

**ON THE SPECTRUM OF SECOND ORDER $(\tilde{p}, \tilde{q})$ –DIFFERENCE
OPERATOR OVER THE SEQUENCE SPACE ℓ_p ($1 < p < \infty$)**

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Abstract: This article presents a detailed analysis of the spectral and fine spectral properties of the second-order $(\tilde{p}, \tilde{q})$ –difference operator $\Delta_{\tilde{p}, \tilde{q}}^2$, where $(0 < \tilde{q} < \tilde{p} \leq 1)$. The operator extends the classical $\tilde{q}$ –difference operator and is examined within the Banach space ℓ_p ($1 < p < \infty$). In this setting, $\Delta_{\tilde{p}, \tilde{q}}^2$ admits a matrix representation characterized by a lower triangular structure with three nonzero diagonals.

Keywords and Phrases: Spectrum, infinite matrices, sequence space, $(\tilde{p}, \tilde{q})$ –analog.

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1. Introduction

Spectral theory is a fundamental area of modern functional analysis with numerous applications. Mainly, it focuses on certain inverse operators, their essential characteristics, and how they relate to the original operators. These inverse operators often appear naturally when solving various types of equations, such as systems

of linear equations, differential equations or integral equations. The spectrum of a bounded linear operator is a generalization of eigen values of matrix related with that operator. The fundamental principle of the modern spectral theorem is that certain linear operators on infinite dimensional spaces can be represented in a “diagonal” matrix form. From this diagonal form, we can determine the spectrum of the operator. The spectrum of an operator can be classified into three parts: the point spectrum, the continuous spectrum and the residual spectrum. These three disjoint parts together are refereed to as the “fine spectrum”.

Spectral analysis of bounded linear operators like the Cesàro, Toeplitz, and difference operators has been a rich area of research in functional analysis and operator theory. The spectrum of these operators and their different forms on various sequence spaces have been comprehensively studied. The calculation of spectrum of the difference operator Δ ($\Delta y_k = y_k - y_{k+1}$) were initially investigated by Altay and Başar [6], Kayaduman and Furkan [27], and Akhmedov and Başar [1, 2] in the sequence spaces c_o , c ; ℓ_1 , bv and ℓ_p , bv_p respectively. Moreover, the spectrum of various kinds of difference operators like Δ^2 [15], $B(r, s)$ [7, 10, 16, 19], $B(r, s, t)$ [11, 20, 21], Δ_v [3, 36, 37], Δ_{ab} [4, 17, 18], Δ_{uv}^2 [33], Δ_{uvw}^2 [34], and $A(p_1, p_2; q_1, q_2; r_1, r_2)$ [21] were determined by many researchers over different sequence spaces. Yeşilkayagil and Başar [46, 47] determined the fine spectrum of the Lambda matrix over the sequence spaces c_0 and c . Dündar and Başar [14] calculated the fine spectrum of the upper triangular double band matrix Δ^+ on the sequence space c_0 . Yeşilkayagil and Başar [48] wrote a research paper on the survey for the spectrum of upper triangle over sequence spaces. Ayman-Mursaleen [29] studied about the Quadratic function preserving wavelet type Baskakov operators for enhanced function approximation. This work introduces advanced operator constructions that could inspire further spectral studies of generalized difference operator. Hon et al [23] studied about the spectral distribution of symmetrized Toeplitz sequences. It deals with spectral distributions of structured matrices, directly complementing our spectral analysis of banded infinite matrices. Subsequently, considerable attention has been devoted to the computation of the spectrum of quantum versions of different matrix operators. The spectrum of $\tilde{q}$ -Cesàro matrix was calculated by Yildirim [49], Durna and Turkay [39], Kalita and Dutta [25] over the sequence spaces c_0 , c and ℓ_p respectively. Spectral study of second order $\tilde{q}$ -difference operator was done by Yaying et al. [5, 43], Kalita and Dutta [26]. Yaying et al. [44] determined the spectrum of weighted $\tilde{q}$ -difference operator over the sequence space c_0 . Recently, the spectral analysis of $\tilde{q}$ -analog of well-known operators, such as the difference operator, has attracted growing attention. However, to the best of our knowledge, no attempt has been made to analysis the spectrum of a second order $(\tilde{p}, \tilde{q})$ -difference operator. Thus,

the aim of this paper is to investigate the spectral properties of a second order $(\tilde{p}, \tilde{q})$ - difference operator in the space ℓ_p ($1 < p < \infty$).

The concept of the $(\tilde{p}, \tilde{q})$ -analog is a two parameter generalization of the classical $\tilde{q}$ -analog and $\tilde{q}$ -analog is a one parameter generalization of classical mathematical objects like numbers, functions, and operators. The $(\tilde{p}, \tilde{q})$ -analog of an expression reduces to its $\tilde{q}$ -analog when $\tilde{p} = 1$ and the $\tilde{q}$ -analog further converges to the original expression as $\tilde{q} \rightarrow 1$. In recent mathematical research across diverse areas such as combinatorics, approximation theory, difference and integral equations, quantum algebra, operator theory etc., both $(\tilde{p}, \tilde{q})$ and $\tilde{q}$ -calculus have been used extensively. The paper written by Ayman-Mursaleen and Serra-Capizzano [30] connects $\tilde{q}$ - calculus with approximation theory and convergence methods, providing a broader context for $\tilde{q}$ -difference and $(\tilde{p}, \tilde{q})$ - difference operators. Ayman-Mursaleen et al. [31] studied about the shape preserving approximation properties to family of α - Bernstein shifted knot operators. This article demonstrates the utility of parameter-dependent difference operators in approximation theory, relevant to the $(\tilde{p}, \tilde{q})$ - framework.

