

**SOME FIXED POINT THEOREMS FOR MULTIVALUED
INTERPOLATIVE MAPPING IN HYPERBOLIC METRIC SPACE
WITH SOME APPLICATIONS**

Lucas Wangwe and Santosh Kumar*

Department of Mathematics,
College of Science and Technical Education,
Mbeya University of Science and Technology,
P.O.BOX 131, Mbeya, TANZANIA

E-mail : wangwelucas@gmail.com

*Department of Mathematics,
School of Physical Sciences,
North-Eastern Hill University,
Shillong - 793022, Meghalaya, INDIA

E-mail : drsengar2002@gmail.com

(Received: Oct. 10, 2025 Accepted: Jul. 20, 2026 Published: Aug. 30, 2026)

Abstract: This paper aims to investigate the multivalued fixed point results for various interpolative contraction mappings in hyperbolic metric spaces. We also provide an illustrative example to support our main results. The results presented here will be illustrated with some applications to functional integral inclusion and linear partial differential equations.

Keywords and Phrases: Fixed point theorems, multivalued mapping and interpolative mapping, hyperbolic metric space, Functional integral inclusion, Linear partial differential equations.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Takahashi's research established results on convexity in metric spaces and non-expansive mappings [45]. Building on this foundation, Kohlenbach [33] explored

logical metatheorems with applications in functional analysis. By engaging in proof mining to extract computational information from non-computational proofs, Kohlenbach's work aims to quantify the computational content of theorems concerning set-valued operators, aligning them with proof-theoretic notions. This methodology unifies recent case studies and highlights the quantitative assumptions necessary for applying proof mining in functional analysis. Kohlenbach's approach underscores the interdisciplinary and practical relevance of these mathematical theories in various applied problems.

A similar paper to Reich and Shafrir's [41] work on nonexpansive iterations in hyperbolic spaces could be one by Uddin *et al.* [46], which focuses on the convergence analysis of M-iteration involving α -Reich-Suzuki nonexpansive mappings in a hyperbolic metric space, proving strong and Δ -convergence theorems. Additionally, Valappil *et al.* [48] presented fixed-point results for Reich-Suzuki-type nonexpansive mappings in uniformly convex W-hyperbolic spaces, introducing a new iterative scheme that converges faster than existing methods for such mappings. Ullah *et al.* [47] also contributed to this area by proving strong and Δ convergence results for Reich-Suzuki-type nonexpansive mappings through an M iterative process in a uniformly convex hyperbolic metric space. Guran *et al.* [17] offered insights into the Picard-Thakur hybrid iterative scheme for generalized α -nonexpansive mappings in hyperbolic spaces, demonstrating its faster convergence compared to other methods. Lastly, Xiong and Lan introduce a modified iterative approximation process for total asymptotically nonexpansive nonself mappings in hyperbolic spaces, proving Δ -convergence and strong convergence, thus improving and generalizing existing results in the field. Kohlenbach *et al.* [32] proved results on asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. Niazi Asil *et al.* [38] developed some hyperbolic iterative methods for linear systems. Zidan *et al.* [54] investigated new quicker implicit iterations in hyperbolic spaces.

Markin [35] and Nadler [37] initiated the investigation into fixed point theorems for multivalued mappings addressing a gap that had existed since Banach's work in 1922 [3]. These theorems have numerous applications, including nonlinear analysis, control theory, convex optimization, and differential inclusion. Berinde and Berinde [7] demonstrated results on a broad class of multivalued weakly Picard mappings. Khan *et al.* [31] provided results on common fixed points of two multivalued nonexpansive maps in Kohlenbach hyperbolic spaces. Bello *et al.* [6] established an approximating common fixed point for families of multivalued mean nonexpansive mappings in hyperbolic spaces. Dewangan [10] explored fixed points of multivalued Suzuki nonexpansive mappings in hyperbolic spaces.

Later, Karapinar [20] introduced the result for interpolative Kannan-type contraction in metric spaces. The theory is extended to various settings including metric spaces, partial metric spaces, Branciari distance spaces, and ordered structures, often using tools such as simulation functions, $(\alpha, \beta, \psi, \phi)$ -frameworks, and vector-valued (Perov-type) metrics for more details one can see [4, 5, 12, 21, 22, 23, 24, 25, 26, 27, 30] These works progressively unify and generalize multiple contraction principles, showing that weaker nonlinear contractive conditions are still sufficient to guarantee fixed points, with applications in nonlinear analysis and differential equations. Overall, the research establishes a broad and modern framework of interpolative contractions that significantly extends classical fixed point theory.

Further, Alansari and Ali [1] proved the unified multivalued interpolative Reich-Rus-Ćirić-type contractions. Prithvi *et al.* [39] proved an interpolative operator: Fractal to multivalued fractal chaos. Konwar *et al.* [34] proved some new results for a class of multivalued interpolative Kannan-type contractions. Karapinar *et al.* [29] gave the results on interpolative Hardy-Rogers type multivalued contractions via a simulation function. Debnath *et al.* [9] proved a set-valued interpolative Hardy-Rogers and set-valued Reich-Rus-Ćirić-type contractions in b -metric spaces. El Bekri *et al.* [13] proved a cyclic coupled fixed point via interpolative Kannan-type contractions in metric spaces. For more details, one can refer to [14, 15, 53] and the references therein. Wangwe [49] demonstrated fixed point theorems for interpolative Kannan contraction mappings in Busemann spaces and applied these theorems to matrix equations. Additionally, Wangwe [50] explored fixed point theorems for interpolative Ćirić-type contraction mappings in the convex hull combination of Busemann spaces, with applications to fractional differential equations. Furthermore, Wangwe's work on fixed point theorems for interpolative contraction mappings of the Ćirić type in CAT(0) spaces has found applications in hyperbolic rotation matrices [51]. Fixed point theorems for implicit relations in hyperbolic metric spaces with practical applications were also provided by Wangwe [52]. Furthermore, Hammad *et al.* [18], demonstrated hybrid interpolative mappings for solving fractional Navier-Stokes and functional differential equations.

The novelty of this study can be summarized as follows: Introduction of multivalued interpolative Ćirić-type contraction mappings in the setting of hyperbolic metric spaces. Extension of classical fixed point results from metric spaces to hyperbolic metric spaces, thereby enriching the geometric framework. Development of new Kannan-type and Reich-Ćirić-type contraction conditions within a multivalued interpolative framework. Application of the obtained theoretical results to functional integral inclusion problems, nonlinear mixed integral equations, and lin-

ear partial differential equations. Construction of illustrative examples validating the applicability and correctness of the main results. Unification and generalization of several earlier works (e.g., Debnath *et al.* [9], Konowar *et al.* [34], Salimi *et al.* [42], Karapinar and Samet [28], Samet *et al.* [43]) under a more general framework.

2. Preliminaries

We provide the following definitions, lemmas, and some results that will help develop our main results.

Alfuraidan and Ansari [2] provided the following proposition.

Proposition 2.1. [2] *Let (X, d) be a metric space of hyperbolic type. Let $x, y \in X$ for each $\beta_1, \beta_2 \in [0, 1]$ and $\beta_2 = 1 - \beta_1$, there is a unique point $z \in [x, y]$ this point will be denoted by $z = \beta_2 x \oplus \beta_1 y$ such that $d(x, z) = \beta_1 d(x, y)$ and $d(z, v) = \beta_2 d(u, v)$. If $x \neq y$, we have*

$$d(x, z) + d(z, y) = d(x, y).$$

Kohlenbach [33] established the following definition in relation to convex concepts raised by Takahashi [45].

Definition 2.1. [33] *Let (X, d, W) be a hyperbolic metric space. Assume that d is a metric and $W : X \times X \times [0, 1] \rightarrow X$ is a mapping satisfying the following:*

- (i) $d(z, W(x, y, \beta_1)) \leq \beta_2 d(z, u) + \beta_1 d(z, y)$,
- (ii) $d(W(x, y, \beta_1), W(x, y, \beta_2)) \leq |\beta_1 - \beta_2| d(x, y)$,
- (iii) $W(x, y, \beta_1) = W(x, y, 1 - \beta_1)$,
- (iv) $d(W(u, z, \beta_1), W(v, w, \beta_1)) \leq \beta_2 d(x, y) + \beta_1 d(z, w)$, $\forall u, v \in X$ and $\forall \beta_1, \beta_2 \in [0, 1]$.

The above axioms can be converted by using the following notion:

$$W(x, y, \beta_1) = \beta_1 x \oplus \beta_2 y.$$

For $\beta_1 = \frac{1}{2}$ in above we obtain

$$W\left(x, y, \frac{1}{2}\right) = \frac{1}{2}x \oplus \frac{1}{2}y.$$

With this notion (i), (ii) and (iv) become, respectively,

$$(i) \quad d(z, \beta_1 x \oplus \beta_2 y) \leq \beta_1 d(z, x) + \beta_2 d(z, y).$$

$$(ii) \quad d(\beta_1 x \oplus \beta_2 y, \beta_2 x \oplus \beta_1 y) \leq |\beta_1 - \beta_2| d(x, y).$$

$$(iii) \quad d(\beta_1 x \oplus \beta_2 z, \beta_1 y \oplus \beta_2 w) \leq \beta_2 d(x, y) + \beta_1 d(z, w).$$

If $z = w$, we have

$$d(\beta_1 x \oplus \beta_2 z, \beta_1 y \oplus \beta_2 z) \leq \beta_2 d(x, y). \quad (2.1)$$

If only condition (i) is satisfied, then (X, d, W) is a convex metric space by Takahashi's concepts [45].

