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# MULTIVARIATE SYMMETRIZED AND PERTURBED $A$-GENERALIZED LOGISTIC COMPLEX VALUED TRIGONOMETRIC AND HYPERBOLIC NEURAL NETWORK ACCELERATED APPROXIMATION

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**Abstract:** Here we research the multivariate quantitative symmetrized and perturbed approximation of complex valued continuous functions on a box by complex valued symmetrized and perturbed multivariate neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the used function's high order partial derivatives. The types of our approximations are trigonometric and hyperbolic. Our multivariate symmetrized operators are defined by using a multivariate density function generated by a $q$-deformed and $\lambda$-parametrized $A$-generalized logistic function. These dynamic approximations are pointwise and of the uniform norm. The related complex valued feed-forward neural networks are with one hidden layer.

**Keywords and Phrases:** $q$-deformed and $\lambda$-parametrized $A$-generalized logistic, complex valued symmetrized multivariate neural network approximation, complex valued multivariate quasi-interpolation operator, modulus of continuity, trigonometric and hyperbolic dynamic approximation.

**2020 Mathematics Subject Classification:** 41A17, 41A25, 41A99, 42A10.

## 1. Introduction

The author in [1] and [2], see Chapters 2-5 was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

Again the author inspired by [10], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], by treating both the univariate and multivariate cases.

The brain assymetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, developmental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this article is a specific study under this philosophy of approaching reality as close as possible.

Consequently the author here performs multivariate symmetrized $q$-deformed and $\lambda$-parametrized $A$-generalized logistic activated high order multivariate neural network approximations to continuous functions over boxes with complex values. All convergences are with rates expressed via the moduli of continuity of the involved functions high order partial derivatives, deriving by very tight multivariate Jackson type inequalities.

The basics of our higher order approximations here are some newly discovered by the author trigonometric and hyperbolic type multivariate Taylor's formulae, see [8].

Our boxes are not necessarily symmetric to the origin. The applied symmetrization technique and the newly introduced related multivariate operators cut in half the feed to neural networks, thus enhancing immensely their convergence speed to the unit operator.

A multilayer feed-forward neural network can be defined as follows (with $m \in \mathbb{N}$ hidden layers):

Let $x \in \mathbb{R}^s$; $s \in \mathbb{N}$, where $x = (x_1, ..., x_s)$; $\alpha_j, c_j \in \mathbb{R}^s$; $b_j \in \mathbb{R}$, with $0 \leq j \leq n$, $n \in \mathbb{N}$.

Here $\langle \alpha_j \cdot x \rangle$ is the inner product, thus $\sigma\left(\langle \alpha_j \cdot x \rangle + b_j\right) \in \mathbb{R}$; and $N_n(x) \in \mathbb{R}^s$, by $c_j \in \mathbb{R}^s$, as it is coming from $N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot x \rangle + b_j\right)$.

We define:

$$N_n^{(2)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot N_n(x) \rangle + b_j\right) =$$

$$\sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot \left(\sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot x \rangle + b_j\right)\right)\right\rangle + b_j\right).$$

Furthermore, we can define

$$N_n^{(3)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(2)}(x) \right\rangle + b_j\right).$$

And, in general we define:

$$N_n^{(m)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(m-1)}(x) \right\rangle + b_j\right), \quad \text{for } m \in \mathbb{N}.$$

For more studies in neural networks read [11] - [15].

## 2. Basics

Initially we follow [7], pp. 395-471.

Our activation function here to be used is the $q$-deformed and $\lambda$-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R},\ q, \lambda > 0,\ A > 1. \tag{1}$$

This is the $A$-generalized logistic function.

For more read Chapter 16 of [7]: "Banach space valued ordinary and fractional neural network approximation based on $q$-deformed and $\lambda$-parametrized $A$-generalized logistic function".

The chapters 15, 16 of [7] motivate our current work.

The proposed "symmetrization technique" aims to use half data feed to our neural networks.

We will employ the following density function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R},\ q, \lambda > 0. \tag{2}$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{3}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \ \ \forall\, x \in \mathbb{R}. \tag{4}$$

Adding (3) and (4) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \ \ \forall\, x \in \mathbb{R}, \tag{5}$$

the key to this work.