The $(\tilde{p}, \tilde{q})$ -analog $[m]_{\tilde{p}, \tilde{q}}$ of m for $0 < \tilde{q} < \tilde{p} \leq 1$ can be determined as

$$[m]_{\tilde{p}, \tilde{q}} = \begin{cases} \frac{(\tilde{p})^m - (\tilde{q})^m}{\tilde{p} - \tilde{q}}, & m = 1, 2, 3, \dots \\ 0, & m = 0. \end{cases}$$

One might observe that $[m]_{\tilde{p}, \tilde{q}} = [m]_{\tilde{q}}$ whenever $\tilde{p} = 1$ and $[m]_{\tilde{q}} = m$ whenever $\tilde{q} \rightarrow 1^-$. The $(\tilde{p}, \tilde{q})$ -analog $\binom{m}{k}_{\tilde{p}, \tilde{q}}$ of binomial coefficient $\binom{m}{k}$ can be determined as

$$\binom{m}{k}_{\tilde{p}, \tilde{q}} = \begin{cases} \frac{[m]_{\tilde{p}, \tilde{q}}!}{[m-k]_{\tilde{p}, \tilde{q}}! [k]_{\tilde{p}, \tilde{q}}!}, & m \geq k \\ 0, & k > m. \end{cases}$$

Where, the $(\tilde{p}, \tilde{q})$ -analog of the factorial, i.e. $(\tilde{p}, \tilde{q})$ -factorial, is defined as

$$[m]_{\tilde{p}, \tilde{q}}! = \begin{cases} \prod_{k=1}^m [k]_{\tilde{p}, \tilde{q}}, & m = 1, 2, 3, \dots \\ 1, & m = 0. \end{cases}$$

For an in-depth study of quantum calculus, we refer to the book [24] and the articles [13, 40, 41, 45]. The $(\tilde{p}, \tilde{q})$ -analog of the second-order difference operator is defined as

$$(\Delta_{\tilde{p}, \tilde{q}}^2 u)_k = \tilde{p}u_k - (\tilde{p} + \tilde{q})u_{k-1} + \tilde{q}u_{k-2}, \quad \text{for all } k \in \mathbb{N}$$

and any term with negative indices are zero. The matrix representation of this operator is given by

$$\Delta_{\tilde{p}, \tilde{q}}^2 = \begin{bmatrix} \tilde{p} & 0 & 0 & 0 & \dots \\ -(\tilde{p} + \tilde{q}) & \tilde{p} & 0 & 0 & \dots \\ \tilde{q} & -(\tilde{p} + \tilde{q}) & \tilde{p} & 0 & \dots \\ 0 & \tilde{q} & -(\tilde{p} + \tilde{q}) & \tilde{p} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We note that the $(\tilde{p}, \tilde{q})$ - difference operator reduces to the $\tilde{q}$ - difference operator when $\tilde{p} = 1$, and the additional parameter $\tilde{p}$ introduces genuinely new behavior in the spectrum and fine spectrum of the operator. This two-parameter freedom allows finer control over the spectral geometry than is possible in the one-parameter setting, which is the main novelty of the present work.

The n_{th} order $(\tilde{p}, \tilde{q})$ - difference operator for any function $f(x)$ is defined as

$$\Delta_{\tilde{p}, \tilde{q}, h}^n f(x) = \left[\prod_{j=0}^{n-1} (\tilde{p}^j E_h - \tilde{q}^j) \right] f(x),$$

where E is the shift operator which was introduced by Bustoz and Gordillo [12].

2. Definitions and Preliminaries

Let U and V be two Banach spaces and $M : U \rightarrow V$ a linear operator. The operator M is said to be bounded if for all $u \in U$ there is a real number r such that $\|Mu\| \leq r\|u\|$. Again, the collections, $\mathcal{R}(M) = \{v \in V : v = Mu, u \in U\}$ and $\mathcal{B}(U, V) = \{M : U \rightarrow V : M \text{ is continuous and linear}\}$, are termed as the range of the operator M and the set of all bounded linear operators from U to V respectively.

The operator $M^* : V^* \rightarrow U^*$ is called as the adjoint operator of M , where U^* and V^* are the dual spaces of U and V and the operator is defined as $(M^*g)(u) = g(Mu)$ for all $g \in V^*$ and $u \in U$.

Again consider $M : \mathcal{D}(M) \rightarrow U$ where $\mathcal{D}(M)$ denotes the domain of the operator M , i.e. the subset of U on which M is defined. From M we can construct a new operator, $M_\xi = M - \xi I$, where $\xi \in \mathbb{C}$ and I is the identity operator. If M_ξ has an inverse, then the inverse will be $M_\xi^{-1} = (M - \xi I)^{-1}$.

Definition 2.1. A regular value $\xi \in \mathbb{C}$ of M is such that M_ξ^{-1} exists, bounded and defined on a set which is dense in U . The collection of such ξ is called the resolvent set and is denoted by $\rho(M, U)$.

Definition 2.2. The spectrum of M denoted $\sigma(M)$ is the collection of all $\xi \in \mathbb{C}$ which belong to complement of the resolvent set, i.e. $\sigma(M) = \mathbb{C} - \rho(M)$.

Further, $\sigma(M)$ is classified into three disjoint subsets, namely, the point spectrum $\sigma_p(M)$, the continuous spectrum $\sigma_c(M)$, and the residual spectrum $\sigma_r(M)$ which are defined as follows:

- $\sigma_p(M) = \{\xi \in \mathbb{C} : M_\xi^{-1} \text{ does not exist}\},$
- $\sigma_c(M) = \{\xi \in \mathbb{C} : M_\xi^{-1} \text{ exist, unbounded and defined on a set that is dense in } U\},$
- $\sigma_r(M) = \{\xi \in \mathbb{C} : M_\xi^{-1} \text{ exist, may or may not be bounded and is not dense in } U\}.$

There are more subdivision of the spectrum of a bounded operator such as approximate point spectrum $\sigma_{ap}(M)$, defect spectrum $\sigma_\delta(M)$ and compression spectrum $\sigma_{co}(M)$, which are defined as follows:

- $\sigma_{ap}(M) = \{\xi \in \mathbb{C} : (M - \xi I) \text{ is not bounded below}\},$
- $\sigma_\delta(M) = \{\xi \in \mathbb{C} : (M - \xi I) \text{ is not surjective}\},$
- $\sigma_{co}(M) = \{\xi \in \mathbb{C} : \overline{R(M - \xi I)} \neq U\}.$

To know more about the fundamental Theorems on functional analysis, sequence space, summability theory and spectrum of linear operators we refer to the books [9, 32].

