2.6) satisfying the required conditions is determined, and then our problem is solved.

3. Derivation of (2.7)

Owing to the definitions and the fractional integrals and derivatives (I.1) and (I.2) we may proceed as follows,

$$\begin{aligned}
& I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1} v \\
&= \frac{d}{dx} I_{0x}^{-a+\beta-1,-b-\alpha-\beta,-1} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1} v \qquad (3.1)\\
&= \frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)} \frac{d}{dx}\left[(\exp x)^{a+b+\alpha}\int_0^x (\exp x - \exp t)^{-a+\beta-1}\right. \\
&\quad F\left(-a-b-\alpha, 1; -a+\beta; 1 - \frac{\exp t}{\exp x}\right) \\
&\quad \left.\int_t^1 (\exp u - \exp t)^{a-\beta}(1-\exp u)^{-a-c-\alpha} v(u)dudt\right]
\end{aligned}$$

$$= \frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)} \frac{d}{dx} \left[(\exp x)^{a+b+\alpha} \left(\int_0^x \int_0^u + \int_x^1 \int_0^x \right) \right.$$

$$(\exp x - \exp t)^{-a+\beta-1} . (\exp u - \exp t)^{a-\beta}$$

$$\left. (1-\exp u)^{-a-c-\alpha} F\left(-a-b-\alpha; 1; -a+\beta; 1 - \frac{\exp t}{\exp x} \right) v(u) dt du \right]$$

$$\equiv \frac{d}{dx} \left[\int_0^x \Xi_1(x,u) v(u) du + \int_x^1 \Xi_2(x,u) v(u) du \right]$$

First we shall treat the integral $\Xi_1(x,u)$. Formulas (II.2), (II.3) and (II.4) imply that

$$F\left(-a-b-\alpha; 1; -a+\beta; 1 - \frac{\exp t}{\exp x} \right) = \frac{\Gamma(-a+\beta)}{\Gamma(b+\alpha+\beta)} . \frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(-a+\beta-1)}$$

$$\left(1 - \frac{\exp t}{\exp x} \right)^{a-\beta+1} F\left(a-\beta+2, -b-\alpha-\beta+1; -b-\alpha-\beta+2; \frac{\exp t}{\exp x} \right)$$

$$+ \frac{\Gamma(-a+\beta)\Gamma(-b-\alpha-\beta+1)}{\Gamma(-a-b-\alpha)} \left(\frac{\exp t}{\exp x} \right)^{b+\alpha+\beta-1} \left(1 - \frac{\exp t}{\exp x} \right)^{a-\beta+1}$$

Then by the formula (II.7), we have

$$\Xi_1(x,u) = \frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(b+\alpha+\beta)\Gamma(a-\beta+2)\Gamma(a+\beta-1)} (\exp x)^{b+\alpha+\beta-1} (\exp u)^{a-\beta+1}$$

$$. (1-\exp u)^{-a-c-\alpha} F\left(1, -b-\alpha-\beta+1; -b-\alpha-\beta+2; \frac{\exp u}{\exp x} \right)$$

$$+ \frac{\Gamma(b+\alpha+\beta)\Gamma(-b-\alpha-\beta+1)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)} (\exp u)^{a+b+\alpha} . (1-\exp u)^{-a-c-\alpha} \tag{3.2}$$

By virtue of the relation (II.5) it is easy to see that $\Xi_1(x,u)$ has a logarithmic singularity at $u = x$. As regards $\Xi_2(x,u)$, the formula (II.8) yields the relation,

$$\Xi_2(x,u) = \frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)}$$

$$(\exp x)^{b+\alpha+\beta} (1-\exp u)^{-a-c-\alpha} (\exp u - \exp x)^{\alpha-\beta} \int_0^1 \upsilon^{-a+\beta-1}$$

$$\left(1 - \frac{\exp x}{\exp x - \exp u} \upsilon \right)^{\alpha-\beta} F(-a, -b-\alpha, 1-a+\beta; \exp \upsilon) d\upsilon \tag{3.3}$$

$$= \frac{\Gamma(b+\alpha+\beta)}{\Gamma(-a+\beta)\Gamma(a-\beta-1)\Gamma(b+\alpha+\beta+1)}(\exp x)^{b+\alpha+\beta}(\exp u)^{a-\beta}$$

$$(1-\exp u)^{-a-c-\alpha}F\left(1, b+\alpha+\beta; b+\alpha+\beta+1; \frac{\exp x}{\exp u}\right)$$

Which has also a logarithmic singularity at $u = x$.