So that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the $y$-axis.

The global maximum of $G_{q,\lambda}$ is given by (16.18), p. 401 of [7] as

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}. \tag{7}$$

And, the global max of $G_{\frac{1}{q},\lambda}$ is

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}, \tag{8}$$

both sharing the same maximum at symmetric points.

By Theorem 16.1, p. 401 of [7], we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x - i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0, \ A > 1, \tag{9}$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x - i) = 1, \ \ \forall\, x \in \mathbb{R},\ \lambda, q > 0, \ A > 1. \tag{10}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} W(x - i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{11}$$

By Theorem 16.2, p. 402 of [7], we have that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0, \ A > 1, \tag{12}$$

similarly it holds

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x)\,dx = 1, \tag{13}$$

so that

$$\int_{-\infty}^{\infty} W(x)\,dx = 1, \tag{14}$$

therefore $W$ is a density function.

By Theorem 16.3, p. 402 of [7], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then

$$\sum_{\substack{k=-\infty \\ :\, |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k) < 2\max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{15}$$

where $\lambda, q > 0$, $A > 1$; $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Similarly, we get that

$$\sum_{\substack{k=-\infty \\ :\, |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{\frac{1}{q},\lambda}(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}. \tag{16}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ :\, |nx-k| \geq n^{1-\alpha}}}^{\infty} W(nx-k) < \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{17}$$

where $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$.

Here $\lceil\cdot\rceil$ denotes the ceiling of the number, and $\lfloor\cdot\rfloor$ its integral part.

We mention

**Theorem 16.4 (p. 402, [7])** *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Then*

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K(q). \tag{18}$$

*Similarly, we consider* $\lambda_{\frac{1}{q}} > z_1 > 0$, *such that* $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, *and* $\lambda_{\frac{1}{q}} > 1$. *Thus*

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}(nx-k)} < \max\left\{\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{G_{q,\lambda}(\lambda_q)}\right\} = K(q). \tag{19}$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) > \frac{1}{K(q)}, \tag{20}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}(nx-k) > \frac{1}{K(q)}. \tag{21}$$

Consequently it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k)\right)}{2} > \frac{2}{2K(q)} = \frac{1}{K(q)}, \tag{22}$$

so that

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k)+G_{\frac{1}{q},\lambda}(nx-k)\right)}{2}} < K(q), \tag{23}$$

that is

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q). \tag{23}$$

We have proved.

**Theorem 1.** *Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na\rceil \leq \lfloor nb\rfloor$. *For* $q, \lambda > 0$, $A > 1$, *we consider* $\lambda_q > z_0 > 0$ *with* $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, *and* $\lambda_q > 1$. *Also consider* $\lambda_{\frac{1}{q}} > z_1 > 0$, *such that* $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, *and* $\lambda_{\frac{1}{q}} > 1$. *Then*

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q). \tag{25}$$

We make,

**Remark 2.** *I) By Remark 16.5, p. 402 of [7], we have*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}\left(nx_1-k\right)\neq 1,\ \textit{for some } x_1\in[a,b]\,, \tag{26}$$

*and*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\neq 1,\ \textit{for some } x_2\in[a,b]\,. \tag{27}$$

*Therefore it holds*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_2-k\right)\right)}{2}\neq 1. \tag{28}$$

*Hence*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}\left(nx_1-k\right)+G_{\frac{1}{q},\lambda}\left(nx_1-k\right)\right)}{2}\neq 1, \tag{29}$$

*even if*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}\left(nx_1-k\right)=1, \tag{30}$$

*because then*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{G_{q,\lambda}\left(nx_1-k\right)}{2}+\frac{1}{2}\neq 1, \tag{31}$$

*equivalently*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{G_{q,\lambda}\left(nx_1-k\right)}{2}\neq \frac{1}{2}, \tag{32}$$

*true by*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}\left(nx_1-k\right)\neq 1. \tag{33}$$

*II) Let $[a,b] \subset \mathbb{R}$. For large $n$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. So in general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W\left(nx-k\right) \leq 1. \tag{34}$$

Next, we move on to the multivariate case, see Chapter 15 of [7], pp. 365-394, as a model of action.