3. Goldberg's Classification of Spectrum

Based on the nature of the set $\mathcal{R}(M_\xi)$ and the operator M_ξ^{-1} , Goldberg [22] gave a detailed classification of the spectrum of an operator which is discussed.

For the set $\mathcal{R}(M_\xi)$ and the operator M_ξ^{-1} there are various possibilities:

P : $\mathcal{R}(M_\xi) = U$	1 : M_ξ^{-1} exist and is continuous.
Q : $\overline{\mathcal{R}(M_\xi)} = U$, but $\mathcal{R}(M_\xi) \neq U$	2 : M_ξ^{-1} exist but is not continuous.
R : $\overline{\mathcal{R}(M_\xi)} \neq U$	3 : M_ξ^{-1} doesn't exist.

By combining these possibilities, we get nine various states and they can be identified as $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2$ and R_3 . If $M_\xi \in P_1$ or $M_\xi \in Q_1$, then $\xi \in \rho(M, U)$. If $M_\xi \in R_2$ then M_ξ^{-1} exist and unbounded and $\overline{R(M_\xi)} \neq U$ and we get $\xi \in R_2\sigma(M, U)$. These classifications can be summarized in the following Table-[1].

	1	2	3
P	$\xi \in \rho(M, U)$	—	$\xi \in \sigma_p(M, U)$ $\xi \in \sigma_{ap}(M, U)$
Q	$\xi \in \rho(M, U)$	$\xi \in \sigma_c(M, U)$ $\xi \in \sigma_\delta(M, U)$ $\xi \in \sigma_{ap}(M, U)$	$\xi \in \sigma_p(M, U)$ $\xi \in \sigma_\delta(M, U)$ $\xi \in \sigma_{ap}(M, U)$
R	$\xi \in \sigma_r(M, U)$ $\xi \in \sigma_\delta(M, U)$ $\xi \in \sigma_{co}(M, U)$	$\xi \in \sigma_r(M, U)$ $\xi \in \sigma_\delta(M, U)$ $\xi \in \sigma_{co}(M, U)$ $\xi \in \sigma_{ap}(M, U)$	$\xi \in \sigma_p(M, U)$ $\xi \in \sigma_\delta(M, U)$ $\xi \in \sigma_{co}(M, U)$ $\xi \in \sigma_{ap}(M, U)$

Table 1: Goldberg's classification of spectrum of linear operator

We use this classification for the proof of the Theorems 4.11, 4.12 and 4.13.

Proposition 3.1. ([8], page 28) *Spectral and sub-spectral relationships of an operator M and its adjoint operator M^* are provided below.*

- (a) $\sigma(M^*, V^*) = \sigma(M, V)$,
- (b) $\sigma_{ap}(M^*, V^*) = \sigma_\delta(M, V)$,
- (c) $\sigma_\delta(M^*, V^*) = \sigma_{ap}(M, V)$,
- (d) $\sigma_p(M^*, V^*) = \sigma_{co}(M, V)$,
- (e) $\sigma(M, V) = \sigma_{ap}(M, V) \cup \sigma_p(M^*, V^*) = \sigma_p(M, V) \cup \sigma_{ap}(M^*, V^*)$.

Lemma 3.2. ([22], p. 60) *The adjoint operator M^* of M is onto if and only if M has a bounded inverse.*

Lemma 3.3. ([22], p. 59) *The bounded linear operator $M : U \longrightarrow V$ has dense range if and only if M^* is one to one.*

Before going to the main results we state a remark, “if z is a complex number then $\sqrt{z}$ means the square root of z with non-negative real part. If $Re\sqrt{z} = 0$, then $\sqrt{z}$ means the square root of z with $Im(z) \geq 0$ ”.

4. Spectrum of $\Delta_{\tilde{p}, \tilde{q}}^2(\tilde{p}, -(\tilde{p} + \tilde{q}), \tilde{q})$ on ℓ_p ($1 < p < \infty$)

Theorem 4.1. $\Delta_{\tilde{p}, \tilde{q}}^2 \in B(\ell_p)$ with $(\tilde{p}^p + (\tilde{p} + \tilde{q})^p + \tilde{q}^p)^{1/p} \leq \|\Delta_{\tilde{p}, \tilde{q}}^2\|_{\ell_p} \leq 2(\tilde{p} + \tilde{q})$ for $0 < \tilde{q} < \tilde{p} \leq 1$.

