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ON THE BICOMPLEX VERSION OF GERSHGORIN CIRCLE THEOREM AND BOUNDS FOR ZEROS OF BICOMPLEX POLYNOMIALS

Tanchar Molla

Department of Mathematics,
Dumkal College, Basantapur,
Dumkal, Murshidabad - 742406, West Bengal, INDIA

E-mail : tanumath786@gmail.com

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Abstract: This paper develops a bicomplex Gershgorin circle theorem for locating eigenvalues of $n \times n$ bicomplex matrices and applies it to derive classical bounds for zeros of bicomplex monic polynomials.

Keywords and Phrases: Bicomplex number, Gershgorin theorem, eigenvalue, companion matrix, zero, polynomial.

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1. Introduction

Determining all eigenvalues of a complex square matrix of higher order is a challenging task. Therefore, it is highly desirable to accurately locate all of these eigenvalues. The precise location of the eigenvalues of a matrix plays a crucial role in determining its properties such as singularity, non-singularity, positive definiteness, or negative definiteness. In 1931, Gershgorin [5] introduced the following remarkable result known as the Gershgorin circle theorem, which provides valuable insights into the eigenvalue distribution of a complex square matrix.

Theorem A. [5] *Let $A = (a_{rs})$ be a complex $n \times n$ matrix. Let $R_r(A) = \sum_{s \neq r} |a_{rs}|$, $r =$*

$1, 2, 3, \ldots$. *Then every eigenvalue of A lies within at least one of the discs* $|z-a_{rr}| \leq R_r(A)$, $r = 1, 2, \ldots$.

A complex monic polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ of degree n may be written as

$$P(z) = \begin{vmatrix} -z & 0 & \ldots & 0 & -a_0 \\ 1 & -z & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a_{n-1} - z \end{vmatrix}.$$

Therefore, $P(z)$ becomes the characteristic polynomial of the $n \times n$ matrix

$$C_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}.$$

The matrix C_n is known as the companion matrix of $P(z)$. Obviously, eigenvalues of the matrix C_n are the zeros of the polynomial $P(z)$.

The problem of determining the location of all the zeros of a polynomial in the field of complex numbers $\mathbb{C}$ has a rich historical background [4, 10, 12]. One of the classical results in this field was introduced by Cauchy in 1829 [10], which is as follows.

Theorem B. [10] *If* $P(z) = \sum_{r=0}^{n} a_r z^r$ *is a polynomial of degree* n *in* $\mathbb{C}$ *with complex coefficients, then all the zeros of* $P(z)$ *lie in* $|z| \leq 1 + \max\limits_{0 \leq r \leq n-1} |\frac{a_r}{a_n}|$.

While Theorem A has been improved for complex square matrices [11, 17] and Theorem B extensively developed for polynomials in $\mathbb{C}$ [10, 12], comprehensive extensions of Theorem A to the bicomplex setting are lacking, though some reflections of Theorem B appear in [16].

This paper revisits Theorem A for bicomplex square matrices and derives bounds for the zeros of bicomplex polynomials, analogous to Theorem B. We demonstrate the results with illustrative examples. Standard definitions and notations in bicomplex analysis are assumed and can be found in

2. Preliminary Definitions and Notations

In order to address various problems in physics that couldn't be solved in three dimensions, W. R. Hamilton introduced quaternions in 1844 [9]. Quaternions are an

extension of complex numbers in four-dimensional settings and are defined by three anticommute imaginary units: $i, j,$ and k, satisfying the relation $ij = k$. However, due to their lack of commutativity, several challenges arise when attempting to extend theories of holomorphicity from the complex plane to the skew field of quaternions.

Bicomplex numbers, on the other hand, provide a compatible four-dimensional extension of complex numbers. They have been widely utilized in various fields such as digital image processing, geometry, and theoretical physics [1, 2]. Bicomplex numbers offer a useful framework for tackling problems that require a higher-dimensional extension beyond complex numbers. In 1892, Segre [15] defined bicomplex numbers as $\mathbb{C}_2 = \{z : z = z_1 + jz_2, z_1, z_2 \in \mathbb{C}\} = \{a_1 + ia_2 + ja_3 + ka_4 : a_r \in \mathbb{R}, r = 1, 2, 3, 4\}$ where $ij = ji = k$ and $i^2 = j^2 = -k^2 = -1$. Thus bicomplex numbers can be considered as the complex numbers with complex coefficients.