We make,

**Remark 3.** *We introduce*

$$Z\left(x_1,...,x_N\right) := Z\left(x\right) := \prod_{i=1}^{N} W\left(x_i\right) = \frac{1}{2^N} \prod_{i=1}^{N} \left(G_{q,\lambda} + G_{\frac{1}{q},\lambda}\right)\left(x_i\right), \tag{35}$$

$x = \left(x_1,...,x_N\right) \in \mathbb{R}^N$, $\lambda, q > 0$, $N \in \mathbb{N}$.

*Properties:*

*(i)*

$$Z\left(x\right) > 0, \ \forall \ x \in \mathbb{R}^N, \tag{36}$$

*(ii)*

$$\sum_{k=-\infty}^{\infty} Z\left(x-k\right) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(x_1-k_1,...,x_N-k_N\right) = 1, \tag{37}$$

$k := \left(k_1,...,k_n\right) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

*hence*

*(iii)*

$$\sum_{k=-\infty}^{\infty} Z\left(nx-k\right) = 1, \tag{38}$$

$\forall \ x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

*(iv)*

$$\int_{\mathbb{R}^N} Z\left(x\right) dx = 1, \tag{39}$$

*that is $Z$ is a multivariate density function.*

*Here denote* $\left\|x\right\|_{\infty} := \max\left\{\left|x_1\right|,...,\left|x_N\right|\right\}$, $x \in \mathbb{R}^N$, $\infty := \left(\infty,...,\infty\right)$, $-\infty := \left(-\infty,...,-\infty\right)$, $\lceil na \rceil := \left(\lceil na_1 \rceil,...,\lceil na_N \rceil\right)$, $\lfloor nb \rfloor := \left(\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor\right)$, $a := \left(a_1,...,a_N\right)$, $b := \left(b_1,...,b_N\right)$.

*We obviously see that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z\left(nx-k\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N} W\left(nx_i-k_i\right)\right)=\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} W\left(nx_i-k_i\right)\right). \tag{40}$$

*(v) We derive that*

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z\left(nx-k\right)<\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \quad \text{where } 0<\beta<1, \tag{41}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $x\in\prod_{i=1}^{N}\left[a_i,b_i\right]$.

*(vi) It holds*

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z\left(nx-k\right)}<\left(K\left(q\right)\right)^N, \tag{42}$$

$\forall\ x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right),\ \ n\in\mathbb{N}$.

*It is clear that*

*(vii)*

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\infty} Z\left(nx-k\right)<\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \tag{43}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $x\in\prod_{i=1}^{N}\left[a_i,b_i\right]$.

*Furthermore it holds*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z\left(nx-k\right)\neq 1, \tag{44}$$

*for at least some* $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$.

*Here* $\left(X,\left\|\cdot\right\|_\gamma\right)$ *is a Banach space.*

*Let* $f\in C\left(\prod_{i=1}^{N}\left[a_i,b_i\right],X\right)$, $x=\left(x_1,...,x_N\right)\in\prod_{i=1}^{N}\left[a_i,b_i\right]$, $n\in\mathbb{N}: \lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

*We introduce and define the following multivariate linear normalized symmetrized neural network operator, let* $x := (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i]$*:*

$$\psi_n^s (f, x_1, ..., x_N) := \psi_n^s (f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z (nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z (nx - k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \left(G_{q,\lambda} (nx_i - k_i) + G_{\frac{1}{q},\lambda} (nx_i - k_i)\right)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \left(G_{q,\lambda} (nx_i - k_i) + G_{\frac{1}{q},\lambda} (nx_i - k_i)\right)\right)}. \tag{45}$$

*For large enough* $n \in \mathbb{N}$ *we always obtain* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$*,* $i = 1, ..., N$*. Also* $a_i \leq \frac{k_i}{n} \leq b_i$*, iff* $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$*,* $i = 1, ..., N$*.*

*When* $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *we define the companion operator*

$$\widetilde{\psi}_n^s (g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z (nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z (nx - k)}. \tag{46}$$

*Clearly* $\widetilde{\psi}_n^s$ *is a positive linear operator. We have that*

$$\widetilde{\psi}_n^s (1, x) = 1, \ \ \forall\ x \in \prod_{i=1}^{N} [a_i, b_i].$$

*Notice that* $\psi_n^s (f) \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$ *and* $\widetilde{\psi}_n^s (g) \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$*.*