Proof. The linearity of $\Delta_{\tilde{p}, \tilde{q}}^2$ is straightforward to prove, so it is omitted. Now, consider $e^{(0)} = (1, 0, 0, \dots)$ in ℓ_p . Then $(\Delta_{\tilde{p}, \tilde{q}}^2)e^{(0)} = (\tilde{p}, -(\tilde{p} + \tilde{q}), \tilde{q}, 0, 0, \dots)$ and it

is obtained that

$$\frac{\|(\Delta_{\tilde{p}, \tilde{q}}^2)e^{(0)}\|_{\ell_p}}{\|e^{(0)}\|_{\ell_p}} = (\tilde{p}^p + (\tilde{p} + \tilde{q})^p + \tilde{q}^p)^{1/p}.$$

From this, we get, $(\tilde{p}^p + (\tilde{p} + \tilde{q})^p + \tilde{q}^p)^{1/p} \leq \|(\Delta_{\tilde{p}, \tilde{q}}^2)e^{(0)}\|$. Again, for any $u = (u_k) \in \ell_p$ and using the Minkowaski inequality we get

$$\begin{aligned} \|\Delta_{\tilde{p}, \tilde{q}}^2 u\|_{\ell_p} &= \left(\sum_k \left| \tilde{q}u_{k-1} + (-(\tilde{p} + \tilde{q}))u_k + \tilde{p}u_{k+1} \right|^p \right)^{1/p} \\ &\leq \left(\sum_k \left| \tilde{q}u_{k-1} \right|^p \right)^{1/p} + \left(\sum_k \left| (\tilde{p} + \tilde{q})u_k \right|^p \right)^{1/p} + \left(\sum_k \left| \tilde{p}u_{k+1} \right|^p \right)^{1/p} \\ &= (\tilde{q} + (\tilde{p} + \tilde{q}) + \tilde{p}) \|u\|_{\ell_p} = 2(\tilde{p} + \tilde{q}) \|u\|_{\ell_p}. \end{aligned}$$

As a result we get, $(\tilde{p}^p + (\tilde{p} + \tilde{q})^p + \tilde{q}^p)^{1/p} \leq \|\Delta_{\tilde{p}, \tilde{q}}^2\|_{\ell_p} \leq 2(\tilde{p} + \tilde{q})$.

Theorem 4.2. $\sigma_p(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \emptyset$.

Proof. We prove this theorem by the method of contradiction. Consider, $\sigma_p(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) \neq \emptyset$. Then for any $\theta \neq u \in \ell_p$ with $(\Delta_{\tilde{p}, \tilde{q}}^2)u = \xi u$, we get the following equalities;

$$\begin{aligned} \tilde{p}u_0 &= \xi u_0 \\ -(\tilde{p} + \tilde{q})u_0 + \tilde{p}u_1 &= \xi u_1 \\ \tilde{q}u_0 - (\tilde{p} + \tilde{q})u_1 + \tilde{p}u_2 &= \xi u_2 \\ &\vdots \\ \tilde{q}u_{m-2} - (\tilde{p} + \tilde{q})u_{m-1} + \tilde{p}u_m &= \xi u_m \\ &\vdots \end{aligned}$$

Since u is assumed nonzero, some u_m is the first non zero entry; substituting into the recurrence forces $\xi = \tilde{p}$. Putting the value of $\xi = \tilde{p}$ in the proceeding equation for u_m forces $u_m = 0$, contradicting the choice of u_m as nonzero entry. The contradiction establishes that no such nonzero u exist. Thus $\sigma_p(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \emptyset$.

Lemma 4.3. ([42], p.126) *The matrix $A = (a_{nk})$ defines a bounded linear operator $T \in B(\ell_1)$, mapping ℓ_1 to itself, if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 4.4. ([42], p.126) *The matrix $A = (a_{nk})$ defines a bounded linear operator*

$T \in B(\ell_\infty)$, mapping ℓ_∞ to itself, if and only if the supremum of ℓ_1 norms of the rows of A is bounded.

Lemma 4.5. ([28], p.174, Theorem 9) *If $1 < p < \infty$ and $A \in (\ell_1, \ell_1) \cap (\ell_\infty, \ell_\infty)$. Then $A \in (\ell_p, \ell_p)$.*

Theorem 4.6.

$$\sigma(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| \leq \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

Proof. Let us consider the set

$$S = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| \leq \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

At the first part of this proof we will show that $(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1} \in (\ell_p, \ell_p)$ for $\xi \in \mathbb{C} \setminus S$ and at the second part we will establish that $(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1}$ fails to be bounded on ℓ_p for $\xi \in S$.

Now, we proceed to first part. let $\xi \notin S$. So, we get $\xi \neq \tilde{p}$ and this implies $(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)$ has an inverse and it is a lower triangular matrix, its inverse can be obtained easily which is as follows

$$(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \dots \\ m_2 & m_1 & 0 & 0 & \dots \\ m_3 & m_2 & m_1 & 0 & \dots \\ m_4 & m_3 & m_2 & m_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Where

$$m_1 = \frac{1}{\tilde{p} - \xi}, \quad m_2 = \frac{\tilde{p} + \tilde{q}}{(\tilde{p} - \xi)^2}, \quad m_3 = \frac{(\tilde{p} + \tilde{q})^2 - \tilde{q}(\tilde{p} - \xi)}{(\tilde{p} - \xi)^3}, \dots$$

Here the sequence (m_k) satisfies the following recurrence relation

$$m_k = \frac{(\tilde{p} + \tilde{q})m_{k-1} - \tilde{q}m_{k-2}}{\tilde{p} - \xi} \quad \text{for } k \geq 3.$$

From this recurrence relation, we get the characteristic equation as

$$(\tilde{p} - \xi)w^2 - (\tilde{p} + \tilde{q})w + \tilde{q} = 0,$$

whose solutions are

$$w_1 = \frac{(\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}}{2(\tilde{p} - \xi)} \quad \text{and} \quad w_2 = \frac{(\tilde{p} + \tilde{q}) - \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}}{2(\tilde{p} - \xi)}.$$

From elementary calculations on recurrence sequence, we get

$$m_k = \frac{w_1^k - w_2^k}{\sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}}. \quad (1)$$

Now, we can proceed in two cases.

Case 1: If $(\tilde{p} + \tilde{q})^2 = 4\tilde{q}(\tilde{p} - \xi)$, then we get

$$m_k = \left(\frac{2k}{\tilde{p} + \tilde{q}} \right) \left[\frac{(\tilde{p} + \tilde{q})}{2(\tilde{p} - \xi)} \right]^k.$$

It can be easily proved that $(m_k) \in \ell_p$ iff $\left| \frac{(\tilde{p} + \tilde{q})}{2(\tilde{p} - \xi)} \right| < 1$. So, $\xi \notin S$ implies that $(m_k) \in \ell_p$.