In order to compute the R.H.S. of (3.1), we have to evaluate its principal value, since the functions $\Xi_1(x, u)$ and $\Xi_2(x, u)$ have logarithmic singularities at $u = x$. Then we shall begin to calculate the expression,

$$\Psi(x;\rho) \equiv \frac{d}{dx}\left[\int_0^{x-\rho} \Xi_1(x, u)v(u)du + \int_{x+\rho}^{1} \Xi_2(x, u)v(u)du\right]$$

For sufficiently small ρ.

Making use of the formulas (II.2) and (II.6), we obtain

$$\frac{d}{dx}x^{b+\alpha+\beta-1}\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}$$

$$F\left(1, -b-\alpha-\beta+1; -b-\alpha-\beta+2; \frac{\exp u}{\exp x}\right)v(u)du$$

$$= (\exp x)^{b+\alpha+\beta-1}(\exp x-\rho)^{a-\beta+1}(1-\exp x+\rho)^{-a-c-\alpha}$$

$$F\left(1, -b-\alpha-\beta+1; -b-\alpha-\beta+2; \frac{\exp x-\rho}{\exp x}\right)v(\exp x-\rho) + (b+\alpha+\beta+1)$$

$$(\exp x)^{b+\alpha+\beta-1}\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp x-\exp u)^{-1}v(u)du.$$

Then we have

$$\frac{d}{dx}\int_0^{x-\rho}\Xi_1(x, u)v(u)du = \frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(b+\alpha+\beta)\Gamma(a-\beta+2)\Gamma(-a+\beta-1)}$$

$$(\exp x)^{b+\alpha+\beta-1}(\exp x-\rho)^{a-\beta+1}(1-\exp x+\rho)^{-a-c-\alpha}$$

$$F\left(1, -b-\alpha-\beta+1; -b-\alpha-\beta+2; 1-\frac{\rho}{\exp x}\right)$$

$$v(\exp x-\rho) + \frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+2)\Gamma(-a+\beta-1)}$$

$$\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp x-\exp u)^{-1}v(u)du$$

$$+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}$$

$$(\exp x-\rho)^{a+b+\alpha}(1-\exp x+\rho)^{-a-c-\alpha}v(\exp x-\rho) \tag{3.4}$$

Similarly it follows that

$$\frac{d}{dx}\int_{x+\rho}^{1}\Xi_2(x,u)v(u)du=\frac{-\Gamma(b+\alpha+\beta)}{\Gamma(-\alpha+\beta)\Gamma(a-\beta+1)\Gamma(b+\alpha+\beta+1)}$$

$$(\exp x)^{b+\alpha+\beta}(\exp x+\rho)^{a-\beta}(1-\exp x-\exp\rho)^{-a-c-\alpha}$$

$$F\left(1,b+\alpha+\beta;b+\alpha+\beta+1;\frac{\exp x}{\exp x+\rho}\right)v(\exp x+\rho)+\frac{(\exp x)^{b+\alpha+\beta+1}}{\Gamma(a-\beta+1)\Gamma(-\alpha+\beta)}$$

$$\int_{x+\rho}^{1}(\exp u)^{a-\beta+1}(1-\exp u)^{a-c-\alpha}(\exp u-\exp x)^{-1}v(u)du \tag{3.5}$$

From the relations (3.4) and (3.5) we obtain,

$$\Psi(x;\rho)=\frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}\left[\frac{1}{-b-\alpha-\beta+1}(\exp x-\rho)^{a-\beta+1}\right.$$

$$(1-\exp x+\rho)^{-a-c-\alpha}F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;1-\frac{\rho}{\exp x}\right)$$

$$v(\exp x-\rho)-\frac{1}{b+\alpha+\beta}\exp x(\exp x+\rho)^{a-\beta}(1-\exp x-\rho)^{-a-c-\alpha}$$

$$\left.F\left(1,b+\alpha+\beta;b+\alpha+\beta+1;\frac{\exp x}{\exp x+\rho}\right)v(\exp x+\rho)\right]$$

$$+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}(\exp x-\rho)^{a+b+\alpha}$$

$$(1-\exp x+\rho)^{-a-c-\alpha}v(\exp x-\rho)+\frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}$$

$$\left[\int_0^{x-\rho}+\int_{x+\rho}^{1}\right](\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}v(u)du \tag{3.6}$$