Addition and multiplication on $\mathbb{C}_2$ are defined as follows as on $\mathbb{C}$:

If $w_1 = z_1 + jz_2 = a_1 + ia_2 + ja_3 + ka_4$ & $w_2 = z_3 + jz_4 = b_1 + ib_2 + jb_3 + kb_4$, then

- $w_1 + w_2 = (z_1 + z_3) + j(z_2 + z_4) = (a_1 + b_1) + (a_2 + b_2)i + (a_3 + b_3)j + (a_4 + b_4)k$
- $w_1.w_2 = (z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)j = (a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4) + (a_1b_2 + a_2b_1 - a_3b_4 - a_4b_3)i + (a_1b_3 + a_3b_1 - a_2b_4 - a_4b_2)j + (a_1b_4 + a_4b_1 + a_3b_2 + a_2b_3)k$

Indeed, it is observed that multiplication in $\mathbb{C}_2$ is commutative, associative, and distributive over addition, making $\mathbb{C}_2$ a commutative algebra. However, it is important to note that $\mathbb{C}_2$ is not a field due to the presence of zero-divisors. Specifically, the set $\mathcal{O} = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}\}$ consists of zero-divisors.

In $\mathbb{C}_2$, an element $z_1 + jz_2$ is referred to as non-singular if it does not belong to the set $\mathcal{O}$.

2.1. Idempotent Representation

The idempotent representation of a bicomplex number is a significant form of representation. Two bicomplex numbers, namely $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$, are linearly independent in the linear space $\mathbb{C}_2$ over $\mathbb{C}$. These numbers have the following properties:

- $e_1 + e_2 = 1$
- $e_1 - e_2 = ij$
- $e_1 \cdot e_2 = 0$

- $e_1^2 = e_1$
- $e_2^2 = e_2$

For any bicomplex number $z = z_1 + jz_2 \in \mathbb{C}_2$, it can be uniquely written as $z = \alpha e_1 + \beta e_2$, where $\alpha = z_1 - iz_2$ and $\beta = z_1 + iz_2$. This representation is known as the idempotent representation of z. In this representation, α and β are complex numbers that correspond to the coefficients of e_1 and e_2, respectively. It provides a unique and convenient way to express bicomplex numbers using the linear combination of the idempotent elements e_1 and e_2.

2.2. Auxiliary Complex Spaces

The complex spaces $A_1 = \{\alpha : \alpha = z_1 - iz_2,\ z_1,\ z_2 \in \mathbb{C}\}$ and $A_2 = \{\beta : \beta = z_1 + iz_2,\ z_1,\ z_2 \in \mathbb{C}\}$ are referred to as the auxiliary complex spaces. It is worth noting that the spaces A_1 and A_2 contain the same elements as the complex field $\mathbb{C}$, but they are denoted as A_1 and A_2 to emphasize their role in the idempotent representation.

Each bicomplex number $z_1 + jz_2 = \alpha e_1 + \beta e_2$ associates with a pair of points, $\alpha \in A_1$ and $\beta \in A_2$. These auxiliary complex spaces act as the carriers of the coefficients in the idempotent representation of bicomplex numbers.

Conversely, for every pair of points $(\alpha, \beta) \in A_1 \times A_2$, there exists a unique bicomplex number in $\mathbb{C}_2$. Thus, there is a one-to-one correspondence between the pairs of points in $A_1 \times A_2$ and the bicomplex numbers in $\mathbb{C}_2$, allowing us to uniquely associate a bicomplex number with a pair of points in the auxiliary complex spaces.