Case 2: If $(\tilde{p} + \tilde{q})^2 \neq 4\tilde{q}(\tilde{p} - \xi)$, here we analyze two sub-cases.

Sub-case 1: If $(\tilde{p} + \tilde{q})^2 < 4\tilde{q}(\tilde{p} - \xi)$ i.e. the discriminant is negative.

We now note that w_1 and w_2 are complex conjugates, so $|w_1| = |w_2|$. Since $\xi \notin S$, we have $|w_1| < 1$ and it implies $|w_2| < 1$.

Sub-case 2: If $(\tilde{p} + \tilde{q})^2 > 4\tilde{q}(\tilde{p} - \xi)$ i.e. the discriminant is positive.

Since $\xi \notin S$, we have $|w_1| < 1$. Again using the inequality $|1 - \sqrt{z}| \leq |1 + \sqrt{z}|$ for any $z \in \mathbb{C}$ we get,

$$\left| \frac{(\tilde{p} + \tilde{q})}{2(\tilde{p} - \xi)} - \frac{\sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}}{2(\tilde{p} - \xi)} \right| \leq \left| \frac{(\tilde{p} + \tilde{q})}{2(\tilde{p} - \xi)} + \frac{\sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}}{2(\tilde{p} - \xi)} \right|$$

$$\implies |w_2| \leq |w_1| < 1.$$

So, for both the sub-cases we have $|w_1| < 1$, $|w_2| < 1$ and using this in equality (1), it is obtained that $\lim_{k \rightarrow \infty} m_k = 0$. Now, we determine the ℓ_1 norms of the inverse matrix :

$$\begin{aligned} \|(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1}\|_{(\ell_1; \ell_1)} &= \sup_{k \in \mathbb{N}} \sum_{i=k}^{\infty} |m_i| = \sum_{i=1}^{\infty} |m_i| \\ &\leq \frac{1}{|\sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}}|} \left(\sum_{i=1}^{\infty} |w_1|^i + \sum_{i=1}^{\infty} |w_2|^i \right) < \infty, \end{aligned}$$

since $|w_1| < 1$ and $|w_2| < 1$. It results in $(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1} \in (\ell_1, \ell_1)$ ([42], p.126). Again, since $(m_k) \in \ell_1$, the supremum of the ℓ_1 norms of the rows of $(\Delta_{\tilde{p}, \tilde{q}}^2 - \xi I)^{-1}$

is finite. This lead us to the consequence $(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \in (\ell_\infty, \ell_\infty)$ ([42], p.126). Now, using Lemma 4.5 we get

$$(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \in (\ell_1, \ell_1) \cap (\ell_\infty, \ell_\infty) \implies (\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \in (\ell_p, \ell_p).$$

This proves that

$$\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \subseteq S. \quad (2)$$

Now, we proceed to the second part. Let $\xi \in S$. If $\xi = \tilde{p}$ then, we get

$$(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I) = (\Delta_{\tilde{p},\tilde{q}}^2 - I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ -(\tilde{p} + \tilde{q}) & 0 & 0 & 0 & 0 & \dots \\ \tilde{q} & -(\tilde{p} + \tilde{q}) & 0 & 0 & 0 & \dots \\ 0 & \tilde{q} & -(\tilde{p} + \tilde{q}) & 0 & 0 & \dots \\ 0 & 0 & \tilde{q} & -(\tilde{p} + \tilde{q}) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now $(\Delta_{\tilde{p},\tilde{q}}^2 - I)u = \theta \implies u = \theta$. So, we get $(\Delta_{\tilde{p},\tilde{q}}^2 - I) : \ell_p \longrightarrow \ell_p$ is one-one but not onto. This implies $(\Delta_{\tilde{p},\tilde{q}}^2 - I)$ is not invertible.

If we take ξ from S other $\tilde{p}$ then from **Case 1**, it is obtained that $\left| \frac{(\tilde{p} + \tilde{q})}{2(\tilde{p} - \xi)} \right| \geq 1$.

It implies that $(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \notin B(\ell_p)$. Again, from **Case 2**, it is obtained that $|w_1| \geq 1$ and the inequality $|w_1| > |w_2|$, which directly implies that $(w_k) \not\rightarrow 0$ and so $\sum_{i=1}^{\infty} |w_i|^p$ diverges. Consider $v = (1, 0, 0, \dots) \in \ell_p$. Then $(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1}v = (m_1, m_2, m_3, \dots)$, which doesn't belong to ℓ_p . Therefore $(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \notin B(\ell_p)$, as a result we get

$$S \subseteq \sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p). \quad (3)$$

From (2) and (3), it is obtained that $\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = S$.

Lemma 4.7. ([38], p. 215) *For any $A \in B(\ell_p)$ ($1 < p < \infty$), the adjoint operator $A^* \in B(\ell_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q < \infty$ and can be represented by the transpose of the matrix A .*

Theorem 4.8.

$$\sigma_p((\Delta_{\tilde{p},\tilde{q}}^2)^*, \ell_p^* \cong \ell_q) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| < \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

Proof. Let us consider the set

$$S_1 = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| < \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

We first show that

$$S_1 \subseteq \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_p^* \cong \ell_q).$$

Let $\xi \in S_1$. Now solving $(\Delta_{\tilde{p}, \tilde{q}}^2)^* u = \xi u$, for $\theta \neq u \in \ell_p^* \cong \ell_q$, we get the following linear equations

$$\begin{aligned} \tilde{p}u_0 - (\tilde{p} + \tilde{q})u_1 + \tilde{q}u_2 &= \xi u_0 \\ \tilde{p}u_1 - (\tilde{p} + \tilde{q})u_2 + \tilde{q}u_3 &= \xi u_1 \\ &\vdots \\ \tilde{p}u_k - (\tilde{p} + \tilde{q})u_{k+1} + \tilde{q}u_{k+2} &= \xi u_k \\ &\vdots \end{aligned}$$