*Furthermore it holds*

$$\left\| \psi_n^s (f, x) \right\|_\gamma \leq \widetilde{\psi}_n^s \left( \|f\|_\gamma, x \right), \tag{47}$$

$\forall\ x \in \prod_{i=1}^{N} [a_i, b_i]$*,*

*and*

$$\psi_n^s (cg, x) = c\widetilde{\psi}_n^s (g, x), \ \ \forall\ x \in \prod_{i=1}^{N} [a_i, b_i], \tag{48}$$

*and*

$$\psi_n^s (c) = c, \ \ \forall\ c \in X. \tag{49}$$

*Set also*

$$^*\psi_n^s (f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z (nx - k). \tag{50}$$

*Consequently we derive (by (42))*

$$|\psi_n^s(f,x)-f(x)| \leq (K(q))^N \, |{}^*\psi_n^s(f,x)-f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)|, \tag{51}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$.

*We will estimate the right hand side of (51), when $X=(\mathbb{C},|\cdot|)$, the complex numbers.*

For the last and others we need,

**Definition 4.** ([6], p. 274) *Let $M$ be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p\in[1,\infty]$, and $\left(X,\|\cdot\|_\gamma\right)$ be a Banach space. Let $f\in C(M,X)$. We define the first modulus of continuity of $f$ as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq\delta}} \|f(x)-f(y)\|_\gamma, \quad 0<\delta\leq diam(M). \tag{52}$$

*If $\delta > diam(M)$, then*

$$\omega_1(f,\delta)=\omega_1(f,diam(M)). \tag{53}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$. For $f\in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

**Lemma 5.** ([6], p. 274) *We have $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, iff $f\in C(M,X)$, where $M$ is a convex compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$.*

In this article (see (52)) we work with $p=\infty$.

**(II) Multivariate Alternative Taylor formulae**

We will use,

**Theorem 6.** ([8]) *Let $f\in C^2([c,d],\mathbb{C})$, where $a,x\in[c,d]$. Then*

$$f(x)-f(a)=f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+ \tag{54}$$

$$\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt.$$

We make,

**Remark 7.** *Let now $Q$ be an open convex subset of $\mathbb{R}^k$, $k \geq 2$ ; $z = (z_1, ..., z_k)$, $x_0 := (x_{01}, ..., x_{0k}) \in Q$. We consider $f \in C^2(Q, \mathbb{C})$ each second order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., k$ and $|\alpha| := \sum_{i=1}^{k} \alpha_i = 2$. We consider $g_z(t) := f(x_0 + t(z - x_0))$, $0 \leq t \leq 1$. Clearly $x_0 + t(z - x_0) \in Q$. Then*

$$g_z(0) = f(x_0), \quad g_z(1) = f(z),$$

$$g_z'(t) = \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i} (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})), \tag{55}$$

$$g_z'(0) = \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i} (x_{01}, ..., x_{0k}),$$

*and*

$$g_z''(t) = \left[ \left( \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^2 f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})), \tag{56}$$

$$g_z''(0) = \left[ \left( \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^2 f \right] (x_{01}, ..., x_{0k}).$$

*Notice above the second order partials commute.*

*Clearly $g_z \in C^2([0,1], \mathbb{C})$, and by Theorem 6 we obtain*

$$f(z_1, ..., z_k) - f(x_{01}, ..., x_{0k}) = g_z(1) - g_z(0) =$$

$$g_z'(0) \sin(1) + 2 g_z''(0) \sin^2\left(\frac{1}{2}\right) + \int_0^1 [(g_z''(t) + g_z(t)) - (g_z''(0) + g_z(0))] \sin(1 - t)\, dt. \tag{57}$$

We also mention,

**Theorem 8.** ([8]) *Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a) \sinh(x - a) + 2 f''(a) \sinh^2\left(\frac{x - a}{2}\right) + \tag{58}$$

$$\int_a^x [(f''(t) - f(t)) - (f''(a) - f(a))] \sinh(x - t)\, dt.$$

We make,

**Remark 9.** *Consequently, we get that*

$$f(z_1,...,z_k)-f(x_{01},...,x_{0k})=g_z(1)-g_z(0)=g_z'(0)\sinh(1)$$

$$+2g_z''(0)\sinh^2\left(\frac{1}{2}\right)+\int_0^1\left[\left(g_z''(t)-g_z(t)\right)-\left(g_z''(0)-g_z(0)\right)\right]\sinh(1-t)\,dt. \quad (59)$$

We make,

**Remark 10.** *Let* $f\in C^2\left(\prod_{i=1}^N[a_i,b_i],\mathbb{C}\right)$, $N\in\mathbb{N}$. *Here* $f_\alpha$ *denotes a partial derivative of* $f$, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}_+$, $i=1,...,N$, *and* $|\alpha|:=\sum_{i=1}^N\alpha_i=l$, *where* $l=0,1,2$.

We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order $l$.

We denote

$$\omega_{1,2}^{\max}(f_\alpha,h):=\max_{\alpha:|\alpha|=2}\omega_1(f_\alpha,h). \quad (60)$$

Call also

$$\|f_\alpha\|_{\infty,2}^{\max}:=\max_{|\alpha|=2}\left\{\|f_\alpha\|_\infty\right\}, \quad (61)$$

where $\|\cdot\|_\infty$ is the supremum norm.

Above the mixed partials commute.

Here $\frac{k}{n}:=\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)$, and $x:=(x_1,...,x_N)$, with $\frac{k}{n},x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$, then (by (57), where $g_{\frac{k}{n}}(t):=f\left(x+t\left(\frac{k}{n}-x\right)\right)$, $0\le t\le 1$) we have

$$f\left(\frac{k}{n}\right)-f(x)=\left(\sum_{i=1}^N\left(\frac{k_i}{n}-x_i\right)\frac{\partial f}{\partial x_i}(x)\right)\sin(1)+$$

$$2\left\{\left[\left(\sum_{i=1}^N\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right)+$$

$$\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^N\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x+t\left(\frac{k}{n}-x\right)\right)+f\left(x+t\left(\frac{k}{n}-x\right)\right)\right\}-\right.$$

$$\left.\left\{\left[\left(\sum_{i=1}^N\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\right\}\sin(1-t)\,dt. \quad (62)$$

Denote the remainder

$$R := \int_0^1 \left\{ \left\{ \left[ \left( \sum_{i=1}^{N} \left( \frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^2 f \right] \left( x + t \left( \frac{k}{n} - x \right) \right) + f \left( x + t \left( \frac{k}{n} - x \right) \right) \right\} \right.$$

$$\left. - \left\{ \left[ \left( \sum_{i=1}^{N} \left( \frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^2 f \right] (x) + f(x) \right\} \right\} \sin(1-t)\, dt = \tag{63}$$

$$\int_0^1 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \left( \prod_{i=1}^{N} \left( \frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \left[ f_\alpha \left( x + t \left( \frac{k}{n} - x \right) \right) - f_\alpha(x) \right] \right.$$

$$\left. + \left( f \left( x + t \left( \frac{k}{n} - x \right) \right) - f(x) \right) \right\} \sin(1-t)\, dt.$$

Therefore it holds

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \right.$$

$$\left( \prod_{i=1}^{N} \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \left| f_\alpha \left( x + t \left( \frac{k}{n} - x \right) \right) - f_\alpha(x) \right|$$

$$\left. + \left| f \left( x + t \left( \frac{k}{n} - x \right) \right) - f(x) \right| \right\} |\sin(1-t)|\, dt \leq \tag{64}$$

$$\int_0^1 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \left( \prod_{i=1}^{N} \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \omega_1 \left( f_\alpha, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right.$$

$$\left. + \omega_1 \left( f, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right\} |\sin(1-t)|\, dt \leq (*).$$

Notice here that $(0 < \beta < 1)$

$$\left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \Leftrightarrow \left| \frac{k_i}{n} - x_i \right| \leq \frac{1}{n^{\beta}}, \ \ i = 1, ..., N. \tag{65}$$

We further see that

$$(*) \leq \left\{ \omega_{1,2}^{\max} \left( f_{\alpha}, \frac{1}{n^{\beta}} \right) \left( \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, |\alpha|:=\sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \left( \prod_{i=1}^{N} \frac{1}{n^{\beta \alpha_i}} \right) \right) \right.$$

$$\left. + \omega_1 \left( f, \frac{1}{n^{\beta}} \right) \right\} \int_0^1 |\sin(1-t)| \, dt =$$

$$\left[ \omega_{1,2}^{\max} \left( f_{\alpha}, \frac{1}{n^{\beta}} \right) \left( \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, |\alpha|:=\sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \right) \frac{1}{n^{2\beta}} \right.$$

$$\left. + \omega_1 \left( f, \frac{1}{n^{\beta}} \right) \right] (1 - \cos(1)) = \tag{66}$$

$$(1 - \cos(1)) \left\{ \frac{\omega_{1,2}^{\max} \left( f_{\alpha}, \frac{1}{n^{\beta}} \right) N^2}{n^{2\beta}} + \omega_1 \left( f, \frac{1}{n^{\beta}} \right) \right\}.$$

We have proved that

$$|R| \leq (1 - \cos(1)) \left\{ \frac{\omega_{1,2}^{\max} \left( f_{\alpha}, \frac{1}{n^{\beta}} \right) N^2}{n^{2\beta}} + \omega_1 \left( f, \frac{1}{n^{\beta}} \right) \right\}, \tag{67}$$

given that $\left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}$.

We notice also that

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, |\alpha|:=\sum_{i=1}^{N} \alpha_i = 2}} \left( \frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \right.$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)2\left\|f_\alpha\right\|_\infty+2\left\|f\right\|_\infty\Bigg\}\left|\sin(1-t)\right|dt\leq \tag{68}$$

$$\left\{\left(\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right)\right.$$

$$\left.2\left\|b-a\right\|_\infty^2\left\|f_\alpha\right\|_{\infty,2}^{\max}+2\left\|f\right\|_\infty\right\}\left(\int_0^1\left|\sin(1-t)\right|dt\right)=$$

$$\left(2\left\|b-a\right\|_\infty^2\left\|f_\alpha\right\|_{\infty,2}^{\max}N^2+2\left\|f\right\|_\infty\right)(1-\cos(1)),$$

where $a:=(a_1,...,a_N)$, $b=(b_1,...,b_N)$.

We have proved that

$$|R|\leq\left(2\left\|b-a\right\|_\infty^2\left\|f_\alpha\right\|_{\infty,2}^{\max}N^2+2\left\|f\right\|_\infty\right)(1-\cos(1))=:\rho. \tag{69}$$

## 3. Main Results

Here we discuss the trigonometric symmetrized and perturbed approximation by using the smoothness of $f$.

**Theorem 11.** *Let* $f\in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $0<\beta<1$, $n,N\in\mathbb{N}$, $n^{1-\beta}>2$, $x,x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a:=(a_1,...,a_N)$, $b:=(b_1,...,b_N)$. *Then*

*(i)*

$$\left|\psi_n^s(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\psi_n^s((\cdot-x_i),x)\right)\sin(1)-\right.$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\psi_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|\leq$$

$$(K(q))^N\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \tag{70}$$

$$\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\Big\},$$

*(ii) assume that* $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,...,N$, *and* $f_{\alpha}\left(x_0\right)=0$, $\alpha:\left|\alpha\right|=2$, *we have that*

$$\left|\psi_{n}^{s}\left(f,x\right)-f\left(x\right)\right|\leq$$

$$\left(K\left(q\right)\right)^{N}\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right. \tag{71}$$

$$\left.\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\},$$

*(iii)*

$$\left|\psi_{n}^{s}\left(f,x\right)-f\left(x\right)\right|\leq\left(K\left(q\right)\right)^{N}$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f\left(x\right)}{\partial x_{i}}\right|\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}\sin\left(1\right)+\right.\right.$$

$$4\left\{\sum_{\alpha:\left|\alpha\right|=2}\left|f_{\alpha}\left(x\right)\right|\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\Bigg\}+$$

$$\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.$$

$$\left.\left.+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}, \tag{72}$$

*and*

*(iv)*

$$\left\|\psi_{n}^{s}\left(f\right)-f\right\|_{\infty}\leq\left(K\left(q\right)\right)^{N}$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}\sin\left(1\right)+\right.\right.$$

$$4\left\{\sum_{\alpha:\left|\alpha\right|=2}\left\|f_{\alpha}\right\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\Bigg\}$$

$$\left.+\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right. \tag{73}$$

$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\Big\}\Big\}=:\xi_{n}^{s}\left(f\right).$$

*We observe that $\psi_n^s \to I$ (unit operator), as $n \to \infty$, pointwise and uniformly.*

**Proof.** Here $R$ is as in (63). We see that

$$\widetilde{U}_{n}:=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)R= \tag{74}$$

$$\sum_{\left\{\begin{array}{c}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nb\rfloor}Z\left(nx-k\right)R+\sum_{\left\{\begin{array}{c}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nb\rfloor}Z\left(nx-k\right)R.$$

Therefore

$$\left|\widetilde{U}_{n}\right|\leq\left(\sum_{\left\{\begin{array}{c}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nb\rfloor}Z\left(nx-k\right)\right)$$

$$\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\leq \tag{75}$$

$$\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}.$$

We have established that

$$\left|\widetilde{U}_{n}\right|\leq\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]$$

$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}. \tag{76}$$

By (62) we observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z\left(nx-k\right)-f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)\right)=$$

$$(\sum_{i=1}^{N}\left(\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)\left(\frac{k_i}{n}-x_i\right)\right)\frac{\partial f}{\partial x_i}\left(x\right)\right))\sin\left(1\right)+$$

$$2\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum\limits_{i=1}^{N}\alpha_i=2}}f_\alpha\left(x\right)\left(\frac{2}{\prod\limits_{i=1}^{N}\alpha_i!}\right)(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)$$

$$\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right))\Bigg\}\sin^2\left(\frac{1}{2}\right)+\widetilde{U}_n. \tag{77}$$

The last says

$$^*\psi_n^s\left(f,x\right)-f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)\right)-$$

$$\left(\sum_{=1}^{N}\frac{\partial f\left(x\right)}{\partial x_i}\ ^*\psi_n^s\left(\left(\cdot-x_i\right),x\right)\right)\sin\left(1\right)-$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum\limits_{i=1}^{N}\alpha_i=2}}f_\alpha\left(x\right)\left(\frac{2}{\prod\limits_{i=1}^{N}\alpha_i!}\right)\ ^*\psi_n^s\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)=\widetilde{U}_n. \tag{78}$$

We notice that

$$|^*\psi_n^s\left(\left(\cdot-x_i\right),x\right)|\leq\ ^*\psi_n^s\left(|\cdot-x_i|,x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z\left(nx-k\right)=$$

$$\sum_{\substack{k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z\left(nx-k\right)+$$

$$\sum_{\substack{k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z\left(nx-k\right)\leq$$

$$\frac{1}{n^{\beta}}+(b_i-a_i)\sum_{\substack{k=\lceil na\rceil \\ \left\{ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor} Z\left(nx-k\right)\leq \tag{79}$$

$$\frac{1}{n^{\beta}}+\left(b_i-a_i\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}.$$

We have proved that

$$\left|{}^{*}\psi_n^s\left(\left(\cdot-x_i\right),x\right)\right|\leq\frac{1}{n^{\beta}}+\left(b_i-a_i\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \tag{80}$$

$i=1,...,N$.

Next we see that

$$\left|{}^{*}\psi_n^s\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right|\leq\ {}^{*}\psi_n^s\left(\prod_{i=1}^{N}\left|\cdot-x_i\right|^{\alpha_i},x\right)=$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z\left(nx-k\right)=$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\{ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z\left(nx-k\right)+ \tag{81}$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\{ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z\left(nx-k\right)\leq$$

$$\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_i-a_i\right)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}.$$

We have proved that

$$\left|{}^{*}\psi_n^s\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right|\leq\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_i-a_i\right)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}. \tag{82}$$

At last we observe that

$$\left|\psi_n^s(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\psi_n^s\left(\left(\cdot-x_i\right),x\right)\right)\sin(1)-\right.$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\psi_n^s\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|\leq$$

$$\left(K(q)\right)^N\left|\widetilde{U}_n\right|=$$

$$\left(K(q)\right)^N\left|{}^*\psi_n^s(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)-\right.$$

$$\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\ {}^*\psi_n^s\left(\left(\cdot-x_i\right),x\right)\right)\sin(1)-$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\ {}^*\psi_n^s\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|. \quad (83)$$

Putting all of the above together we have proved the theorem.

We continue with the hyperbolic symmetrized and perturbed approximation.

**Theorem 12.** *Let* $f\in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $0<\beta<1$, $n,N\in\mathbb{N}$, $n^{1-\beta}>2$, $x,x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a:=(a_1,...,a_N)$, $b:=(b_1,...,b_N)$. *Then*

*(i)*

$$\left|\psi_n^s(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\psi_n^s\left(\left(\cdot-x_i\right),x\right)\right)\sinh(1)-\right.$$

$$4\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\psi_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg|\leq$$

$$(K(q))^N\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \tag{84}$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\},$$

*(ii) assume that* $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,...,N$, *and* $f_\alpha(x_0)=0$, $\alpha:|\alpha|=2$, *we have that*

$$|\psi_n^s(f,x)-f(x)|\leq$$

$$(K(q))^N(\cosh(1))\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \tag{85}$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\},$$

*(iii)*

$$|\psi_n^s(f,x)-f(x)|\leq(K(q))^N$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta}+(b_i-a_i)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}\sinh(1)+\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}|f_\alpha(x)|\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right.$$

$$\left.\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}, \tag{86}$$

*and*

*(iv)*

$$\|\psi_n^s(f)-f\|_\infty\leq(K(q))^N$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left\{\frac{1}{n^\beta}+(b_i-a_i)\,\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}\sinh(1)+\right.\right.$$

$$\left.\left.4\left\{\sum_{\alpha:|\alpha|=2}\|f_\alpha\|_\infty\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right]\right\}\sinh^2\left(\frac{1}{2}\right)\right\}\right. \tag{87}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right.$$

$$\left.\left.\left[\|b-a\|_\infty^2\,\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\right\}\right\}=:\widetilde{\varphi}(f).$$

*We observe that $\psi_n^s\to I$ (unit operator), as $n\to\infty$, pointwise and uniformly.*

**Proof.** It is lengthy and similar to [9], Theorem 3.8, pp. 103-108. As such it is omitted.

We make,

**Remark 13.** *By (45) we get that $\|\psi_n^s(f)\|_\infty\le\|f\|_\infty<\infty$, and $\psi_n^s(f)\in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, given that $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$.*

*Clearly then*

$$\left\|\left(\psi_n^s\right)^2(f)\right\|_\infty=\|\psi_n^s(\psi_n^s(f))\|_\infty\le\|\psi_n^s(f)\|_\infty\le\|f\|_\infty, \tag{88}$$

*etc.*

*Therefore we get*

$$\left\|\left(\psi_n^s\right)^k(f)\right\|_\infty\le\|f\|_\infty,\quad\forall\ k\in\mathbb{N}, \tag{89}$$

*the contraction property.*

*Also we see that*

$$\left\|\left(\psi_n^s\right)^k(f)\right\|_\infty\le\left\|\left(\psi_n^s\right)^{k-1}(f)\right\|_\infty\le...\le\|\psi_n^s(f)\|_\infty\le\|f\|_\infty. \tag{90}$$

*Also $\psi_n^s(1)=1$, $\left(\psi_n^s\right)^k(1)=1$, $\forall\ k\in\mathbb{N}$.*

*Following 18.14, pp. 401-402, of [4], similarly we obtain that*

$$\|(\psi_n^s)^r f-f\|_\infty\le r\,\|\psi_n^s(f)-f\|_\infty,\quad r\in\mathbb{N}. \tag{91}$$

We give,

**Theorem 14.** *All as in Theorems 11, 12. Then*

*(i)*

$$\left\| \left( \psi_n^s \right)^r f - f \right\|_\infty \leq r \xi_n^s \left( f \right), \tag{92}$$

*where* $\xi_n^s \left( f \right)$ *as in (73).*

*(ii)*

$$\left\| \left( \psi_n^s \right)^r f - f \right\|_\infty \leq r \widetilde{\varphi}_n \left( f \right), \tag{93}$$

*where* $\widetilde{\varphi}_n \left( f \right)$ *as in (87).*

So that the speed of convergence to the unit operator of $\left( \psi_n^s \right)^r$ is not worse than of $\psi_n^s$.

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