2.3. Cartesian Product

$\mathbb{C}_2$-cartesian product of $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ is defined as follows:

$$X_1 \times_e X_2 = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \alpha e_1 + \beta e_2,\ (\alpha, \beta) \in X_1 \times X_2\}.$$

2.4. Norm

The norm function $\|\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$ denote the set of all non negative real numbers) is defined as follows:

For a bicomplex number $z = z_1 + jz_2 = \alpha e_1 + \beta e_2 \in \mathbb{C}_2$, the norm is given by

$$\|z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\alpha|^2 + |\beta|^2}{2}\right\}^{\frac{1}{2}}.$$

This norm function has the following properties:

If $a \in \mathbb{C},\ w_1 = z_1 + jz_2 = \alpha e_1 + \beta e_2$ & $w_2 = z_3 + jz_4 \in \mathbb{C}_2$, then

i. $\|aw_1\| = |a|.\|w_1\|$.

ii. $\|w_1 + w_2\| \leq \|w_1\| + \|w_2\|$.

iii. $\|w_1 w_2\| \leq \sqrt{2}\|w_1\|.\|w_2\|$.

iv.
$$\begin{cases} |\alpha| \\ |\beta| \end{cases} \leq \sqrt{2}\|w_1\| \leq |\alpha| + |\beta|.$$

2.5. $\mathbb{C}_2$-Open Discus

An open discus $D(a; r_1, r_2)$ with centre $a = a_1e_1 + a_2e_2$ and radii $r_1 > 0, r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) &= B_1(a_1, r_1) \times_e B_1(a_2, r_2) \\ &= \{\alpha e_1 + \beta e_2 \in \mathbb{C}_2 : |\alpha - a_1| < r_1,\ |\beta - a_2| < r_2\} \end{aligned}$$

where $B_1(z, r)$ is an open ball with center $z \in \mathbb{C}$ and radius $r > 0$.

2.6. $\mathbb{C}_2$-Closed Discus

A closed discus $\bar{D}(a; r_1, r_2)$ with centre $a = a_1e_1 + a_2e_2$ and radii $r_1 > 0, r_2 > 0$ is defined by

$$\begin{aligned} \bar{D}(a; r_1, r_2) &= \bar{B}_1(a_1, r_1) \times_e \bar{B}_1(a_2, r_2) \\ &= \{\alpha e_1 + \beta e_2 \in \mathbb{C}_2 : |\alpha - a_1| \leq r_1,\ |\beta - a_2| \leq r_2\} \end{aligned}$$

where $\bar{B}_1(z, r)$ is a closed ball with center $z \in \mathbb{C}$ and radius $r > 0$.

Geometrically, $\bar{D}(a; r_1, r_2)$ represents a duocylinder or double cylinder in 4-dimensional Euclidean space which is analogous to a cylinder in 3-space [18].

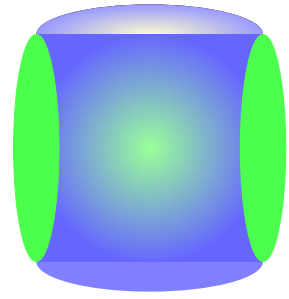


Figure 1: Perspective projection of $\bar{D}(a; r_1, r_2)$

2.7. $\mathbb{C}_2$-Ball

Open ball $B_2(a, r)$ and closed ball $\bar{B}_2(a, r)$ with center $a = a_1 + ja_2 \in \mathbb{C}_2$ and radious $r > 0$ are defined as

$$\begin{aligned} B_2(a, r) &= \{z_1 + jz_2 \in \mathbb{C}_2 : \|z_1 + jz_2 - (a_1 + ja_2)\| < r\}, \\ \bar{B}_2(a, r) &= \{z_1 + jz_2 \in \mathbb{C}_2 : \|z_1 + jz_2 - (a_1 + ja_2)\| \leq r\}. \end{aligned}$$