Now, if we take $\xi = \tilde{p}$ then we get an eigenvector $(1, 0, 0, \dots)$ corresponding to $\xi = \tilde{p}$. We consider $\xi \in S_1$ other than $\tilde{p}$. The above linear equations can also be expressed in terms of u_1 and u_0 as

$$\begin{aligned} u_2 &= \frac{(\tilde{p} + \tilde{q})}{\tilde{q}} u_1 - \frac{\tilde{p} - \xi}{\tilde{q}} u_0 \\ u_3 &= \frac{(\tilde{p} + \tilde{q})^2 - \tilde{q}(\tilde{p} - \xi)}{\tilde{q}^2} u_1 - \frac{(\tilde{p} - \xi)(\tilde{p} + \tilde{q})}{\tilde{q}^2} u_0 \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ u_k &= \frac{m_k(\tilde{p} - \xi)^k}{\tilde{p}^{k-1}} u_1 - \frac{m_{k-1}(\tilde{p} - \xi)^k}{\tilde{q}^{k-1}} u_0; \quad k \geq 2. \end{aligned} \tag{4}$$

Here (m_k) comes from equality (1). Now, we can find the eigenvector (u_k) for $\xi \neq \tilde{p}$. Here we can make a choice for u_0 and u_1 . Let, $u_0 = 1$ and $u_1 = \frac{2(\tilde{p} - \xi)}{(\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4\tilde{q}(\tilde{p} - \xi)}}$. Already, we have obtained that w_1 and w_2 are

roots of the characteristic equation $(\tilde{p} - \xi)w^2 - (\tilde{p} + \tilde{q})w + \tilde{q} = 0$. So, we get $w_1.w_2 = \frac{\tilde{q}}{\tilde{p} - \xi}$ and $w_1 - w_2 = \frac{\sqrt{(\tilde{p} + \tilde{q})^2 - 4\tilde{q}(\tilde{p} - \xi)}}{\tilde{p} - \xi}$.

It can also be seen that $u_1 = \frac{1}{w_1}$. Using, these facts in the relation of the sequence (u_k) , we get

$$\begin{aligned} u_k &= \frac{m_k(\tilde{p} - \xi)^k}{\tilde{q}^{k-1}} u_1 - \frac{m_{k-1}(\tilde{p} - \xi)^k}{\tilde{q}^{k-1}} u_0 = \left(\frac{\tilde{p} - \xi}{\tilde{q}} \right)^{k-1} (\tilde{p} - \xi)(-m_{k-1}u_0 + m_k u_1) \\ &= \frac{1}{(w_1 w_2)^{k-1}} (\tilde{p} - \xi) \left(\frac{-w_1^{k-1} + w_2^{k-1}}{\sqrt{(\tilde{p} + \tilde{q})^2 - 4\tilde{q}(\tilde{p} - \xi)}} + \frac{w_1^k - w_2^k}{\sqrt{(\tilde{p} + \tilde{q})^2 - 4\tilde{q}(\tilde{p} - \xi)}} w_1^{-1} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{w_1^{k-1}w_2^{k-1}} \frac{\tilde{p} - \xi}{\sqrt{(\tilde{p} + \tilde{q})^2 - 4\tilde{q}(\tilde{p} - \xi)}} (-w_1^{k-1} + w_2^{k-1} + w_1^{k-1} - w_2^k w_1^{-1}) \\
&= \frac{1}{w_1^{k-1}w_2^{k-1}} \left(\frac{1}{w_1 - w_2} \right) w_2^{k-1} \left(1 - \frac{w_2}{w_1} \right) = \frac{1}{w_1^{k-1}} \frac{1}{w_1} = \frac{1}{w_1^k} = u_1^k.
\end{aligned}$$

Finally, we obtained that the sequence $u_k = u_1^k$ for all $k \geq 2$. If we consider $w_1 = w_2$ i.e. for the case $(\tilde{p} + \tilde{q})^2 = 4\tilde{q}(\tilde{p} - \xi)$, we get the same result. Here, it is clearly seen that the sequence $u = (u_k) \in \ell_p^*$ since $|u_1| < 1$. As a result, we get

$$S_1 \subseteq \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q). \quad (5)$$

Now we consider $\xi \notin S_1$ i.e. $|2(\tilde{p} - \xi)| \geq \left| (1 + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right|$. It implies $|1/w_1| \geq 1$, or $|w_1| \leq 1$. Here, we have to show that

$$\xi \notin \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q).$$

From equality (4) it is obtained that

$$\frac{u_{k+1}}{u_k} = \left(\frac{\tilde{p} - \xi}{\tilde{q}} \right) \left(\frac{m_k}{m_{k-1}} \right) \left(\frac{-u_0 + \frac{m_{k+1}}{m_k} u_1}{-u_0 + \frac{m_k}{m_{k-1}} u_1} \right).$$

Based on the roots w_1 and w_2 , we will consider three cases.