Lemma 2.1. *Let (X, d, W) be a convex hyperbolic metric space. Then the following is true:*

$$d(u, v) = d(u, W(u, v, \beta_1)) + d(v, W(u, v, \beta_1)) \quad (2.2)$$

for all $(u, v, \beta_1) \in X \times X \times I$.

Nadler [37] introduced the following definition for the Hausdorff concept of the distance between two arbitrary sets as follows.

Definition 2.2. *Let H be a Pompeiu-Hausdorff metric in a complete metric space (X, d) and $CB(X)$ denote the class of all non-empty closed and bounded subsets of X , then for $A, B \in CB(X)$, we define*

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \quad (2.3)$$

where

$$\delta(A, B) = \sup_{u \in A} d(u, B), \quad (2.4)$$

$$d(x, B) = \inf_{y \in B} d(x, y) \quad (2.5)$$

and

$$\delta(B, A) = \sup_{y \in B} d(y, A), \quad (2.6)$$

$$d(y, A) = \inf_{x \in A} d(y, x). \quad (2.7)$$

$(CB(X), H)$ is called the Pompeiu-Hausdorff metric space induced by δ .

Theorem 2.1. [37] *Let (X, d) be a complete metric space and $\mathcal{K} : X \rightarrow X$ be a mapping such that*

$$H(\mathcal{K}x, \mathcal{K}y) \leq \beta d(x, y),$$

where $0 \leq \beta < 1$. Then $\mathcal{K}$ has a fixed point in X .

Remark 2.1. [37] If $A, B \in CB(X)$ and $x \in A$, then for $\alpha > 0$, there exists $y \in B$ such that

$$d(x, y) \leq H(A, B) + \alpha. \quad (2.8)$$

However, there may not be a point $y \in B$ such that

$$d(x, y) \leq H(A, B). \quad (2.9)$$

If B is compact, then such a point y does exist, i.e.,

$$d(x, y) \leq H(A, B). \quad (2.10)$$

Example 2.1. Let $X = \mathbb{R}$ and define the sets A and B as follows:

$$A = \{0\}, B = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

- (i) Hausdorff distance $H(A, B)$: The Hausdorff distance between sets A and B is given by

$$H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\},$$

for $x = 0 \in A$ and $y \in B$,

$$\inf_{y \in B} d(0, y) = \inf_{y \in B} |y| = 0$$

(as y can be arbitrarily small).

However, $\sup_{y \in B} \inf_{x \in A} d(x, y) = \sup_{y \in B} |y| = 1$. Therefore,

$$H(A, B) = 1.$$

- (ii) Existence of $y \in B$ such that $d(x, y) \leq H(A, B) + \alpha$: For any $\alpha > 0$, we can choose $y = \frac{1}{n}$ for sufficiently large n , such that $\frac{1}{n} \leq \alpha$. Hence,

$$d\left(0, \frac{1}{n}\right) = \frac{1}{n} \leq \alpha$$

which implies $d(0, y) \leq H(A, B) + \alpha$.

- (iii) Non-existence of $y \in B$ such that $d(x, y) \leq H(A, B)$: Since $H(A, B) = 1$, there is no $y \in B$ such that $d(0, y) = 1$. The elements of B are all of the form $\frac{1}{n}$ for $n \in \mathbb{N}$, and these distances are always less than 1 but can get arbitrarily close to 0.
- (iv) Suppose B is a compact subset of $\mathbb{R}$. For example, let $B = \{\frac{1}{n} \mid n \in \mathbb{N}, n \leq N\} \cup \{0\}$ for some large N . Now B includes 0, making $d(0, y) = 0$, which is clearly less than or equal to $H(A, B)$.

Compactness of B and Existence of $y \in B$ such that $d(x, y) \leq H(A, B)$: If B is compact, every sequence in B has a convergent subsequence, ensuring that the infimum distance is achieved.

In this case, for any $x \in A$, there exists $y \in B$ such that $d(x, y) \leq H(A, B)$. Therefore, the sets $A = \{0\}$ and $B = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ provide an example where for any $\alpha > 0$, there exists a $y \in B$ such that $d(x, y) \leq H(A, B) + \alpha$. However, there is no point $y \in B$ such that $d(x, y) \leq H(A, B)$ unless B is compact.

Let us consider an example to illustrate this remark involving the Hausdorff distance in the context of compact metric spaces.

Example 2.2. Let $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$. Consider the following two compact subsets of X : $A = \{0, 1, 2, 3\}$, $B = \{0.5, 2.5\}$

- (i) Compute the Hausdorff distance $H(A, B)$: The Hausdorff distance $H(A, B)$ is defined as:

$$H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(y, x) \right\}$$

$$\sup_{x \in A} \inf_{y \in B} d(x, y):$$

$$\text{For } x = 0, \inf_{y \in B} d(0, 0.5) = 0.5$$

$$\text{For } x = 1, \inf_{y \in B} d(1, 0.5) = 0.5$$

$$\text{For } x = 2, \inf_{y \in B} d(2, 2.5) = 0.5$$

$$\text{For } x = 3, \inf_{y \in B} d(3, 2.5) = 0.5$$

$$\text{For } y = 0.5, \inf_{x \in A} d(0.5, x) = 0.5$$

$$\text{For } y = 2.5, \inf_{x \in A} d(2.5, x) = 0.5$$

$$\text{Thus, } H(A, B) = 0.5.$$

(ii) Existence of $y \in B$ such that $d(x, y) \leq H(A, B)$: Consider $x = 1 \in A$.

$d(1, 0.5) = 0.5$, which is exactly $H(A, B)$.

Therefore, $y = 0.5 \in B$ satisfies $d(1, y) = 0.5 = H(A, B)$.

(iii) For $\alpha > 0$: Let $\alpha = 0.1$. For $x = 1 \in A$, we have $d(1, y) = 0.5$ for $y = 0.5 \in B$, which satisfies $d(x, y) \leq H(A, B) + \alpha = 0.6$.

This example demonstrates that when B is compact, it is possible to find a point $y \in B$ such that $d(x, y) \leq H(A, B)$, validating the statement of the remark.

Kannan [19] extended the Banach contraction principle by proving the following theorem:

Theorem 2.2. [19] *Let (X, d) be a complete metric space and $\mathcal{K} : X \rightarrow X$ be a self-mapping such that*

$$d(\mathcal{K}x, \mathcal{K}y) \leq \lambda[d(x, \mathcal{K}x) + d(y, \mathcal{K}y)], \quad (2.11)$$

for all $x, y \in X$ and $0 \leq \lambda \leq \frac{1}{2}$. Then $\mathcal{K}$ has a unique fixed point $z \in X$ and for any $x \in X$ the sequence of iterate $\{\mathcal{K}^n x\}$ converges to z .

Karapinar [20] introduced the following results by extending Kannan [19] for interpolative Kannan-type contraction as follows:

Definition 2.3. [20] *Let (X, d) be a metric space, a mapping $\mathcal{K} : X \rightarrow X$ is said to an interpolative Kannan contraction if*

$$d(\mathcal{K}x, \mathcal{K}y) \leq \lambda[d(x, \mathcal{K}x)]^\delta \cdot [d(y, \mathcal{K}y)]^{1-\delta}, \quad (2.12)$$

for all $x, y \in X$ with $x \neq \mathcal{K}x$, where $\lambda \in [0, 1)$ and $\delta \in (0, 1)$.

Theorem 2.3. [20] *Let (X, d) be a complete metric space and $\mathcal{K}$ be an interpolative Kannan-type contraction. Then $\mathcal{K}$ has a unique fixed point in X .*

Konowar *et al.* [34] proved the multivalued interpolative Kannan-type contraction as below:

Definition 2.4. [34] *Suppose that (X, d) is a metric space. A map $\mathcal{K} : X \rightarrow CB(X)$ is called a multivalued interpolative Kannan-type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$H(\zeta x, \mathcal{K}y) \leq \lambda \left[d(x, \mathcal{K}x) \right]^\alpha \cdot \left[d(y, \mathcal{K}y) \right]^{1-\alpha}, \quad (2.13)$$

for all $x, y \in X \setminus \text{Fix}(\mathcal{K})$.

Using the above Definition Konowar *et al.* [34] proved the following theorem:

Theorem 2.4. [34] Suppose that (X, d) is a complete metric space and $\mathcal{K}$ is a multivalued interpolative Kannan-type contraction such that $\mathcal{K}$ is compact for each $x \in X$. Then, $\text{Fix}(\mathcal{K}) \neq \emptyset$ in X .

Debnath *et al.* [9] proved the multivalued version of interpolative Hardy-Rogers-type contraction in b -metric spaces as below:

Definition 2.5. [9] Let (X, d, s) be a b -metric space. A mapping $\mathcal{K} : X \rightarrow CB(X)$ is called a multi-valued interpolative Hardy-Rogers-type contraction if there exist $\lambda \in [0, 1)$ and $a, b, c \in (0, 1)$ with $a + b + c < 1$ such that

$$H(\mathcal{K}x, \mathcal{K}y) \leq \lambda \left[d(x, y) \right]^a \cdot \left[d(x, \mathcal{K}x) \right]^b \cdot \left[d(y, \mathcal{K}y) \right]^c \cdot \left[\frac{1}{2s} (d(x, \mathcal{K}y) + d(y, \mathcal{K}x)) \right]^{1-a-b-c}, \quad (2.14)$$

for all $x, y \in X \setminus \text{Fix}(\mathcal{K})$.

Debnath *et al.* [9] proved the following theorem:

Theorem 2.5. [9] Let (X, d, s) be a complete b -metric space whose all closed and bounded subsets are compact and $\mathcal{K}$ be a multi-valued interpolative Hardy-Rogers-type contraction. Then $\mathcal{K}$ has a fixed point.