Theorem 2.1. [15] *If* $0 < r_1 \leq r_2$, *then*

$$i)\ B_2(a, \tfrac{r_1}{\sqrt{2}}) \subsetneqq D(a; r_1, r_2) \subsetneqq B_2\left(a, \left[\tfrac{r_1^2+r_2^2}{2}\right]^{\frac{1}{2}}\right)$$

$$ii)\ \bar{B}_2(a, \tfrac{r_1}{\sqrt{2}}) \subsetneqq \bar{D}(a; r_1, r_2) \subsetneqq \bar{B}_2\left(a, \left[\tfrac{r_1^2+r_2^2}{2}\right]^{\frac{1}{2}}\right).$$

2.8. Bicomplex matrix

The set of all $m \times n$ bicomplex matrices with entries as bicomplex numbers is denoted by $M_{m\times n}(\mathbb{C}_2)$. When the number of rows is equal to the number of columns, it is denoted by $M_n(\mathbb{C}_2)$. In $M_{m\times n}(\mathbb{C}_2)$, the ordinary matrix addition and multiplication are defined as usual.

Any matrix $A = (a_{rs}) \in M_{m\times n}(\mathbb{C}_2)$ with $a_{rs} = \alpha_{rs}e_1 + \beta_{rs}e_2$ can be expressed as $A = Be_1 + Ce_2$, where $B = (\alpha_{rs}) \in M_{m\times n}(A_1)$ and $C = (\beta_{rs}) \in M_{m\times n}(A_2)$.

There are several other important properties and characteristics of bicomplex matrices, which are further explored in the reference [13]. These properties help in understanding the algebraic structure and operations of bicomplex matrices.

2.9. Zeros of a bicomplex polynomial

Let $P(z) = \sum_{t=0}^{n} a_t z^t$ be a bicomplex polynomial of degree n with $z = z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2$ and $a_t = \alpha_t e_1 + \beta_t e_2, t = 0, 1, 2, ..., n$. Then $P(z)$ can be written as

$$\begin{aligned} P(z) &= \sum_{t=0}^{n} \alpha_t \omega_1^t e_1 + \sum_{t=0}^{n} \beta_t \omega_2^t e_2 \\ &= \phi(\omega_1)e_1 + \psi(\omega_2)e_2. \end{aligned}$$

The set of zeros S of $P(z)$ can be fully determined by the sets S_1 and S_2, which are the distinct zeros of the respective polynomials $\phi(\omega_1)$ and $\psi(\omega_2)$. In other words, we have $S = S_1e_1 + S_2e_2$, where S_1 and S_2 represent the sets of zeros of $\phi(\omega_1)$ and $\psi(\omega_2)$, respectively.

To study further about the zeros of bicomplex polynomials, one can see the articles mentioned in the references, namely [6, 14]. These articles provide more insights into the properties and characteristics of zeros of bicomplex polynomials.

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let* $A = (a_{rs}) = (\alpha_{rs}e_1 + \beta_{rs}e_2)$ *be a bicomplex square matrix of order* n *with* $R_{1r}(A) = \left(\sum_{s\neq r} |\alpha_{rs}|^2\right)^{\frac{1}{2}}$ *and* $R_{2r}(A) = \left(\sum_{s\neq r} |\beta_{rs}^2|\right)^{\frac{1}{2}}$. *Then each eigenvalue of* A *lies within at least one of the closed discuses* $\bar{D}(\alpha_{rr}e_1 +$

$\beta_{rr}e_2; R_{1r}(A), R_{2r}(A))$, $r = 1,\ 2, ...,\ n$.

Proof. If $X = \begin{pmatrix} x_{11}e_1 + x_{12}e_2 \\ x_{21}e_1 + x_{22}e_2 \\ ..... \\ x_{n1}e_1 + x_{n2}e_2 \end{pmatrix}$ be an eigenvector corresponding to the eigen value $\lambda = \lambda_1 e_1 + \lambda_2 e_2$, then

$$AX = \lambda X$$

$$\text{i.e, } \sum_{s=1}^{n}(\alpha_{rs}e_1 + \beta_{rs}e_2)(x_{s1}e_1 x_{s2}e_2) = (\lambda_1 e_1 + \lambda_2 e_2)(x_{r1}e_1 + x_{r2}e_2),\ r = 1,\ 2, ...,\ n$$

$$\text{i.e, } \sum_{s=1}^{n}(\alpha_{rs}x_{s1})e_1 + \sum_{s=1}^{n}(\beta_{rs}x_{s2})e_2 = (\lambda_1 x_{r1})e_1 + (\lambda_2 x_{r2})e_2,\ r = 1,\ 2, ...,\ n.$$