Case 1: $|w_2| < |w_1| \leq 1$. For this case, we get $(\tilde{p} + \tilde{q})^2 \neq 4\tilde{q}(\tilde{p} - \xi)$ and

$$\lim_{k \rightarrow \infty} \frac{m_k}{m_{k-1}} = \lim_{k \rightarrow \infty} \frac{m_{k+1}}{m_k} = \lim_{k \rightarrow \infty} \frac{w_1^{k+1} - w_2^{k+1}}{w_1^k - w_2^k} = w_1.$$

Again, from equality (4) we get

$$u_k = \left(\frac{\tilde{p} - \xi}{\tilde{q}} \right)^{k-1} (\tilde{p} - \xi)(-m_{k-1}u_0 + m_k u_1). \quad (6)$$

If $-u_0 + w_1 u_1 = 0$, then from equality (6) we get, $(u_k) = \left(\frac{u_0}{w_1^k} \right)$ which doesn't belong to ℓ_q since $|w_1| \leq 1$. Otherwise,

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right|^q = \frac{1}{|w_1|^q |w_2|^q} |w_1|^q = \frac{1}{|w_2|^q} > 1.$$

Case 2: $|w_2| = |w_1| < 1$. For this case, we get $(\tilde{p} + \tilde{q})^2 = 4\tilde{q}(\tilde{p} - \xi)$ and using the formula

$$m_k = \left(\frac{2k}{\tilde{p} + \tilde{q}} \right) \left[\frac{\tilde{p} + \tilde{q}}{2(\tilde{p} - \xi)} \right]^k, \text{ for all } k \geq 1.$$

we get that

$$\lim_{k \rightarrow \infty} \left| \frac{m_k}{m_{k-1}} \right|^q = \left| \frac{\tilde{p} + \tilde{q}}{2(\tilde{p} - \xi)} \right|^q = |w_1|^q.$$

which leads to

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right|^q = \frac{1}{|w_1|^q |w_2|^q} |w_1|^q = \frac{1}{|w_2|^q} > 1.$$

This implies (u_k) doesn't belongs to ℓ_q .

Case 3: $|w_1| = |w_2| = 1$. For this case, we get $(\tilde{p} + \tilde{q})^2 = 4\tilde{q}(\tilde{p} - \xi)$ and $\left| \frac{\tilde{p} + \tilde{q}}{2\tilde{q}} \right| = 1$.

Assume that $\xi \in \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q)$. This implies there exist $\theta \neq u \in \ell_q$. Now, rewriting equality (4) we get

$$u_k = \left(\frac{\tilde{p} + \tilde{q}}{2\tilde{q}} \right)^{k-1} \left[k u_1 - (k-1) \frac{\tilde{p} + \tilde{q}}{2\tilde{q}} u_0 \right].$$

Since $\lim_{k \rightarrow \infty} u_k = 0 \implies \lim_{k \rightarrow \infty} \left[k u_1 - (k-1) \frac{\tilde{p} + \tilde{q}}{2\tilde{q}} u_0 \right] = 0$ and we must have $u_0 = u_1 = 0$. Consequently, it implies $u = \theta$, a contradiction. So, we get $\xi \notin \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q)$ and it implies

$$\sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q) \subseteq S_1. \quad (7)$$

From (5) and (7) it is obtained that $\sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q) = S_1$.

Theorem 4.9.

$$\sigma_r(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| < \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

Proof. From Lemma (3.3), we get $\sigma_r(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \sigma_p((\Delta_{\tilde{p}, \tilde{q}}^2)^*, \ell_q) \setminus \sigma_p(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p)$. Now applying the Theorems 4.2 and 4.8 we get the required result.

Theorem 4.10.

$$\sigma_c(\Delta_{\tilde{p}, \tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| = \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$$

Proof. We have $\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \sigma_p(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \cup \sigma_r(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \cup \sigma_c(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p)$ and the corresponding sets are pair-wise disjoint. Now applying Theorems 4.6, 4.8 and 4.9, we get the required result.

Theorem 4.11. $P_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = Q_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = R_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \emptyset$.

Proof. From Table-[1], $\sigma_p(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = P_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \cup Q_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \cup R_3\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p)$. Again, from Theorem 4.2, we get $\sigma_p(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \emptyset$ and consequently obtain the required result.

Theorem 4.12. *The operator $\Delta_{\tilde{p},\tilde{q}}^2$ satisfies the following relations*

- (a) $Q_2\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| = \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\},$
- (b) $R_2\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \supseteq \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| < \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\} \setminus \{1\},$
- (c) $R_1\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \subseteq \{1\}.$

Proof. We have from Table-[1] that $\sigma_c(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = Q_2\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p)$. Now using Theorem 4.10, we get the result (a).

Again, from Theorem 4.6, if for any $\xi \in \sigma_r(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \setminus \{1\}$ then the operator $(\Delta_{\tilde{p},\tilde{q}}^2 - \xi I)^{-1} \notin \Delta_{\tilde{p},\tilde{q}}^2(\ell_p)$. From this, we get $\sigma_r(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \setminus \{1\} \subseteq R_2\sigma(\Delta_{\tilde{p},\tilde{q}}^2, c_o)$ and $R_1\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \subseteq \{1\}.$

Theorem 4.13. *For the operator $\Delta_{\tilde{p},\tilde{q}}^2$ the following results hold.*

- (a) $\sigma_{ap}(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \supseteq \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \mu)| \leq \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\} \setminus \{1\},$
- (b) $\sigma_{ap}((\Delta_{\tilde{p},\tilde{q}}^2)^*, \ell_p^*) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| \leq \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\},$
- (c) $\sigma_\delta(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| \leq \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\},$
- (d) $\sigma_{co}(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \left\{ \xi \in \mathbb{C} : |2(\tilde{p} - \xi)| < \left| (\tilde{p} + \tilde{q}) + \sqrt{(\tilde{p} + \tilde{q})^2 - 4(\tilde{p} - \xi)\tilde{q}} \right| \right\}.$

Proof. From Table-[1], we get $\sigma_{ap}(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) = \sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p) \setminus R_1\sigma(\Delta_{\tilde{p},\tilde{q}}^2, \ell_p)$. Now applying the Theorems 4.6 and 4.12 we get the result (a). The result (b) is obtained from the relation (e) in Proposition 3.1. The result (c) is obtained from the relation (b) in Proposition 3.1. The result (d) is obtained from the relation (d) in Proposition 3.1.