By virtue of the formula (II.5), the terms in the first brackets on the right hand side of (3.5) can be written in the form,

$$(\exp x-\rho)^{a-\beta+1}(1-\exp x+\rho)^{-a-c-\alpha}\sum_{n=0}^{\infty}\frac{(-b-\alpha-\beta+1)_n}{n!}$$

$$.[\Psi(n+1)-\Psi(-b-\alpha-\beta+1+n)-\log\rho+\log\exp x]\left(\frac{\rho}{\exp x}\right)^n$$

$$\exp v(\exp x-\rho)-\exp x(\exp x+\rho)^{a-\beta}(1-\exp x-\rho)^{-a-c-\alpha}\sum_{n=0}^{\infty}\frac{(b+\alpha+\beta)_n}{n!}$$

$$.[\Psi(n+1)-\Psi(b+\alpha+\beta+n)-\log\rho+\log(\exp x+\rho)]\left(\frac{\rho}{\exp x+\rho}\right)v(\exp x+\rho),$$

where $\Psi(z)=\Gamma'(z)/\Gamma(z)$. Hence, by letting $\rho\to 0$ in the relation (3.6), we obtain,

$$\lim_{\rho\to 0}\Psi(x;\rho)=\frac{1}{\Gamma(a-\beta+1)\Gamma(-a+\beta+\alpha)}$$

$$[\Psi(b+\alpha+\beta)-\Psi(-b-\alpha-\beta+1)](\exp x)^{a+b+\alpha}$$

$$(1-\exp x)^{-a-c-\alpha}v(x)+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}$$

$$.(\exp x)^{a+b+\alpha}(1-\exp x)^{-a-c-\alpha}v(x)+\frac{(\exp x)^{b+\alpha+\beta+1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}$$

$$\int_0^1(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}v(u)du \tag{3.7}$$

In view of the formulas $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi\exp z$ and $\Psi(z)=\Psi(1-z)-\pi\cot\pi\exp z$ Magnus, Oberhettinger and Soni (1966), the relation (2.7) can be derived.

4. Holder Continuity of f(x)

To find the Holder Continuity $f(x)$ we shall refer to Lemmas 1 and 2 in Sethi and Mahmoud (2003 a). Investigating the proof of Lemma 1, we find that the Lemma holds valid without the assumption $\varphi(a)=\Psi(b)=0$ when $\alpha+\beta=0$. Then due to the assumptions $a-\beta+1>0$ and $k_1>1-\alpha-\beta>0$, we have;

$$(\exp x)^{a-\beta+1}R_{0,x}^{\alpha+\beta-1}\varphi_1\in H^{k_1+\alpha+\beta-1}(\bar{U})$$

Lemma 1 is also valid by replacing the assumption $0>\alpha>-k$ by $0<\alpha<1-k$. Thus from $0<a+\alpha<1-k_1$, $c<k_1$, $a+c-\beta<0$ and $\varphi_1(1)=0$, we see the relation,

$$g(x)\equiv J_{x_1}^{a+\alpha,c,-a+\beta-1}\varphi_1\in H^{k_1+a+\alpha}(\bar{U}).$$

By virtue of Lemma 2 we obtain,

$$(\exp x)^{a-\beta+1}I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}g\in H^{k_1+\alpha+\beta-1}(\bar{U})$$

where the assumptions $a-\beta+1>0, a+b+\alpha>0, b+\alpha+\beta>0$ and $k_1+\alpha+\beta-1>0$ are used, similarly, by noting $k_2-a+\beta-1>0$, we have,

$$(\exp x)^{a-\beta+1} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} \varphi_2 \in H^{k_2-a+\beta-1}(\bar{U})$$

Therefore we have established the relation,

$$f(x) \in H^{\min(k_1+\alpha+\beta-1, k_2-a+\beta-1)}(\bar{U})$$

Generalized fractional calculus and Hypergeometric Function

We shall summarize definitions and formulas of the generalized fractional calculus and the Gauss hypergeometric function that have been used in the preceding sections.

(I) The generalized functional calculus Saigo (1978), Saxena and Sethi (1975), Saxena and Banerji (1975), Sethi and Mahmound (2003 a).

$$I_{0x}^{\alpha,\beta,\eta} f =\equiv \frac{(\exp x)^{-a-\beta}}{\Gamma(\alpha)} \int_0^x (\exp x - \exp t)^{\alpha-1}$$

$$F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\exp t}{\exp x}\right) f(t)dt, \ (a>0) \quad (I.1)$$

$$\equiv \frac{d^n}{dx^n} I_{0x}^{a+n,\beta-n,\eta-n} f, \quad (0<\alpha+n \leq, \text{n is a positive integer})$$

$$J_{x1}^{\alpha,\beta,\eta} f \equiv \frac{(1-\exp x)^{-\alpha-\beta}}{\Gamma(\alpha)} \int_x^1 (\exp t - \exp x)^{\alpha-1}$$

$$F\left(\alpha+\beta, -\eta; \alpha; \frac{\exp t - \exp x}{1-\exp x}\right) f(t)dt, \quad (a>0)$$

$$\equiv (-1)^n \frac{d^n}{dx^n} J_{x1}^{\alpha+n,\beta-n,\eta-n} f, \quad (0<\alpha+n \leq, \text{n is a positive integer}) \quad (I.2)$$

$$R_{0x}^{\alpha} f \equiv I_{0x}^{\alpha,-\alpha,\eta} f, \quad (I.3)$$

$$I_{0x}^{\alpha,\beta,\eta} I_{0x}^{\gamma,\delta,\alpha+\eta} f = I_{0x}^{\alpha+\gamma,\beta+\delta,\eta} f, \quad (\gamma>0) \quad (I.4)$$

$$J_{x1}^{\alpha,\beta,\eta} J_{x1}^{\gamma,\delta,\alpha+\eta} f = J_{x1}^{\alpha+\gamma,\beta+\delta,\eta} f, \quad (\gamma>0) \quad (I.5)$$

$$I_{0x}^{0,0,\eta} f = f(x) \quad (I.6)$$

$$\left(I_{0x}^{\alpha,\beta,\eta}\right)^{-1} = I_{0x}^{-\alpha,-\beta,\alpha+\eta} \quad (I.7)$$

(II) The Gauss hypergeometric function is the series

$$F(a,b;c;\exp z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}(\exp z)^n \qquad (II.1)$$

Together with its analytic continuation, where $(a)_n=\Gamma(a+n)/\Gamma(a)$. In the following, formulas (II.2) to (II.6) are found in Magnus, Oberhettinger and Soni (1966), and formulas (II.7) and (II.8) are deduced respectively

$$F(a,b;a;\exp z)=(1-\exp z)^{-b} \qquad (II.2)$$

$$F(a,b;c;\exp z)=(1-\exp z)^{c-a-b}F(c-a,c-b;c;\exp z) \qquad (II.3)$$

$$F(a,b;c;\exp z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-\exp z)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-\exp z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-\exp z), \quad (II.4)$$

$(c-a-b\neq \text{integer}, |\arg(1-\exp z)|<\pi)$

$$F(a,b;a+b;\exp z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}$$

$$.[2\Psi(n+1)-\Psi(a+n)-\Psi(b+n)-\log(1-\exp z)](1-\exp z)^n, \qquad (II.5)$$

$(|\arg(1-\exp z)|<\pi, |1-\exp z|<1)$

$$\frac{d}{dz}[(\exp z)^a F(a,b;c;\exp z)]=a(\exp z)^{a-1}F(a+1;b;c;\exp z) \qquad (II.6)$$

$$\int_0^1 \upsilon^{\alpha-1}(1-\upsilon)^{d-a-1}F(b,d;c,\upsilon\exp z)d\upsilon=\frac{\Gamma(a)\Gamma(d-a)}{\Gamma(d)}F(a,b;c;\exp z) \quad (II.7)$$

$(Red>Rea>0, |\arg(1-\exp z)|<\pi)$

$$\int_0^1 \upsilon^{\delta-1}(1-\upsilon)^{c-1}(1-\upsilon\exp z)^{b-d-c}F(b,d;c;\upsilon)d\upsilon=\frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b)}{\Gamma(d+e-a)\Gamma(d+e-b)}$$

$$(1-\exp z)^{b-d-c}\ F\left(e,d+e-a-b;d+e-a;d+e-b;\frac{\exp z}{\exp z-1}\right) \qquad (II.8)$$

$(Red>0, Ree>0, Re(d+e-a-b)>0, |\arg(1-\exp z)|<\pi)$

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