By uniqueness property of the idempotent representation of a bicomplex number, we obtain for each $r = 1,\ 2, ...,\ n$ that

$$\sum_{s=1}^{n}\alpha_{rs}x_{s1} = \lambda_1 x_{r1}$$

$$\text{and } \sum_{s=1}^{n}\beta_{rs}x_{s2} = \lambda_2 x_{r2}$$

$$\text{i.e, } \sum_{s\neq r}\alpha_{rs}x_{s1} = (\lambda_1 - \alpha_{rr})x_{r1} \tag{3.1}$$

$$\text{and } \sum_{s\neq r}\beta_{rs}x_{s2} = (\lambda_2 - \beta_{rr})x_{r2}. \tag{3.2}$$

The Schwarz inequality for complex numbers $c_s, d_s (s = 1, 2, ..., n)$ states that:

$$\left|\sum_{s=1}^{n} c_s\overline{d_s}\right|^2 \leq \left(\sum_{s=1}^{n}|c_s|^2\right)\left(\sum_{s=1}^{n}|d_s|^2\right).$$

Applying this, it follows from (3.1) that

$$|\lambda_1 - \alpha_{rr}||x_{r1}| = |\sum_{s\neq r}\alpha_{rs}x_{s1}|$$

$$\leq \left(\sum_{s\neq r}|\alpha_{rs}|^2\right)^{\frac{1}{2}}\left(\sum_{s\neq r}|x_{s1}|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{s\neq r}|\alpha_{rs}|^2\right)^{\frac{1}{2}}\left(\sum_{s=1}^{n}|x_{s1}|^2\right)^{\frac{1}{2}}.$$

Since $|x_{r1}| \leq (\sum_{s=1}^{n}|x_{s1}|^2)^{\frac{1}{2}}$, we obtain that

$$|\lambda_1 - \alpha_{rr}| \leq \left(\sum_{s\neq r}|\alpha_{rs}|^2\right)^{\frac{1}{2}} = R_{1r}(A).$$

Similarly, we get from (3.2) that

$$|\lambda_2 - \beta_{rr}| \leq \left(\sum_{s\neq r}|\beta_{rs}|^2\right)^{\frac{1}{2}} = R_{2r}(A).$$

Hence all the eigenvalues of A lie in

$$\bigcup_{r=1}^{n}\bar{D}(\alpha_{rr}e_1 + \beta_{rr}e_2; R_{1r}(A), R_{2r}(A)).$$

This completes the proof of the theorem.

Remark 3.1. *The bicomplex analog of the Gershgorin theorem in the finite complex plane $\mathbb{C}$ is represented by Theorem 3.1.*

Remark 3.2. *Taking* $C_{1k}(A) = \left(\sum_{r\neq k}|\alpha_{rk}|^2\right)^{\frac{1}{2}}$, $C_{2k}(A) = \left(\sum_{r\neq k}|\beta_{rk}|^2\right)^{\frac{1}{2}}$, *the following result is obtained immediately.*

Corollary 3.1. *Let $A = (a_{rs}) = (\alpha_{rs}e_1 + \beta_{rs}e_2)$ be a $n \times n$ bicomplex square matrix with* $C_{1k}(A) = \left(\sum_{r\neq k}|\alpha_{rk}|^2\right)^{\frac{1}{2}}$ *and* $C_{2k}(A) = \left(\sum_{r\neq k}|\beta_{rk}|^2\right)^{\frac{1}{2}}$. *Then every eigenvalue of A resides within at least one of the closed discuses $\bar{D}(\alpha_{kk}e_1+\beta_{kk}e_2; C_{1k}(A), C_{2k}(A))$, $k = 1,\ 2, ...,\ n$.*

Proof. Since any matrix A and its transpose A^T have the same eigenvalues, proof is pursued from Theorem 3.1 by considering A^T in place of the matrix A.

Remark 3.3. *Following example ensures the validity of Theorem 3.1.*

Example 3.1. Let $A = \begin{pmatrix} i & 1 & ij \\ 0 & 0 & 3+i \\ 0 & 3-i & 3 \end{pmatrix} \in M_3(\mathbb{C}_2)$.

The eigenvalues $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \in \mathbb{C}_2$ of the matrix A are given by

$$\begin{vmatrix} i-\lambda & 1 & ij \\ 0 & -\lambda & 3+i \\ 0 & 3-i & 3-\lambda \end{vmatrix} = 0$$

i.e, $(i-\lambda)(\lambda-5)(\lambda+2) = 0$

i.e, $\phi(\lambda_1)e_1 + \psi(\lambda_2)e_2 = 0$ where $\phi(\lambda_1) = (i-\lambda_1)(\lambda_1-5)(\lambda_1+2)$ & $\psi(\lambda_2) = (i-\lambda_2)(\lambda_2-5)(\lambda_2+2)$.

Now, the set of distinct zeros of $\phi(\lambda_1)$ is $S_1 = \{\lambda_1^1 = -2,\ \lambda_1^2 = 5,\ \lambda_1^3 = i\}$ and that of $\psi(\lambda_2)$ is $S_2 = \{\lambda_2^1 = -2,\ \lambda_2^2 = 5,\ \lambda_2^3 = i\}$.

Consequently, the set of 9 eigenvalues of A is

$$\begin{aligned} S &= S_1 e_1 + S_2 e_2 \\ &= \{\lambda_1^s e_1 + \lambda_2^t e_2 :\ s,t = 1,\ 2,\ 3\}. \end{aligned}$$

Clearly, each of these eigenvalues of A lies within at least one of the closed discuses $\bar{D}(i;\sqrt{2},\sqrt{2})$, $\bar{D}(0;\sqrt{10},\sqrt{10})$, $\bar{D}(3;\sqrt{10},\sqrt{10})$.

Theorem 3.2. *Let $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a bicomplex monic polynomial of degree n with $a_r = \alpha_r e_1 + \beta_r e_2,\ r = 1,\ 2, ...,\ n-1$. Then all the zeros of $P(z)$ lie in the closed discus $\bar{D}(0; 1+\alpha, 1+\beta)$ where $\alpha = \max\limits_{0\le r\le n-1}|\alpha_r|$ & $\beta = \max\limits_{0\le r\le n-1}|\beta_r|$.*

Proof. Clearly, the companion matrix for the polynomial $P(z)$ is

$$C_n = \begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & -a_{n-1} \end{pmatrix}.$$

Hence in view of Theorem 3.1, all the eigenvalues of C_n lie within

$$\bar{D}(0;|\alpha_0|,|\beta_0|)\bigcup_{r=1}^{n-2}\bar{D}(0;(1+|\alpha_r|^2)^{\frac{1}{2}},(1+|\beta_r|^2)^{\frac{1}{2}})\bigcup\bar{D}(-\alpha_{n-1}e_1-\beta_{n-1}e_2;1,1).$$

Since $|\lambda_1+\alpha_{n-1}| \le 1$ & $|\lambda_2+\beta_{n-1}| \le 1$ imply respectively $|\lambda_1| \le 1+|\alpha_{n-1}|\&|\lambda_2| \le 1+|\beta_{n-1}|$, it follows that all zeros of $P(z)$ which are actually the eigenvalues of C_n lie within the closed discus $\bar{D}(0;1+\alpha,1+\beta)$ where $\alpha = \max\limits_{0\le r\le n-1}|\alpha_r|$ & $\beta = \max\limits_{0\le r\le n-1}|\beta_r|$.