Definition 2.6. [43] Let $\mathcal{K}$ be a self-mapping on a metric space (X, d) and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that $\mathcal{K}$ is an α -admissible mapping if

$$x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(\mathcal{K}x, \mathcal{K}y) \geq 1.$$

Definition 2.7. [42] If $\psi \in \Psi$, denote Ψ the family of nondecreasing functions: $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (1) $\psi(t) < t$ for all $t > 0$,
- (2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$,

where ψ^n is the n^{th} iteration of ψ .

Definition 2.8. [43] Let $\mathcal{K}$ be a self-mapping on a metric space (X, d) . We say that $\mathcal{K}$ is an α - ψ -contractive mapping if there exist two functions $\alpha : X \times X \rightarrow [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(\mathcal{K}x, \mathcal{K}y) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

Definition 2.9. [42] Let $\mathcal{K}$ be a self-mapping on a metric space (X, d) and let $\alpha, \eta : X \times X \rightarrow [0, \infty)$ be two functions. We say that $\mathcal{K}$ is an α -admissible mapping with respect to η if

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \implies \alpha(\mathcal{K}x, \mathcal{K}y) \geq \eta(\mathcal{K}x, \mathcal{K}y).$$

Theorem 2.6. [42] Let (X, d) be a complete metric space and $\mathcal{K} : X \rightarrow X$ be an α -admissible mapping with respect to η . Assume that

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \implies d(\mathcal{K}x, \mathcal{K}y) \leq \psi(M_H(x, y)),$$

where $\psi \in \Psi$ and

$$M_H(x, y) = \max \left\{ d(x, y), \frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2}, \frac{d(x, \mathcal{K}y) + d(y, \mathcal{K}x)}{2} \right\}.$$

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{K}x_0) \geq \eta(x_0, \mathcal{K}x_0)$,
- (ii) either $\mathcal{K}$ is continuous or for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $\mathcal{K}$ has a fixed point.

3. Main results

We prove the following theorem.

Theorem 3.1. Let (X, d) be a complete hyperbolic metric space and define the sets A and B whose all closed and bounded subsets are compact. The map $\mathcal{K} : X \rightarrow 2^X \in CB(X)$ is called an α - ψ -multivalued interpolative Ćirić type contraction. If there exists $a \in (0, 1)$ satisfying the following conditions.

- (i) $\mathcal{K}$ is α - ψ -admissible,
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{K}x_0) \geq 1$,
- (iii) $\mathcal{K}$ is continuous and satisfies

$$\alpha(x, y)H(\mathcal{K}x, \mathcal{K}y) \leq \psi(\mathcal{N}_H(x, y)), \quad (3.1)$$

where

$$\mathcal{N}_H(x, y) = [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^{1-a}, \quad (3.2)$$

for each $x, y \in X$ $\text{Fix}(\mathcal{K})$.

Then $\mathcal{K}$ has a fixed point, that is, there exists $z \in X$ such that $\mathcal{K}z = z$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, \mathcal{K}x_0) \geq 1$. Define a sequences $\{x_n\}$ in X by

$$x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2}, \forall n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $z = x_n$ is a unique fixed point of $\mathcal{K}$.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{K}$ is α -admissible we have

$$\alpha(x_n, x_{n+1}) \geq 1 \implies \alpha(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \geq 1.$$

Let $x_n, x_{n+1} \in X$ and $\beta_1, \beta_2 \in [0, 1]$ satisfying $\beta_1 + \beta_2 = 1$, assume that $z = \beta_1 x_n \oplus \beta_2 x_{n+1}$ denote the unique point of X , by proposition (2.1) we have

$$\begin{aligned} d(x_n, \beta_1 x_n \oplus \beta_2 x_{n+1}) &\leq \beta_1 d(x_n, x_n) + \beta_2 d(x_n, x_{n+1}), \\ &\leq \beta_2 d(x_n, x_{n+1}), \\ &= \beta_2 d(x_n, x_{n+1}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(x_{n+1}, \beta_1 x_n \oplus \beta_2 x_{n+1}) &\leq \beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+1}), \\ &\leq \beta_1 d(x_n, x_{n+1}), \\ &= \beta_1 d(x_n, x_{n+1}). \end{aligned} \tag{3.4}$$

Let X be a hyperbolic metric space.

For $x_0 \in X$, we construct a sequence $\{x_n\}$ such that

$$\begin{aligned} x_n &= W(x_n, x_n, \beta_1). \\ x_{n+1} &= W(x_n, \mathcal{K}x_n, \beta_1), \end{aligned} \tag{3.5}$$

for $n \in \mathbb{N} \cup \{0\}$.

Applying Definition 2.1 and (2.1), we have

$$W(x_n, \mathcal{K}x_n, \beta_1) = \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n. \tag{3.6}$$

Using (3.5) and (3.6), we obtain

$$x_{n+1} = \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n. \tag{3.7}$$

Taking $\beta_1 = \frac{1}{2}$ and $\beta_2 = 1 - \beta_1 = \frac{1}{2}$ in (3.7), gives

$$x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2}. \tag{3.8}$$

From (3.7), leads to

$$x_{n+1} = \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n. \quad (3.9)$$

Which follows that

$$x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2} = \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n, \quad (3.10)$$

where $n = 0, 1, 2, 3 \dots$

Suppose $x = x_n$ and $y = x_{n+1}$, using (3.3) and (3.4) to calculate the metric in (3.1) we obtains

$$\begin{aligned} d(x, y) = d(x_n, x_{n+1}) &= d(x_n, \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n) = \beta_1 d(x_n, x_n) + \beta_2 d(x_n, \mathcal{K}x_n) \\ &= \beta_2 d(x_n, x_{n+1}). \\ d(x, \mathcal{K}x) &= d(x_n, \mathcal{K}x_n) = d(x_n, x_{n+1}) = d(x_n, \beta_1 x_n \oplus \beta_2 x_{n+1}), \\ &= \beta_1 d(x_n, x_n) + \beta_2 d(x_n, x_{n+1}) = \beta_2 d(x_n, x_{n+1}). \\ d(y, \mathcal{K}y) &= d(x_{n+1}, \mathcal{K}x_{n+1}) = d(x_{n+1}, x_{n+2}) \\ &= d(x_{n+1}, \beta_1 x_{n+1} \oplus \beta_2 x_{n+2}), \\ &= \beta_1 d(x_{n+1}, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+2}), \\ &= \beta_2 d(x_{n+1}, x_{n+2}), \end{aligned}$$

and

$$\begin{aligned} \frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} &= \frac{\beta_2 d(x_n, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+2})}{2}, \\ &= \beta_2 \left[\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right]. \end{aligned}$$

Using $x_n \in X$ and $x_{n+1} \in \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n$, Remark 2.1 and inequality (2.1), there must exists $x_{n+2} \in \beta_1 x_{n+1} \oplus \beta_2 \mathcal{K}x_{n+1}$ such that

$$d(\beta_1 x_n \oplus \beta_2 \mathcal{K}x_n, \beta_1 x_{n+1} \oplus \beta_2 \mathcal{K}x_{n+1}) < \beta_2 d(x_{n+1}, x_{n+2}) \leq H(\mathcal{K}x_n, \mathcal{K}x_{n+1}). \quad (3.11)$$

From inequality (2.10), we get

$$\beta_2 d(x_{n+1}, x_{n+2}) \leq H(\mathcal{K}x_n, \mathcal{K}x_{n+1}). \quad (3.12)$$

Let $x = x_n$ and $y = x_{n+1}$ in (3.1), we have

$$\begin{aligned} H(\mathcal{K}x, \mathcal{K}y) = H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) &\leq \alpha(x_n, x_{n+1}) H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \\ &\leq \psi(\mathcal{N}_H(x_n, x_{n+1})), \end{aligned} \quad (3.13)$$

where

$$\begin{aligned}\mathcal{N}_H(x_n, x_{n+1}) &\leq [\beta_2 d(x_n, x_{n+1})]^a \cdot \left[\frac{\beta_2 d(x_n, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+2})}{2} \right]^{1-a}, \\ \mathcal{N}_H(x_n, x_{n+1}) &\leq [d(x_n, x_{n+1})]^a \cdot \left[\beta_2 \left(\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right) \right]^{1-a}.\end{aligned}\quad (3.14)$$

Suppose that

$$\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \leq d(x_{n+1}, x_{n+2}). \quad (3.15)$$

Using (3.15) in (3.14), we get

$$\begin{aligned}\mathcal{N}_H(x_n, x_{n+1}) &\leq [\beta_2 d(x_n, x_{n+1})]^a \cdot [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}, \\ &\leq [\beta_2 d(x_n, x_{n+1})]^a [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}.\end{aligned}\quad (3.16)$$

By applying (3.16) in (3.13), gives

$$\alpha(x_n, x_{n+1}) H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \leq \psi([\beta_2 d(x_n, x_{n+1})]^a [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}). \quad (3.17)$$

From (3.12), we obtain

$$\alpha(x_n, x_{n+1}) \beta_2 d(x_{n+1}, x_{n+2}) \leq \psi([\beta_2 d(x_n, x_{n+1})]^a [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}). \quad (3.18)$$

Consequently, for $\alpha(x_n, x_{n+1}) \geq 1$ using Definition 2.6, we obtain

$$\beta_2 d(x_{n+1}, x_{n+2}) \leq \psi([\beta_2 d(x_n, x_{n+1})]^a [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}). \quad (3.19)$$

By the property of ψ using Definition 2.7, we get

$$\begin{aligned}\beta_2 d(x_{n+1}, x_{n+2}) &< [\beta_2 d(x_n, x_{n+1})]^a [\beta_2 d(x_{n+1}, x_{n+2})]^{1-a}, \\ \left[\beta_2 d(x_{n+1}, x_{n+2}) \right]^{1-(1-a)} &< \beta_2^a [d(x_n, x_{n+1})]^a, \\ \beta_2^a d(x_{n+1}, x_{n+2})^a &< \beta_2 [d(x_n, x_{n+1})]^a, \\ \beta_2 d(x_{n+1}, x_{n+2}) &< \beta_2^{\frac{1}{a}} d(x_n, x_{n+1}), \\ d(x_{n+1}, x_{n+2}) &< \beta_2^{\frac{1-a}{a}} d(x_n, x_{n+1}).\end{aligned}\quad (3.20)$$

Letting $\beta_2^{\frac{1-a}{a}} \leq 1$, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})).$$

Repeating the above argument several times, we get

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_0, x_1)). \quad (3.21)$$

Since $n \rightarrow \infty$, we conclude that

$$d(x_{n+1}, x_{n+2}) = 0.$$

Hence $\{u_n\}$ is a Cauchy sequence.

Next, we verify that $\{x_n\}$ is a Cauchy sequence. Fix $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ such that $\sum_{n>n(\epsilon)} \psi^n(d(x_0, x_1)) < \epsilon$. Consider $n, m \in \mathbb{N}_0$, for $m > n > n(\epsilon)$, $p \geq 2$, using hyperbolic inequality (3.21), Definition 2.1 and Lemma 2.1, we have

$$\begin{aligned} d(x_{n+1}, x_{m+p}) &\leq d(x_{n+1}, W(x_{n+1}, x_{n+2}, \beta_1)) + \dots, \\ &\leq \beta_1 d(x_{n+1}, x_{n+1}) + \beta_2 d(x_{n+1}, x_{m+2}), \\ &\leq \beta_2 d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x_{n+2}) + \dots, \\ &\leq \sum_{k=n}^{m-1} d(x_{k+1}, x_{k+2}), \\ &\leq \sum_{k=n}^{m-1} \psi^n(d(x_0, x_1)), \\ &\leq \sum_{n>n(\epsilon)} \psi^n(d(x_0, x_1)) < \epsilon, \end{aligned} \quad (3.22)$$

taking limits of $n, m \rightarrow \infty$, we get

$$d(u_{n+1}, u_{m+p}) < \epsilon.$$

Thus, this prove that $\{x_n\}$ is a Cauchy sequence.

For completeness of X , there is $z \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = z.$$

Since $\mathcal{K}$ is continuous, we have

$$\lim_{n \rightarrow \infty} \mathcal{K}x_{n+1} = \mathcal{K}z = z.$$

Also $\mathcal{K}$ is α -admissible, we have

$$x_{n+1}, z \in X, \alpha(x_{n+1}, z) \geq 1 \implies \alpha(\mathcal{K}x_n, \mathcal{K}z) \geq 1.$$

And $\mathcal{K}$ is α - ψ -contractive, we have

$$\alpha(x_{n+1}, z)d(\mathcal{K}x_{n+1}, \mathcal{K}z) \leq \psi(d(x_{n+1}, z)),$$

now, we show that z is a fixed point of $\mathcal{K}$. From Definition 2.1, it follows that

$$\begin{aligned} d(z, \mathcal{K}z) &\leq d(z, W(z, \mathcal{K}z, \beta_1)) \leq \beta_1 d(z, u_{n+2}) + \beta_2 d(u_{n+2}, \mathcal{K}z), \\ &\leq \beta_1 d(z, \mathcal{K}x_{n+1}) + \beta_2 H(\mathcal{K}x_{n+1}, \mathcal{K}z). \end{aligned} \quad (3.23)$$

Let $x = x_{n+1}$ and $y = z$ in (3.1), we obtains

$$\alpha(x_{n+1}, z)H(\mathcal{K}x_{n+1}, \mathcal{K}z) \leq \lambda \mathcal{N}_H(x_{n+1}, z), \quad (3.24)$$

where

$$\begin{aligned} \mathcal{N}_H(x_{n+1}, z) &= [d(x_{n+1}, z)]^a \cdot \left[\frac{d(x_{n+1}, \mathcal{K}x_{n+1}) + d(z, \mathcal{K}z)}{2} \right]^{1-a}, \\ &= [d(z, z)]^a \cdot \left[\frac{d(z, \mathcal{K}z) + d(z, \mathcal{K}z)}{2} \right]^{1-a}, \\ \mathcal{N}_H(u_{n+1}, z) &= 0. \end{aligned} \quad (3.25)$$

Using (3.25) in (3.24) gives

$$\alpha(x_{n+1}, z)H(\mathcal{K}u_{n+1}, \mathcal{K}z) = 0. \quad (3.26)$$

Consequently, using (3.26) in (3.23), leads to

$$\begin{aligned} d(z, \mathcal{K}z) &\leq \beta_1 d(z, \mathcal{K}z) + 0, \\ d(z, \mathcal{K}z) &\leq \beta_1 d(z, \mathcal{K}z), \\ \beta_2 d(z, \mathcal{K}z) &\leq 0, \\ d(z, \mathcal{K}z) &= 0, \end{aligned} \quad (3.27)$$

which shows that $z = \mathcal{K}z$. Hence, z is a fixed point of $\mathcal{K}$.

The following results extend Theorem 2.6 to α - η - ψ -set valued interpolative mapping concerning η in hyperbolic metric space.

Theorem 3.2. *Let (X, d) be a complete hyperbolic metric space and define the sets A and B whose all closed and bounded subsets are compact and $\mathcal{K} : X \rightarrow X$ be an α - η - ψ -admissible set-valued interpolative mapping with constant a, b such that $a + b < 1$. Suppose that the following assertions hold:*

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{K}x_0) \geq \eta(x_0, \mathcal{K}x_0)$,
- (ii) either $\mathcal{K}$ is continuous or for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$ and satisfy

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \implies H(\mathcal{K}x, \mathcal{K}y) \leq \psi(M_H(x, y)), \quad (3.28)$$

where $\psi \in \Psi$ and

$$M_H(x, y) = [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^b \cdot \left[\frac{d(x, \mathcal{K}y) + d(y, \mathcal{K}x)}{2} \right]^{1-a-b},$$

for all $x, y \in X$.

Then $\mathcal{K}$ has a fixed point.

Proof. Let x_0 be a randomly point in X and $\mathcal{K}$ be α -admissible with respect to η such that

$$\alpha(x_0, \mathcal{K}x_0) \geq \eta(x_0, \mathcal{K}x_0).$$

Define a sequence $\{x_n\}$ in X by

$$x_n = \mathcal{K}^n x_0 = \mathcal{K}x_{n-1}$$

and

$$x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2},$$

for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point of $\mathcal{K}$. It completes the proof. Otherwise, assume $x_{n+1} \neq x_n$. For continuity of $\mathcal{K}$ with respect to α - η , we have

$$\alpha(x_0, \mathcal{K}x_0) \geq \eta(x_0, \mathcal{K}x_0),$$

leads to

$$\alpha(x_1, x_2) = \alpha(\mathcal{K}x_0, \mathcal{K}^2 x_0) \geq \eta(\mathcal{K}x_0, \mathcal{K}^2 x_0) = \eta(x_1, x_2).$$

Repeating this procedure, we get

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N} \cup \{0\}$. Let $x = x_n$, $y = x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2} = \beta_1 x_n \oplus \beta_2 \mathcal{K}x_n$ using inequality (3.28) we get

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \implies H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \leq \psi(M_H(x_n, x_{n+1})), \quad (3.29)$$

where

$$\begin{aligned}
M_H(x_n, x_{n+1}) &= [d(x_n, x_{n+1})]^a \cdot \left[\frac{d(x_n, \mathcal{K}x_n) + d(x_{n+1}, \mathcal{K}x_{n+1})}{2} \right]^b \\
&\quad \left[\frac{d(x_n, \mathcal{K}x_{n+1}) + d(x_{n+1}, \mathcal{K}x_n)}{2} \right]^{1-a-b}, \\
&\leq [d(x_n, x_{n+1})]^a \cdot \left[\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right]^b \\
&\quad \left[\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2} \right]^{1-a-b}, \\
&\leq [\beta_2 d(x_n, x_{n+1})]^a \cdot \left[\frac{\beta_2 d(x_n, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+2})}{2} \right]^b \\
&\quad \left[\frac{\beta_1 \beta_2 d(x_n, x_{n+1}) + \beta_1 \beta_2 d(x_{n+1}, x_{n+2})}{2} \right]^{1-a-b}. \tag{3.30}
\end{aligned}$$

Suppose that $\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \leq d(x_{n+1}, x_{n+2})$.

