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A SIMPLE PROOF OF TRIPLE PRODUCT IDENTITY OF JACOBI

P.S. Guruprasad and Pradeep N.*

Department of Studies in Mathematics
University of Mysore, Manasa Gangotri, Mysore-570006, India

*G. S. S. S. Institute of Engineering and Technology for Women
K.R.S. Road, Metagalli, Mysore-570016, India
E-mails: guruprasadps@rediffmail.com; pradeep_bharadwaj83@rediffmail.com

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: In this note, we give a simple proof of Jacobi's triple product identity using q-binomial theorem.

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1. Introduction

Jacobi triple product identity states that

$$\sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} z^n \;=\; (q)_\infty(-zq)_\infty(-1/z)_\infty, \qquad z \neq 0, \quad |q| < 1. \tag{1.1}$$

Andrews [1] gave a proof of (1.1) using two identities of Euler. Combinatorial proofs of Jacobi's triple identity were given by Wright [7], Cheema [2] and Sudler [6]. We can also find a proof of (1.1) in [3]. Hirschhorn [4,5] has proved Jacobi's two-square and four-square theorems using Jacobi's triple product identity. The main purpose of this note is to give a simple proof of (1.1) using only q-binomial theorem:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{q_n} t^n \;=\; \frac{(at)_\infty}{(t)_\infty}, \qquad |t| < 1, \quad |q| < 1. \tag{1.2}$$

Changing a to a/b, t to bt, and letting $b \to 0$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \;=\; (at)_\infty, \qquad |q| < 1. \tag{1.3}$$

Putting $a = -1$ in the above identity, we deduce

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \;=\; (-t)_\infty, \qquad |q| < 1. \tag{1.4}$$