Remark 3.4. *The bicomplex adaptation of Theorem B can be seen as Theorem 3.2.*

Remark 3.5. *Following theorem can be stated as a consequence of Corollary 3.1.*

Theorem 3.3. *If $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ is a monic polynomial of degree n in $\mathbb{C}_2$ with $a_r = \alpha_r e_1 + \beta_r e_2,\ r = 1,\ 2, ...,\ n-1$, then the closed discus $\bar{D}(0; \gamma, \delta)$ contains all the zeros of $P(z)$ where $\gamma = \max\{1, \sum_{r=0}^{n-1} |\alpha_r|\}$ & $\delta = \max\{1, \sum_{r=0}^{n-1} |\beta_r|\}$.*

Proof. Using Corollary 3.1, it is obvious that all the zeros of $P(z)$ reside within

$$\bar{D}(0; 1, 1) \bigcup \bar{D}(-\alpha_{n-1}e_1 - \beta_{n-1}e_2; \sum_{r=0}^{n-2} |\alpha_r|, \sum_{r=0}^{n-2} |\beta_r|).$$

Now,

$$|\lambda_1 + \alpha_{n-1}| \leq \sum_{r=0}^{n-2} |\alpha_r| \implies |\lambda_1| \leq \sum_{r=0}^{n-1} |\alpha_r|$$

and

$$|\lambda_2 + \beta_{n-1}| \leq \sum_{r=0}^{n-2} |\beta_r| \implies |\lambda_2| \leq \sum_{r=0}^{n-1} |\beta_r|.$$

Hence all the zeros of $P(z)$ lie in the closed discus $\bar{D}(0; \gamma, \delta)$ where
$\gamma = \max\{1, \sum_{r=0}^{n-1} |\alpha_r|\}$ & $\delta = \max\{1, \sum_{r=0}^{n-1} |\beta_r|\}$.

Remark 3.6. *Following example ensures the validity and sharpness of both Theorem 3.2 & Theorem 3.3.*

Example 3.2. Let $P(z) = z^3 + \frac{1}{3}z^2 - \frac{2}{3}(1 - ij)z - \frac{1}{3},\ z = z_1 + jz_2$.

Here, $a_2 = \frac{1}{3}e_1 + \frac{1}{3}e_2,\ a_1 = -\frac{2}{3}(1 - ij) = 0.e_1 - \frac{4}{3}e_2,\ a_0 = -\frac{1}{3}e_1 - \frac{1}{3}e_2$.

Hence all the zeros of $P(z)$ by Theorem 3.2 lie in $\bar{D}(0; 1.33, 2.33)$ whereas by Theorem 3.3 in $\bar{D}(0; 1, 2)$.

Again, if we take the bicomplex polynomial $Q(z) = z^3 + z^2 - 2(1 - ij)z - 1$, all the zeros of $Q(z)$ contain by Theorem 3.2 within $\bar{D}(0; 2, 5)$ whereas by Theorem 3.3 within $\bar{D}(0; 2, 6)$.

Thus, each of Theorem 3.2 and Theorem 3.3 provides more precise bounds than the other for certain bicomplex polynomials.

4. Conclusion

In this paper, we have established a bicomplex extension of the classical Gershgorin circle theorem. This extension provides a framework for estimating the location of eigenvalues of $n \times n$ bicomplex matrices. By using this theorem, we have also derived classical-type bounds for the zeros of bicomplex monic polynomials. These results contribute to the foundational understanding of spectral theory in the context of bicomplex analysis and offer new tools for addressing polynomial root localization in the bicomplex setting. A promising direction for future research involves extending these findings to higher-dimensional bicomplex systems that utilize idempotent elements including

$$0, \quad 1, \quad \frac{1 \pm i_1 i_2}{2}, \quad \frac{1 \pm i_2 i_3}{2}, \quad \ldots, \quad \frac{1 \pm i_{n-1} i_n}{2}$$

within the framework of the extended bicomplex algebra $\mathbb{C}_n$. Further exploration of this higher-dimensional generalization is needed, as it remains an open and largely unexamined area, prompting further investigation into the structure, spectral properties, and polynomial theory in multidimensional bicomplex spaces.

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