Using (3.30), gives

$$\begin{aligned}
M_H(x_n, x_{n+1}) &\leq [\beta_2 d(x_n, x_{n+1})]^a \cdot \left[\beta_2 d(x_{n+1}, x_{n+2}) \right]^b \\
&\quad \left[\beta_1 \beta_2 d(x_{n+1}, x_{n+2}) \right]^{1-a-b}, \\
&= \beta_2 \cdot \beta_1^{1-a-b} [d(x_n, x_{n+1})]^a \cdot \left[d(x_{n+1}, x_{n+2}) \right]^{1-a}. \tag{3.31}
\end{aligned}$$

Applying (3.31) in (3.29), we obtain

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \implies$$

$$H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \leq \psi \left(\beta_2 \cdot \beta_1^{1-a-b} [d(x_n, x_{n+1})]^a \cdot \left[d(x_{n+1}, x_{n+2}) \right]^{1-a} \right).$$

By Remark 2.1 and Definition 2.7, there must exists $x_{n+2} \in \mathcal{K}x_{n+1}$ such that

$$\beta_2 d(x_{n+1}, x_{n+2}) \leq H(\mathcal{K}x_n, \mathcal{K}x_{n+1}).$$

For the property of ψ implies

$$\begin{aligned}\beta_2 d(x_{n+1}, x_{n+2}) &< \beta_2 \beta_1^{1-a-b} [d(x_n, x_{n+1})]^a \cdot [d(x_{n+1}, x_{n+2})]^{1-a}, \\ [d(x_{n+1}, x_{n+2})]^{1-(1-a)} &< \beta_1^{1-a-b} [d(x_n, x_{n+1})]^a, \\ [d(x_{n+1}, x_{n+2})]^a &< \beta_1^{1-a-b} [d(x_n, x_{n+1})]^a, \\ d(x_{n+1}, x_{n+2}) &< \left(\beta_1^{1-a-b} \right)^{\frac{1}{a}} [d(x_n, x_{n+1})].\end{aligned}$$

Letting $\left(\beta_1^{1-a-b} \right)^{\frac{1}{a}} \leq 1$ and the property of ψ , leads to

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})).$$

By induction, we get

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_0, x_1)).$$

Taking a limit of $n \rightarrow \infty$, leads to

$$d(x_{n+1}, x_{n+2}) = 0.$$

Next, to show that $\{x_n\}$ is a Cauchy sequence. If not, fix $\epsilon > 0$ and there exists integers n_k and m_k with $n_k > m_k > k$ such that

$$d(n_k, m_k) \geq \epsilon.$$

Let $x = n_k$ and $y = m_k$ in (3.28), we get

$$\alpha(n_k, m_k) \geq \eta(n_k, m_k) \implies H(\mathcal{K}n_k, \mathcal{K}m_k) \leq \psi(M_H(n_k, m_k)),$$

where

$$\begin{aligned}M_H(n_k, m_k) &= [d(n_k, m_k)]^a \cdot \left[\frac{d(n_k, \mathcal{K}n_k) + d(m_k, \mathcal{K}m_k)}{2} \right]^b \\ &\quad \left[\frac{d(n_k, \mathcal{K}m_k) + d(m_k, \mathcal{K}n_k)}{2} \right]^{1-a-b} \\ &\leq [d(n_k, m_k)]^a \cdot \left[\frac{d(n_k, n_{k+1}) + d(m_k, m_{k+1})}{2} \right]^b \\ &\quad \left[\frac{d(n_k, m_{k+1}) + d(m_k, n_{k+1})}{2} \right]^{1-a-b}, \\ &= [\epsilon]^a \cdot \left[\frac{0+0}{2} \right]^b \cdot \left[\frac{\epsilon+\epsilon}{2} \right]^{1-a-b} = 0.\end{aligned}$$

Consequently, we have

$$\begin{aligned} d(n_k, m_k) &\leq H(\mathcal{K}n_k, \mathcal{K}m_k) \leq \psi(0), \\ d(n_k, m_k) &= 0, \end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

The proof for existence of fixed point of $\mathcal{K}$ follow similar steps of Theorem 3.1. This complete the proof.

We provide the following example for the demonstration of Theorem 3.1.

Example 3.1. Let $X = [0, \infty]$ with hyperbolic metric $d(x, y) = |x - y|$. Set A : Let $A = \{\frac{x}{3}, \frac{x}{2}\}$ and Set $B = \{\frac{x}{4}, \frac{x^2+2}{3}\}$. Define a mapping $\mathcal{K} : X \rightarrow 2^X$ by

$$\mathcal{K}u = \begin{cases} \{\frac{x}{3}, \frac{x}{2}\}, & 0 \leq x \leq 1, \\ \{\frac{x}{4}, \frac{x^2+2}{3}\}, & x > 1. \end{cases}$$

Also, define $\alpha : X \times X \rightarrow [0, +\infty)$ and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 4, & x, y \in [0, 1], \\ e^{x+y}, & x, y > 1. \\ 0, & \text{otherwise.} \end{cases}$$

where x is an arbitrary point in X and $\mathcal{K}$ is a multivalued map. Since $\mathcal{K}$ is α - ψ -contractive mapping, there exists $x_0 \in X$ such that

$$\alpha(x_0, \mathcal{K}x_0) \geq 1,$$

and so for $x_0 = 1$, we have

$$\begin{aligned} \alpha(1, [\frac{x}{3}, \frac{x}{2}]) &\geq 1, \\ 4 &\geq 1, \end{aligned}$$

That is, $\mathcal{K}$ is α -admissible.

To show that the mapping $\mathcal{K}$ defined on $X = [0, \infty]$ with the hyperbolic metric d satisfies the α - ψ -interpolative multivalued contraction condition of inequality (3.1), follow these steps:

Calculate $\mathcal{K}x$ and $\mathcal{K}y$

For $x, y \in X$:

$$A = \mathcal{K}x = \left[\frac{x}{3}, \frac{x}{2} \right]$$

$$B = \mathcal{K}x = \left[\frac{x}{4}, \frac{x^2 + 2}{3} \right]$$

Calculate $H(\mathcal{K}x, \mathcal{K}y)$. Define the Hausdorff Metric H , for two sets A and B in a metric space (X, d) , the Hausdorff metric H is defined as: The Hausdorff metric is defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

where

$$d(x, B) = \inf_{b \in B} |x - b|$$

We begin by calculating the following Hausdorff metric. Specifically, for any $x = 1 \in A$, we can choose $y = 2 \in B$ such that:

$$\begin{aligned} d(x, y) &= |1 - 2| = 1 \\ d(x, \mathcal{K}x) &= d\left(1, \left[\frac{1}{3}, \frac{1}{2}\right]\right) = \inf \left\{ d\left(1, \frac{1}{3}\right), d\left(1, \frac{1}{2}\right) \right\}, \\ &= \inf \left\{ \left|1 - \frac{1}{3}\right|, \left|1 - \frac{1}{2}\right| \right\}, \\ &= \inf \left\{ \left|\frac{2}{3}\right|, \left|\frac{1}{2}\right| \right\} = \frac{1}{2}. \end{aligned} \tag{3.32}$$

$$\begin{aligned} d(y, \mathcal{K}y) &= d\left(2, \left[\frac{1}{2}, 2\right]\right) = \inf \left\{ d\left(2, \frac{1}{2}\right), d(2, 2) \right\}, \\ &= \inf \left\{ \left|\frac{3}{2}\right|, |0| \right\}, \\ &= 0. \end{aligned} \tag{3.33}$$

$$H(\mathcal{K}x, \mathcal{K}y) = H\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{2}, 2\right]\right)$$

Let

$$A = \left[\frac{1}{3}, \frac{1}{2} \right], \quad B = \left[\frac{1}{2}, 2 \right]$$

$$\sup_{a \in A} d(a, B) = \sup_{a \in A} \left| a - \frac{1}{2} \right| = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$\sup_{b \in B} d(b, A) = \sup_{b \in B} \left| b - \frac{1}{2} \right| = 2 - \frac{1}{2} = \frac{3}{2},$$

$$H(A, B) = \max \left\{ \frac{1}{6}, \frac{3}{2} \right\} = \frac{3}{2},$$

$$H \left(\left[\frac{1}{3}, \frac{1}{2} \right], \left[\frac{1}{2}, 2 \right] \right) = \frac{3}{2}.$$

Existence of $y \in B$: For any $x \in A$ and any $\alpha > 0$, there exists a point $y \in B$ such that

$$d(x, y) \leq H(A, B) + \alpha.$$

$$1 \leq 1.5 + \alpha.$$

However, for certain points $x \in A$, there may not be a $y \in B$ such that

$$d(x, y) \leq H(A, B).$$

For example, for $x = 1 \in A$, the closest point in B is 3, but $d(1, 3) = 2$, which is greater than $H(A, B) = 1.5$.

Compact Set B : Now, if we modify B to be a compact set, the situation changes.

Existence of $y \in B$: For any $x \in A$, there always exists a point $y \in B$ such that

$$d(x, y) \leq H(A, B).$$

Since B is compact, every $x \in A$ will have a corresponding $y \in B$ that meets this condition.

This example demonstrates how the properties change based on the compactness of set B .

Applying all of the above equalities in inequality (3.1), we obtain

$$\alpha(1, 2)1.5 \leq \psi(\mathcal{N}_H(x, y)), \quad (3.34)$$

where

$$\mathcal{N}_H(x, y) = [1]^a \cdot [0.25]^{1-a}. \quad (3.35)$$

implies that

$$0 \leq \psi(0.25^{1-a}).$$

Suppose $a = 0.25$, $n = 1$ and $\psi(t) = \frac{1}{2^t}$ in the above inequality, we get

$$0 \leq \psi(0.25^{0.75}),$$

$$0 \leq \psi(0.35),$$

$$0 \leq \frac{1}{2^{0.35}}.$$

$$0 \leq 0.785.$$

This shows that the conditions in Theorem 3.1 are satisfied. Since $\mathcal{K}$ has two fixed points 0 and 2. If the above steps hold, the mapping $\mathcal{K}$ satisfies the α - ψ -multivalued contraction condition.

We give the following corollary as a generalization of Theorem 3.1.

Corollary 3.1. *Let (X, d) be a complete compact hyperbolic metric space. Consider two compact sets A and B in X with a mapping $\mathcal{K} : X \longrightarrow X$ called an α - ψ multivalued interpolative Reich-Chatterjea such that*

$$\alpha(x, y)H(\mathcal{K}x, \mathcal{K}y) \leq \psi \left(\left[\frac{1}{2}d(x, y) \right]^a \cdot [d(x, \mathcal{K}y)]^b \cdot [d(y, \mathcal{K}x)]^{1-a-b} \right), \quad (3.36)$$

for all $x, y \in X$ and $a + b \leq 1$. Then $\mathcal{K}$ has a fixed point $z \in X$ and for any $x \in X$ the sequence of iterate $\{\mathcal{K}^n x\}$ converges to z .