5. Conclusion

In this article, we have carried out a detailed spectral and fine spectral analysis of the second order $(\tilde{p}, \tilde{q})$ - difference operator $\Delta_{\tilde{p},\tilde{q}}^2$ over the sequence ℓ_p ($1 < p <$

∞). We showed that $\Delta_{\tilde{p}, \tilde{q}}^2$ is bounded on ℓ_p with explicit norm bounds. Applying Goldberg's classification, we obtained the precise sub-classification of the spectrum of the operator $\Delta_{\tilde{p}, \tilde{q}}^2$. Since $\Delta_{\tilde{p}, \tilde{q}}^2$ reduces to the well-studied $\tilde{q}$ - difference operator when $\tilde{p} = 1$, our results recover and generalize the existing one-parameter theory to a genuinely two-parameter setting.

Future scope: The spectral and fine-spectral behavior of $\Delta_{\tilde{p}, \tilde{q}}^2$ could be studied over bv_p , c_0 , c and other sequence spaces. The analysis could be extended to the n^{th} order $(\tilde{p}, \tilde{q})$ - difference operator $\Delta_{\tilde{p}, \tilde{q}, h}^n$ defined in the introduction.

References

- [1] Akhmedov, A. M. & Başar, F., On the fine spectra of the difference operator Δ over the sequence space ℓ_p ($1 \leq p < \infty$), Demonstr. Math., 39(3) (2006), 585-596.
- [2] Akhmedov, A.M. & Başar, F., The fine spectra of the difference operator Δ over the sequence space bv_p ($1 \leq p < \infty$), Acta Math. Sin. Engl. Ser., 23(10) (2007), 1757-1768.
- [3] Akhmedov, A.M. & El-Shabrawy, S. R., On the fine spectrum of the operator Δ_v over the sequence spaces c and ℓ_p ($1 < p < \infty$), Appl. Math. Inf. Sci., 5(3) (2011), 635-654.
- [4] Akhmedov, A. M. & El-Shabrawy, S. R., On the fine spectrum of the operator Δ_{ab} over the sequence space c , Comput. Math. Appl., 61(10) (2011), 2994-3002.
- [5] Alotaibi, A., Yaying, T. and Mohiuddine, S. A., Sequence space and spectrum of q -difference operator of second order, Symmetry, 14(6) (2022) 1155.
- [6] Altay, B. & Başar, F., On the fine spectrum of the difference operator Δ on c_0 and c , Inf. Sci., 168(1-4) (2004), 217-224.
- [7] Altay, B. & Başar, F., On the fine spectrum of the generalised difference operator $B(r, s)$ over the sequence spaces c_0 and c , Int. J. Math. Math. Sci., 2005(18) (2005), 3005-3013.
- [8] Appell, J., Pascale, E. & Vignoli, A., Nonlinear Spectral Theory, Walter de Gruyter, Berlin 2004.
- [9] Başar, F., Summability Theory and its Applications, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton · London · New York, 2022.

- [10] Bilgiç, H. & Furkan, H., On the fine spectrum of the generalised difference operator $B(r, s)$ over the sequence spaces ℓ_p and bv_p ($1 < p < \infty$), *Nonlinear Anal.*, 68(3) (2008), 499-506.
- [11] Bilgiç, H. & Furkan, H., On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces ℓ_1 and bv , *Math. Comput. Model.*, 45(7-8) (2007), 883-891.
- [12] Bustoz, J. & Gordillo, L. F., q -Hausdorff summability, *J. Comput. Anal. Appl.*, 7(1) (2005), 35-48.
- [13] Corcino, R. B. & Montero, C. B., On (p, q) -difference operator, *J. Korean Math. Soc.*, 49(3) (2012), 537-54.
- [14] Dündar, E. & Başar, F., On the fine spectrum of the upper triangle double band matrix Δ^+ on the sequence space c_0 , *Math. Commun.*, 18 (2013), 337-348.
- [15] Dutta, S. & Baliarsingh, P., On the spectrum of 2nd order generalized difference operator Δ^2 over the sequence space c_0 , *Bol. Soc. Parana. Mat.*, 31(2) (2013), 235-244.
- [16] Dutta, A. J. & Tripathy, B. C., Fine spectrum of the generalised difference operator the class of convergent series, *Int. J. Anal.*, 2013(1-4) (2013).
- [17] Dutta, S. & Baliarsingh P., On the spectrum of difference operator Δ_{ab} over the sequence space ℓ_p and bv_p ($1 < p < \infty$), *Math. Sci. Lett.*, 3(2) (2014), 115-120.
- [18] El-Shabrawy, S. R., Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0 , *J. Inequal. Appl.*, 2014(241) (2014).
- [19] Furkan, H., On the fine spectrum of the generalised difference operator $B(r, s)$ over the sequence space ℓ_1 and bv , *Hokkaido Math. J.*, 35(4) (2006), 893-904.
- [20] Furkan, H., Bilgiç, H. & Altay B., On the fine spectrum of the operator $B(r, s, t)$ over c_0 and c , *Comput. Math. Appl.*, 53(6) (2007), 989-998.
- [21] Furkan, H., Bilgiç, H. & Başar, F., On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces ℓ_p and bv_p , ($1 < p < \infty$), *Comput. Math. Appl.*, 60(7) (2010), 2141-2152.

- [22] Golberg, S., Unbounded Linear Operators: Theory & Applications, Dover Publications, New York, 2006.
- [23] Hon, S., Ayman-Mursaleen, M. & Serra-Capizzano, S., A note on the spectral distribution of symmetrized Toeplitz sequences, *Linear Algebra Appl.*, 579, (2019), 32-50.
- [24] Kac, V. G. & Cheung, P., Quantum Calculus Springer, New York, 2002.
- [25] Kalita, N. & Dutta, A. J., Spectral analysis of the quantum Cesàro operator over the sequence space ℓ_p ($1 < p < \infty$), *Bol. