

On Reduction and Product Formulae for q- Series

Ashutosh Singh and Rama Jain*,

Department of Mathematics, T.D.P.G. College,

Jaunpur - 222002,(U.P.) India.

*Department of Mathematics, Mahila P.G. College, Lucknow, India

Address for correspondence: Rama Jain,

Department of Mathematics, Mahila P.G. College, Lucknow, India

Abstract: In this paper we have established certain reduction and product formulae for basic hypergeometric series.

Keywords: Basic hypergeometric series /Reduction formula /Product formula.

2000 Mathematics Subject Classification: Primary 33E05; Secondary 05A30, 33D15

1. Introduction, Notations and Definitions

As usual, let for any positive integer n,

$$[\alpha; q^k]_n = (1 - \alpha)(1 - \alpha q^k) \dots (1 - \alpha q^{k(n-1)}),$$

$$[\alpha; q^k]_0 = 1,$$

$$[\alpha; q^k]_\infty = \prod_{n=0}^{\infty} (1 - \alpha q^{nk}),$$

and

$$[a_1, a_2, \dots, a_r; q^k]_n = [a_1; q^k]_n [a_2; q^k]_n \dots [a_r; q^k]_n.$$

we define a generalized basic or q-hypergeometric series as,

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q^k; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q^k]_n z^n}{[q^k, b_1, b_2, \dots, b_s; q^k]_n} [(-)^n q^{kn(n-1)/2}]^{(1+s-r)}, \quad (1.1)$$

where $q \neq 0$ and $r \leq s + 1$.

The series (1.1) converges absolutely for all z, if $r \leq s$ and for $|z| < 1$ if $r = s + 1$.

Following Agrawal and Verma [1], we define the bibasic hypergeometric series as,

$$\phi \left[\begin{matrix} (a) : (b); x \\ (c) : (d); q^\lambda \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[(a); q^2]_n [(b); q]_n x^n q^{\lambda n(n-1)/2}}{[(c); q^2]_n [(d); q]_n}, \quad (1.2)$$

where (a) stands for the sequence of parameters $a_1, a_2, \dots, a_A$.

The series (1.2) converges for all values of x if $\lambda > 0$ and when $\lambda = 0$, we require the additional convergence condition $|x| < 1$.

We shall make use of the following identity in our analysis

$$\sum_{n=0}^{\infty} \sum_{r=0}^n A(n, r) = \sum_{n, r=0}^{\infty} A(n + r, r). \quad (1.3)$$

[2; (2.1.2) p.100]

2. Main Results

(i) Let us consider the following summation formula due to Verma and Jain [3; (3.4) p. 243]

$$\phi \left[\begin{matrix} q^{-2n}, -q^{2-n}, b^2; \frac{q^{1-n}}{b^2c}, -c; q \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2} : -b^2c, \frac{q^{1-n}}{c} \end{matrix} \right] = \frac{[-1; q]_n [b^2; q]_n [b^2c^2; q^2]_n}{[-b^2c; q]_n [c; q]_n [b^2; q^2]_n}. \quad (2.1)$$

Multiplying both sides of the summation formula (2.1) by $\Omega_n z^n$, summing over n from 0 to ∞ and then applying the identity (1.3) we get,

$$\begin{aligned} \sum_{n,r=0}^{\infty} \frac{\Omega_{n+r} z^{n+r} [b^2; q^2]_r [-c; q]_r [q^2; q^2]_{n+r} [b^2c; q]_{n+r} [b^2; q^2]_n [c; q]_n (q^n + q^r)}{[q^2; q^2]_r [-b^2c; q]_r [b^2; q^2]_{n+r} [c; q]_{n+r} [q^2; q^2]_n [b^2c; q]_n (1 + q^{n+r})} \\ = \sum_{n=0}^{\infty} \frac{\Omega_n z^n [-1, b^2, bc, -bc; q]_n}{[-b^2c, c, b, -b; q]_n}. \end{aligned} \quad (2.2)$$

Now, choosing $\Omega_n = \frac{[b^2; q^2]_n [c; q]_n}{[q^2; q^2]_n [b^2c; q]_n} (1 + q^n)$ in (2.2) we find,

$$\begin{aligned} & {}_3\phi_2 \left[\begin{matrix} b, -b, -c; q; z \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} b, -b, c; q; zq \end{matrix} \right] \\ & + {}_3\phi_2 \left[\begin{matrix} b, -b, -c; q; qz \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} b, -b, c; q; z \end{matrix} \right] \\ & = 2 {}_3\phi_2 \left[\begin{matrix} b^2, bc, -bc; q; z \end{matrix} \right]. \end{aligned} \quad (2.3)$$

(ii) Let us consider the following summation formula [Verma and Jain 3;(3.5) p. 643]

$$\phi \left[\begin{matrix} q^{-2n}, -q^{2-n}, b^2; \frac{q^{-n}}{b^2c}, c; q^2 \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2}; -b^2cq, \frac{q^{1-n}}{c} \end{matrix} \right] = \frac{[-1; q]_n [b^2; q]_n [b^2c^2q; q]_n [b^2c^2; q^2]_n}{[-b^2cq; q]_n [c; q]_n [b^2c^2; q]_n [b^2; q^2]_n}. \quad (2.4)$$

Multiplying both sides of the summation formula (2.4) by $\Omega_n z^n$, summation over n from 0 to ∞ and then applying the identity (1.3) we find after some simplifications,

$$\begin{aligned} \sum_{n,r=0}^{\infty} \Omega_{n+r} z^{n+r} \frac{[b^2; q^2]_r [-c; q]_r [b^2; q^2]_n [c; q]_n}{[q^2; q^2]_r [-b^2cq; q]_r [q^2; q^2]_n [b^2cq; q]_n} \times \\ \times \frac{[q^2; q^2]_{n+r} [b^2cq; q]_{n+r} (q^n + q^r)}{[b^2; q^2]_{n+r} [c; q]_{n+r} (1 + q^{n+r})} = \sum_{n=0}^{\infty} \frac{[-1, b^2, b^2c^2q, bc, -bc; q]_n}{[-b^2cq, c, b^2c^2, b, -b; q]_n}. \end{aligned} \quad (2.5)$$

Now, taking $\Omega_n = \frac{[b^2; q^2]_n [c; q]_n (1 + q^n)}{[q^2; q^2]_n [b^2cq; q]_n}$ in (2.5) we get

$${}_3\phi_2 \left[\begin{matrix} b, -b, -c; q; z \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} b, -b, c; q; zq \end{matrix} \right]$$

$$\begin{aligned}
& + {}_3\phi_2 \left[\begin{matrix} b, -b, -c; q; zq \\ -q, -b^2cq \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} b, -b, c; q; z \\ -q, -b^2cq \end{matrix} \right] \\
& = 2 {}_4\phi_3 \left[\begin{matrix} b^2, b^2c^2q, bc, -bc; q; z \\ -b^2cq, b^2c^2, b^2cq \end{matrix} \right].
\end{aligned} \tag{2.6}$$

(iii) Taking the following summation formula [Verma and Jain 3;(3.6) p. 643]

$$\phi \left[\begin{matrix} q^{-2n}, -q^{2-n}, b^2 : d, -\frac{q^{2-n}}{b^2}; \frac{1}{b^2} \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2} : -\frac{q^{1-n}}{d}, \frac{d}{q} \end{matrix} \right] = \frac{[b^2q; q]_n \left[\frac{d^2}{q}; q \right]_n \left[\frac{d^2}{q^2}; q^2 \right]_n [-1; q]_n}{\left[\frac{d}{q}; q \right]_n \left[\frac{d^2}{q^2}; q \right]_n [b^2q; q^2]_n [-d; q]_n}. \tag{2.7}$$

and doing all as above we get,

$$\begin{aligned}
& \sum_{n,r=0}^{\infty} \Omega_{n+r} z^{n+r} \frac{[q^2; q^2]_{n+r} [-d/q; q]_{n+r} [b^2; q^2]_r [d; q]_r [b^2; q^2]_n [-d; q]_n}{[b^2; q^2]_{n+r} [-d; q]_{n+r} [q^2; q^2]_r [d/q; q]_r [q^2; q^2]_n [-d/q; q]_n} \frac{q^n + q^r}{1 + q^{n+r}} \\
& = \sum_{n=0}^{\infty} \Omega_n z^n \frac{[-1, d/q, b^2q, d^2/q; q]_n}{[d^2/q^2, -d; q]_n [b^2q; q^2]_n}.
\end{aligned} \tag{2.8}$$

Now, choosing $\Omega_n = \frac{[b^2; q^2]_n [-d; q]_n}{[q^2; q^2]_n [-d/q; q]_n} (1 + q^n)$ in (2.8) we find,

$$\begin{aligned}
& {}_3\phi_2 \left[\begin{matrix} -d, b, -b; q; zq \\ -q, -d/q \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} b, -b, d; q; z \\ -q, d/q \end{matrix} \right] \\
& + {}_3\phi_2 \left[\begin{matrix} -d, b, -b; q; z \\ -q, -d/q \end{matrix} \right] {}_3\phi_2 \left[\begin{matrix} d, b, -b; q; zq \\ -q, d/q \end{matrix} \right] \\
& = 2 {}_4\phi_3 \left[\begin{matrix} b^2q, d^2/q, b, -b; q; z \\ d^2/q^2, b\sqrt{q}, -b\sqrt{q} \end{matrix} \right].
\end{aligned} \tag{2.9}$$

(iv) Let us consider the summation formula [Verma and Jain 3;(3.8) p. 643]

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{1-n}, c^2, d^2; q^2; q^2 \\ q^{-2n}, cdq, cdq^2 \end{matrix} \right] = \frac{[c; q]_{n+1} [-d; q]_{n+1} - [-c; q]_{n+1} [d; q]_{n+1}}{(d-c)[-1; q]_{n+1} [cdq; q]_n}. \tag{2.10}$$

Multiplying both sides of the summation formula (2.10) by $\Omega_n z^n$, summing over n from 0 to ∞ and then applying the identity,

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n/2} A(n, r) = \sum_{n,r=0}^{\infty} A(n+2r, r). \tag{2.11}$$

[2; (2.1.4) p.101]

we get

$$\sum_{n,r=0}^{\infty} (-)^r \Omega_{n+2r} z^{n+2r} \frac{[q; q]_{n+2r} [q^2; q^2]_{n+r} [c^2, d^2; q^2]_r q^{r^2+2r}}{[q^2; q^2]_{n+2r} [q; q]_n [q^2; q^2]_r [cdq, cdq^2; q^2]_r}$$

$$= \frac{(1-c)(1+d)}{2(c-d)} \sum_{n=0}^{\infty} \Omega_n \frac{[cq, -dq; q]_n z^n}{[-q, cdq; q]_n} - \frac{(1+c)(1-d)}{2(c-d)} \sum_{n=0}^{\infty} \Omega_n \frac{[-cq, dq; q]_n z^n}{[-q, cdq; q]_n} \quad (2.12)$$

Now, choosing $\Omega_n = [-q; q]_{n+2r}$ in (2.12) we get the following reduction formula,

$$\begin{aligned} & \sum_{n,r=0}^{\infty} \frac{(-)^n [q^2; q^2]_{n+r} [c^2, d^2, q^2]_r q^{r^2+2r} z^{n+2r}}{[q; q]_n [q^2, cdq, cdq^2; q^2]_r} \\ &= \frac{(1-c)(1+d)}{2(c-d)} {}_3\phi_2 \left[\begin{matrix} cq, -dq, q; q; z \\ cdq, 0 \end{matrix} \right] \\ & - \frac{(1+c)(1-d)}{2(c-d)} {}_3\phi_2 \left[\begin{matrix} -cq, dq, q; q; z \\ cdq, 0 \end{matrix} \right]. \end{aligned} \quad (2.13)$$

(v) Let us consider the following summation formula [Verma and Jain 3;(3.12) p. 644]

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, q^{1-2n}, c^2, d^2; q^2; q^2 \\ q^{-4n}, cdq, cdq^2 \end{matrix} \right] = \frac{[c; q]_{2n+1} [-d; q]_{2n+1} - [-c; q]_{2n+1} [d; q]_{2n+1}}{(d-c)[-1; q]_{2n+1} [cdq; q]_{2n}}. \quad (2.14)$$

Multiplying both sides of the summation formula (2.14) by $\Omega_n z^n$, summing over n from 0 to ∞ and then applying the identity (1.3) we get

$$\begin{aligned} & \sum_{n,r=0}^{\infty} \Omega_{n+r} z^{n+r} \frac{[q^2; q^2]_{r+2n} [c^2, d^2; q^2]_r (-)^r q^{r(r+2)}}{[q; q]_{2n} [-q; q]_{2r+2n} [q^2, cdq, cdq^2; q^2]_r} \\ &= \frac{(1-c)(1+d)}{2(d-c)} \sum_{n=0}^{\infty} \Omega_n z^n \frac{[cq, -dq; q]_{2n}}{[-q, cdq; q]_{2n}} - \frac{(1+c)(1-d)}{2(d-c)} \sum_{n=0}^{\infty} \Omega_n z^n \frac{[-cq, dq; q]_{2n}}{[-q, cdq; q]_{2n}}. \end{aligned} \quad (2.15)$$

Now, choosing $\Omega_n = [-q; q]_{2n}$ in (2.15) we get the following reduction formula,

$$\begin{aligned} & \sum_{n,r=0}^{\infty} \frac{[q^2; q^2]_{r+2n} [c^2, d^2; q^2]_r (-)^r q^{r(r+2)}}{[q; q]_{2n} [q^2, cdq, cdq^2; q^2]_r} \\ &= \frac{(1-c)(1+d)}{2(d-c)} {}_5\phi_4 \left[\begin{matrix} -dq, -dq^2, cq, cq^2, q^2; q^2; z \\ cdq, cdq^2, 0, 0 \end{matrix} \right] \\ & - \frac{(1+c)(1-d)}{2(d-c)} {}_5\phi_4 \left[\begin{matrix} dq, dq^2, -cq, -cq^2, q^2; q^2; z \\ cdq, cdq^2, 0, 0 \end{matrix} \right]. \end{aligned} \quad (2.16)$$

Acknowledgement

First author is thankful to the Department of Science and Technology, Government of India, New Delhi, for support under a major research project No. SR/ S4/ MS : 561 / 08 entitled "Investigations of Ramanujan's work in the field of basic hypergeometric series" and the authorities of U.G.C. for sanctioning a minor research project F. No. 8-3 (115)/2011 (MRP/NRCB) under which this work has been done.

References

- [1] Agarwal, R.P. and Verma, A., Generalized basic hypergeometric series with unconnected bases, Proc. Camb. Philos. Soc. 63 (1967), 181-192.
- [2] Srivastava, H. M. and Manocha, H.L., A treatise on Generating Functions, Ellis Horwood limited; Halsted Press; a division of John Wiley & Sons, New York (1984).
- [3] Verma, A. and Jain, V.K., Q-Analogue of a transformation of Whipple, Rocky Mountain Journal of Mathematics, vol.13, no. 4(1983), 639-650.