Proof. Let x_0 be an arbitrary point in X and $\mathcal{K}$ be α -admissible such that

$$\alpha(x_0, \mathcal{K}x_0) \geq 1.$$

Consider the sequence $\{x_n\} \subseteq X$ defined by

$$x_n = \mathcal{K}^n x_0 = \mathcal{K}x_{n-1}$$

and

$$x_{n+1} = \frac{x_n \oplus \mathcal{K}x_n}{2},$$

for all $n \in \mathbb{N}$.

We prove that $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

If for some n we have $d(x_n, x_{n+1}) = 0$, implies that $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ will be a fixed point of $\mathcal{K}$.

So, we suppose that $d(x_n, x_{n+1}) > 0, \forall n \geq 1$. Using (3.36), we get

$$\begin{aligned} \alpha(x_n, x_{n+1})H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) &\leq \psi \left([d(x_n, x_{n+1})]^a \cdot \left[\frac{1}{2}d(x_n, \mathcal{K}x_{n+1}) \right]^b \right. \\ &\quad \left. [d(x_{n+1}, \mathcal{K}x_n)]^{1-a-b} \right). \end{aligned}$$

Since $\alpha(x_n, x_{n+1}) \geq 1$, we have

$$H(\mathcal{K}x_n, \mathcal{K}x_{n+1}) \leq \psi \left([d(x_n, x_{n+1})]^a \cdot \left[\frac{1}{2}d(x_n, x_{n+2}) \right]^b \cdot [d(x_{n+1}, x_{n+1})]^{1-a-b} \right),$$

Applying Remark 2.1 and Definition 2.1, 2.7, we get

$$\begin{aligned}\beta_2 d(x_{n+1}, x_{n+2}) &\leq \psi \left([\beta_2 d(x_n, x_{n+1})]^a \cdot \left[\frac{\beta_2 d(x_n, x_{n+1}) + \beta_2 d(x_{n+1}, x_{n+2})}{2} \right]^b \right. \\ &\quad \left. [d(x_{n+1}, x_{n+1})]^{1-a-b} \right), \\ \beta_2 d(x_{n+1}, x_{n+2}) &\leq \psi(0), \\ d(x_{n+1}, x_{n+2}) &\leq 0,\end{aligned}$$

implies that $d(x_{n+1}, x_{n+2}) = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. The other steps of the proof of this corollary follows a similar proof to Theorem 3.1. It completes the proofs.

The following corollary is provided due to the usefulness of Theorem 3.2.

Corollary 3.2. *Let (X, d) be a complete compact hyperbolic metric space with A and B closed compact subsets in X and $\mathcal{K} : X \rightarrow X$ be an α - ψ multivalued rational interpolative Reich-Kannan type mapping concerning η . Assume that*

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \implies H(\mathcal{K}x, \mathcal{K}y) \leq \psi(M_H(x, y)), \quad (3.37)$$

where $\psi \in \Psi$ and

$$M_H(x, y) = [d(x, y)]^a \cdot \left[\frac{d(y, \mathcal{K}y)(1 + d(x, \mathcal{K}x))}{1 + d(x, \mathcal{K}x)} \right]^{1-a}.$$

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{K}x_0) \geq \eta(x_0, \mathcal{K}x_0)$,
- (ii) either $\mathcal{K}$ is continuous or for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $\mathcal{K}$ has a fixed point.

Proof. The proof of this corollary follows a similar proof to Theorem 3.2. This completes the proofs.

4. Some Applications

This section addresses the application of functional integral inclusion and non-linear mixed integral equations in hyperbolic metric spaces.

4.1. Existence of the Solution for Functional Integral Inclusion in Hyperbolic metric Space

In this subsection, we investigate the existence of a solution to the functional integral inclusion for demonstration of Theorem 3.1 in a hyperbolic metric space.

Consider the following functional integral inclusion from Dhage [8, 11, 36].

$$x(t) \in q(t) + \int_0^{\varsigma(t)} k_1(t, s)F(s, x(\theta(s)))ds + \int_0^{\sigma(t)} k_2(t, s)G(s, v(\eta(s)))ds, \quad (4.1)$$

$t, s \in J$, where $J = [0, 1]$ is the closed and bounded interval in $\mathbb{R}$. $q : J \rightarrow \mathbb{R}$, $k_1, k_2 : J \times J \rightarrow \mathbb{R}$, $F, G : J \times \mathbb{R} \rightarrow L^1(J, \mathbb{R})$ and $\varsigma, \theta, \sigma, \eta : J \rightarrow J$.

Equation (4.1) can be transformed as a function $x \in C(J, \mathbb{R})$ that satisfies:

$$x(t) = q(t) + \int_0^{\varsigma(t)} k_1(t, s)x_1(s)ds + \int_0^{\sigma(t)} k_2(t, s)y_1(s)ds, \quad (4.2)$$

for some $x_1, y_1 \in L^1(J, \mathbb{R})$, $x_1(t) \in F(s, x(\theta(s)))$ and $y_1 \in G(s, y(\eta(s)))$ and $t, s \in J$.

Definition 4.1. [11] A measurable multivalued function $F : J \rightarrow L^1(J, \mathbb{R})$ is said to be integrable bounded if there exists a function $h \in L^1(J, \mathbb{R})$ such that $\|v\| \leq h(t)$ and $t \in J$ for all $v \in F(t)$.

Definition 4.2. [11] Let X be a Banach space with norm $\|\cdot\|$. A multivalued function $\beta : J \times X \rightarrow L^1(J, \mathbb{R})$ is called Caratheodory if

- (i) $t \rightarrow \beta(t, x)$ is measurable for each $x \in X$, and
- (ii) $x \rightarrow \beta(t, x)$ is an upper semi-continuous almost every where for $t \in J$.

Definition 4.3. [11] A Caratheodory multivalued function $\beta : J \times X \rightarrow X$ is called L^1 -Caratheodory if for every real number $r > 0$ there exists a function $hr \in L^1(J, \mathbb{R})$ such that

$$\|F(t, x)\| = \sup\{\|x\| : x \in F(t, x) \leq hr(t)\},$$

for $t \in J$ and $u \in X$ with $|u| \leq r$.

Now, we formulate the following theorem.

Theorem 4.1. Assume that the following hypotheses hold:

- (i) The functions $\varsigma, \theta, \sigma, \eta, \zeta : J \rightarrow J$ are continuous.
- (ii) The function $q : J \rightarrow \mathbb{R}$ is continuous.

(iii) The functions k_1, k_2 are continuous on $J \times J$ with $\lambda_1 = \max_{t,s \in J} |k_1(t, s)|$ and $\lambda_2 = \max_{t,s \in J} |k_2(t, s)|$.

(iv) There exists a function $\lambda \in L^1(J, \mathbb{R})$ such that

$$\begin{aligned} H(F(t, u_1), F(t, u_2)) &\leq \lambda_1 |u_1 - u_2|, \\ H(G(t, v_1), G(t, v_2)) &\leq \lambda_2 |v_1 - v_2|, \end{aligned}$$

for all $u, v \in X$, $t \in J$ and $\lambda = \lambda_1 \varsigma(t) + \lambda_2 \sigma(t) \leq 1$, $\alpha(x, y) \geq 1$,

where

$$\begin{aligned} \alpha(x, y) H(\mathcal{K}x, \mathcal{K}y) &= H(F(t, x_1), F(t, x_2)) + H(G(t, y_1), G(t, y_2)) \\ &\leq \lambda d(x, y) \end{aligned} \quad (4.3)$$

$$\mathcal{N}_H(x, y) = d(x, y) = \left(\lambda [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^{1-a} \right).$$

Then, $\mathcal{K}$ has a fixed point.

Proof. We prove that $\mathcal{K}$ is a contraction in X . By using the hypotheses imposed in Theorem 4.1.

Let $\mathcal{K} : J \rightarrow L^1(J, \mathbb{R})$ be a multivalued mapping. Denoted by

$$\mathcal{K}x(t) = q(t) + \int_0^{\varsigma(t)} k_1(t, s)x_1(s)ds + \int_0^{\sigma(t)} k_2(t, s)y_1(s)ds. \quad (4.4)$$

We have

$$\begin{aligned} |\mathcal{K}x(t) - \mathcal{K}y(t)| &\leq q(t) + \int_0^{x(t)} k_1(t, s)x_1(s)ds + \int_0^{\sigma(t)} k_2(t, s)y_1(s)ds - \\ &\quad q(t) + \int_0^{\varsigma(t)} k_1(t, s)x_2(s)ds + \int_0^{\sigma(t)} k_2(t, s)y_2(s)ds, \\ &\leq \int_0^{\varsigma(t)} k_1(t, s)|x_1 - x_2|(s)ds + \int_0^{\sigma(t)} k_2(t, s)|y_1 - y_2|(s)ds, \\ &\leq \int_0^{\varsigma(t)} \lambda_1 |x_1 - x_2| ds + \int_0^{\sigma(t)} \lambda_2 |y_1 - y_2| ds, \\ &\leq \lambda_1 |x_1 - x_2| \int_0^{\varsigma(t)} ds + \lambda_2 |y_1 - y_2| \int_0^{\sigma(t)} ds, \\ &\leq \lambda_1 |x_1 - x_2| \varsigma(t) + \lambda_2 |y_1 - y_2| \sigma(t), \\ &\leq \lambda_1 \varsigma(t) |x_1 - x_2| + \lambda_2 \sigma(t) |y_1 - y_2|, \\ &\leq \lambda_1 \varsigma(t) + \lambda_2 \sigma(t) |x - y|, \\ H(\mathcal{K}x, \mathcal{K}y) &\leq \lambda |x - y|. \\ H(\mathcal{K}x, \mathcal{K}y) &\leq |x - y|. \end{aligned}$$

Consequently, we have

$$H(\mathcal{K}x, \mathcal{K}y) \leq [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^{1-a}.$$

For the α - ψ property, we have

$$\alpha(x, y)H(\mathcal{K}x, \mathcal{K}y) \leq \psi \left([d(u, v)]^a \cdot \left[\frac{d(u, \varsigma u) + d(v, \varsigma v)}{2} \right]^{1-a} \right). \quad (4.5)$$

Thus, the functional integral equation (4.2) has a fixed point which is also a solution to equation (4.1). Hence, our proof is completed.

4.2. Existence of the Solution for Linear First-order Partial Differential Equation on Hyperbolic metric space

In this subsection, we apply Theorem 3.2 to establish the existence of multi-valued solutions for a class of first-order hyperbolic partial differential equations in conservative form and show that the associated multivalued solution operator satisfies the assumptions of Theorem 3.2 in the framework of a hyperbolic metric space.

Consider the first-order hyperbolic partial differential equations (PDEs) from [40, 44] using fixed point theory approaches. Let $X = \mathbb{R}^d \times \mathbb{R}^+$ be the space-time domain and let $u : X \rightarrow \mathbb{R}$ be a classical solution of the linear first-order partial differential equation in conservative form

$$\partial_t u(x, t) + \nabla \cdot (v(x, t)u(x, t)) = 0, \quad (x, t) \in X. \quad (4.6)$$

Let $v \in [C^1(U)]^d$ be a continuously differentiable vector field and assume the initial condition

$$u(x, 0) = u_0(x).$$

where

$u(x, t)$: A scalar field (unknown function) depending on space $x \in \mathbb{R}^d$ and time $t \geq 0$, interpreted as density, temperature, or concentration.

$X(t; \xi)$: Characteristic curve (trajectory in space), satisfying

$$\frac{dX}{dt}(t; \xi) = v(X(t; \xi), t), \quad X(0; \xi) = \xi, \quad (4.7)$$

where ξ is the initial position of a particle.

$\frac{d}{dt}u(X(t; \xi), t)$: Total derivative of u along the trajectory $X(t; \xi)$, measuring change of u along the flow.

$\partial_t u(X(t; \xi), t)$: Partial derivative with respect to time, describing variation at a fixed spatial point.

$\nabla u(X(t; \xi), t)$: Spatial gradient of u ,

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \right). \quad (4.8)$$

$v(x, t)$: Velocity vector field governing motion of particles in the medium.

$v(X(t; \xi), t) \cdot \nabla u(X(t; \xi), t)$: Advection term representing directional derivative of u along the flow.

$\nabla \cdot v(x, t)$: Divergence of velocity field,

$$\nabla \cdot v = \sum_{i=1}^d \frac{\partial v_i}{\partial x_i}, \quad (4.9)$$

measuring expansion or compression of the flow.

$-(\nabla \cdot v)u$: Reaction term describing compression/expansion effects; positive divergence leads to decay, negative divergence leads to growth.

This identity represents the chain rule along characteristics for transport equations: change of u = local time change + transport by velocity field + modification by divergence of the flow.

Then $u(x, t)$ is given by

$$u(x(t; \xi), t) = u_0(\xi) \exp \left(- \int_0^t \nabla \cdot v(x(s; \xi), s) ds \right), \quad (4.10)$$

along the characteristic curves $X(t; \xi)$, $X : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, defined by

$$\frac{dX}{dt}(t; \xi) = v(X(t; \xi), t), \quad t > 0, \quad X(0; \xi) = \xi. \quad (4.11)$$

We establish the following theorem.

Theorem 4.2. *Let $X = \mathbb{R}^d \times \mathbb{R}^+$ be the space-time domain and let $u : X \rightarrow \mathbb{R}$ be a classical solution of the linear first-order partial differential equation in conservative form (4.6) the following conditions hold:*

(i) *Let $v \in [C^1(U)]^d$ be a continuously differentiable vector field,*

(ii) *for each $x, y \in X$, we have*

$$|\nabla \cdot v(x(s; \xi), s) - \nabla \cdot v(y(s; \xi), s)| \leq e^{-\lambda t} |x - y|,$$

for all $x, y \in X$, $t > 0$,

where

$$M_H(x, y) = d(x, y) = [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^b \cdot \left[\frac{d(x, \mathcal{K}y) + d(y, \mathcal{K}x)}{2} \right]^{1-a-b}. \quad (4.12)$$

(iii) there exists

$$\frac{\|a\|_\infty}{\lambda} e^{-\lambda t} \leq 1.$$

Then, $\mathcal{K}$ has a fixed point which is a solution of the linear first-order partial differential equation (4.6).

Proof. Let $\mathcal{K} : X \rightarrow CB(X)$ be a multivalued mapping given by

$$\mathcal{K}u(x(t; \xi), t) = u_0(\xi) \exp \left(- \int_0^t \nabla \cdot v(x(s; \xi), s) ds \right).$$

We consider the evolution along the characteristic curve $x(t; \xi)$ and define

$$a(t) = \nabla \cdot v(x(t; \xi), t).$$

Then the transport equation reduces to the linear ODE

$$\frac{d}{dt} u(t) = -a(t)u(t), \quad u(0) = u_0(\xi). \quad (4.13)$$

Define the operator $\mathcal{K}$ on $X = C([0, T])$ by

$$(\mathcal{K}u)(t) = u_0(\xi) - \int_0^t a(s)u(s) ds.$$

Then: $\mathcal{K} : X \rightarrow X$ is well-defined. For $u, v \in X$,

$$\begin{aligned} |(\mathcal{K}u - \mathcal{K}v)(t)| &= u_0(\xi) - \int_0^t a(s)u(s) ds - \left(u_0(\xi) - \int_0^t a(s)v(s) ds \right) \\ |(\mathcal{K}u - \mathcal{K}v)(t)| &\leq \int_0^t |a(s)| |u(s) - v(s)| ds. \end{aligned}$$

Using weighted norm $\|u\|_\lambda = \sup_t e^{-\lambda t} |u(t)|$, we get

$$\begin{aligned}\|\mathcal{K}u - \mathcal{K}v\|_\lambda &\leq \frac{\|a\|_\infty}{\lambda} \|u - v\|_\lambda. \\ \|\mathcal{K}u - \mathcal{K}v\| &\leq \frac{\|a\|_\infty}{\lambda} e^{-\lambda t} \|u - v\|.\end{aligned}\tag{4.14}$$

Choosing $\lambda > \|a\|_\infty$, $\mathcal{K}$ becomes a contraction. Hence, by Banach fixed point theorem, $\mathcal{K}$ has a unique fixed point u , i.e.

$$\mathcal{K}u = u.$$

Taking $u = x, v = y$, it follows that

$$\begin{aligned}H(\mathcal{K}x, \mathcal{K}y) &\leq [d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^b \\ &\quad \left[\frac{d(x, \mathcal{K}y) + d(y, \mathcal{K}x)}{2} \right]^{1-a-b}.\end{aligned}$$

By the property of ψ as a non-decreasing sequence, we have

$$\begin{aligned}H(\mathcal{K}x, \mathcal{K}y) &\leq \psi \left([d(x, y)]^a \cdot \left[\frac{d(x, \mathcal{K}x) + d(y, \mathcal{K}y)}{2} \right]^b \right. \\ &\quad \left. \left[\frac{d(x, \mathcal{K}y) + d(y, \mathcal{K}x)}{2} \right]^{1-a-b} \right)\end{aligned}\tag{4.15}$$

Thus, the integral equation (4.10) has a solution which is a solution to the linear first-order partial differential equation (4.6). Hence, our proof is completed.

5. Conclusion

This paper establishes new multivalued fixed point theorems for Ćirić-type contraction mappings in hyperbolic metric spaces. The obtained results extend and generalize several existing works, including those of Debnath et al. [9] and Konwar et al. [34] by incorporating multivalued and interpolative structures within a hyperbolic setting. In addition, the study develops Kannan-type and Reich-Ćirić-type contraction frameworks under multivalued interpolative mappings, thereby broadening classical fixed point theory. Illustrative examples are provided to validate the theoretical results, and applications to functional integral inclusions and linear partial differential equations, demonstrate the applicability and effectiveness of the proposed results. Furthermore, this work extends earlier results from standard metric spaces (e.g., Salimi et al. [42]) to the more general framework of hyperbolic metric spaces.

References

- [1] Alansari, M. and Ali, M.U., Unified multivalued interpolative Reich–Rus–Ćirić-type contractions, *Adv. Diff. Equ.*, 1 (2021), 311.
- [2] Alfuraidan, M. and Ansari, Q. eds., *Fixed point theory and graph theory: Foundations and integrative approaches*, Academic Press, 2016 (2016).
- [3] Banach, S., Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, 3(1922), 133-181.
- [4] Aydi, H., Karapinar, E. and Roldan Lopez de Hierro, A. F., ω -Interpolative Ćirić–Reich–Rus Type Contractions, *Mathematics*, 7(1) (2019), Article 57.
- [5] Aydi, H., Chen, C.-M. and Karapinar, E., Interpolative Ćirić–Reich–Rus Type Contractions via the Branciari Distance, *Mathematics*, 7(1) (2019), Article 84.
- [6] Bello, A.U., Okeke, C. C. and Izuchukw, C., Approximating common fixed point for family of multivalued mean nonexpansive mappings in hyperbolic spaces, *Adv. Fixed Point Theory*, 7(4) (2017), 524-543.
- [7] Berinde, M. and Berinde, V., On a general class of multivalued weakly Picard mappings, *J. Math. Anal. Appl.*, 326 (2007), 772–782.
- [8] Darwish, M. A., Note on stability theorems for nonlinear mixed integral equations, *Korean J. Comput. and Appl. Math.*, 6(1) (1999), 633-637. <https://doi.org/10.1007/BF03009955>
- [9] Debnath, P. and de La Sen, M., Set-valued interpolative Hardy-Rogers and set-valued Reich–Rus–Ćirić-type contractions in b -metric spaces, *Mathematics*, (2019).
- [10] Dewangan, K., Fixed points of multivalued Suzuki nonexpansive mappings in hyperbolic spaces, *Journal of Hyperstructures*, 12(1) (2023), 111-133.
- [11] Dhage, B. C., Multivalued mappings and fixed points II, *Tamkang journal of mathematics*, 37(1) (2006), 27-46.
- [12] Gaba, Y. U. and Karapinar, E., A New Approach to the Interpolative Contractions, *Axioms*, 8(4) (2019), Article 110.

- [13] El Bekri, Y., Edraoui, M., Mouline, J. and Bassou, A., 2023. Cyclic Coupled Fixed Point via Interpolative Kannan Type Contractions, *Mathematical Statistician and Engineering Applications*, 72(2) (2023), 24-30.
- [14] Gautam, P., Singh, S.R., Kumar, S., and Verma, S., On nonunique fixed point theorems via interpolative Chatterjea type Suzuki contraction in quasi-partial b-metric space, *Journal of Mathematics*, Vol. 2022, Article ID 2347294, 10 Pages.
- [15] Gautam, P., Kumar, S., Verma, S. and Gulati, S., On some w-Interpolative contractions of Suzuki type mappings in Quasi-partial b-metric space, *Journal of Function Spaces*, Vol. 2022, Article Id. 9158199, 12 Pages.
- [16] Goebel, K. and Reich, S., *Uniform Convexity, Hyperbolic Geometry, and Non-expansive Mappings*, Marcel Dekker, Inc., 83 (1984).
- [17] Guran, L., Ahmad, K., Shabbir, K., Bota, M., Computational comparative analysis of fixed point approximations of generalized α -nonexpansive mappings in hyperbolic spaces, *AIMS Mathematics*, (2023).
doi: 10.3934/math.2023129
- [18] Hammad H. A., Aydi H and Kattan D. A., Hybrid interpolative mappings for solving fractional Navier–Stokes and functional differential equations, *Boundary Value Problems*, 1(1) (2023), 116.
- [19] Kannan, R., Some results on fixed points, *Bull. Calcutta Math. Soc.*, 60 (1968), 71-76.
- [20] Karapinar, E., Revisiting the Kannan type contractions via interpolation, *Adv. Theory Nonlinear Anal. Appl.*, 2 (2018), 85-87.
- [21] Karapinar, E., Fulga, A. and Roldan Lopez de Hierro, A. F., Fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$ -interpolative contractions, *Adv. Differ. Equ.*, 2021 (2021), Article 339.
- [22] Karapinar, E., Fulga, A. and Yesilkaya, S. S., New Results on Perov-Interpolative Contractions of Suzuki Type Mappings, *J. Funct. Spaces*, 2021 (2021), Article ID 9587604, 7 pp.
- [23] Karapinar, E. and Agarwal, R. P., Interpolative Rus–Reich–Ćirić Type Contractions via Simulation Functions, *An. St. Univ. Ovidius Constanta Ser. Math.*, 27(3) (2019), 137-152.

- [24] Karapinar, E., Aydi, H. and Mitrovic, Z. D., On Interpolative Boyd–Wong and Matkowski Type Contractions, TWMS J. Pure Appl. Math., 11(2) (2020), 204-212.
- [25] Karapinar, E., Alqahtani, O. and Aydi, H., On Interpolative Hardy–Rogers Type Contractions, Symmetry, 11(1) (2019), Article 8.
- [26] Karapinar, E., Agarwal, R. P. and Aydi, H., Interpolative Reich–Rus–Ćirić Type Contractions on Partial Metric Spaces, Mathematics, 6(11) (2018), Article 256.
- [27] Karapinar, E., Interpolative Kannan–Meir–Keeler Type Contraction, Adv. Theory Nonlinear Anal. Appl., (2021).
- [28] Karapinar, E. and Samet, B., September. Generalized α - ψ contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal., (2012).
- [29] Karapinar, E., Alid, A., Hussain, A. and Aydi, H., On Interpolative Hardy–Rogers Type Multivalued Contractions via a Simulation Function, Filomat, 36(8) (2022), 2847-2856.
- [30] Khan, M. S., Singh, Y. M. and Karapinar, E., On the interpolative (φ, ψ) -type Z -contraction, U.P.B. Sci. Bull. Ser. A, 83(2) (2021).
- [31] Khan, S. H. and Abbas, M., Common fixed points of two multivalued non-expansive maps in Kohlenbach hyperbolic spaces, Fixed Point Theory Appl., 9(14) (2014), 1-11.
- [32] Kohlenbach, U., Some logical metatheorems with applications in functional analysis, Trans. Am. Math. Soc., 357(1) (2005), 89-128.
- [33] Kohlenbach, U. and Leuştean, L., Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, J. European Math. Soc., 12(1) (2009), 71-92.
- [34] Konwar, N., Srivastava, R., Debnath, P. and Srivastava, H. M., Some new results for a class of multivalued interpolative Kannan-type contractions, Axioms, 11(2) (2022), 76.
- [35] Markin, J. T., A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc., 74 (1968), 639–640.

- [36] Miller, R. K. Nohel, J. A. and Wong, J. S. W., A stability theorem for nonlinear mixed integral equations, *J. Math. Anal. Appl.*, 25(1969), 446-449.
- [37] Nadler, S. B., Multi-valued contraction mappings, *Pac. J. Math.*, 30 (1969), 475-488.
- [38] Niazi Asil, K. and Ghasemi Kamalvand, M., Some hyperbolic iterative methods for linear systems, *J. Appl. Math.*, 2020(1) (2020), 9874162. doi: 10.1155/2020/9874162
- [39] Prithvi, B. V. and Katiyar, S. K., Interpolative operators: Fractal to multi-valued fractal, *Chaos, Solitons and Fractals*, 164 (2022), 112449.
- [40] Quarteroni, A., and Valli, A., Numerical approximation of partial differential equations. Berlin, Heidelberg: Springer Berlin Heidelberg, (1994).
- [41] Reich, S. and Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.: Theory, Methods Appl.*, 15(6) (1990), 537-558.
- [42] Salimi, P., Latif, A. and Hussain, N., Modified α - ψ -contractive mappings with applications, *Fixed Point Theory Appl.*, (2013), 1-19.
- [43] Samet, B., Vetro, C. and Vetro, P., Fixed point theorems for α - ψ -contractive type mappings, *Nonlinear Anal.: Theory, Methods Appl.*, 75(4) (2012), 2154-2165.
- [44] Smith, G. D., Numerical solution of partial differential equations: finite difference methods. Oxford university press, (1985).
- [45] Takahashi, W., A convexity in metric space and nonexpansive mappings, I. *Kodai Math. Sem. Rep.*, 22 (1970), 142-149.
- [46] Uddin, I., Aggarwal, S. and Abdou, A. A., Approximation of Endpoints for α Reich-Suzuki Non-expansive Mappings in Hyperbolic Metric Spaces, *J. European Math. Soc.*, 12(1) (2009), 71-92.
- [47] Ullah, K., Ahmad, J., De, la, Sen, M., Khan, M. N., Approximating Fixed Points of Reich-Suzuki Type Nonexpansive Mappings in Hyperbolic Spaces, *J. Math.*, (2020). doi: 10.1155/2020/2169652

- [48] Valiya, S, Valappil, and Pulickakunnel, S., Some fixed-point theorems for a pair of Reich-Suzuki-type nonexpansive mappings in hyperbolic spaces, *Topological Algebra and its Applications*, (2023). doi: 10.1515/taa-2022-0132
- [49] Wangwe, L., Fixed point theorems for interpolative Kannan contraction mappings in Busemann space with an application to matrix equation, *The J. Anal.*, 31(2) (2023), 1123-1142.
- [50] Wangwe, L., Fixed point theorems for interpolative Ćirić-type contraction mappings in convex hull combination of Busemann space with an application to fractional differential equations, *J. Comput. Phy.*, 32(2) (2024), 613-636.
- [51] Wangwe, L., Fixed point theorems for interpolative Ćirić-type contraction mappings in CAT (0) space with application to hyperbolic rotation matrix, *The J. Anal.*, 31(4) (2023), 205-231.
- [52] Wangwe, L., Fixed point theorems for implicit relation in hyperbolic metric spaces with applications, *The J. Anal.*, 32(3) (2024), 1909-1927.
- [53] Wangwe, L. and Kumar, S., Fixed point Results for Interpolative ψ -Hardy-Rogers type contraction mappings in quasi-partial b-metric space with some Applications, *The J. Anal.*, (2022) 18 Pages, <https://doi.org/10.1007/s41478-022-00456-4>
- [54] Zidan, A. M., Soliman, A. H., Nabil, T. and Barakat, M. A., An investigation of new quicker implicit iterations in hyperbolic spaces, *Thermal Science*, 24(1) (2020), 199-207. doi: 10.2298/TSCI20S1199Z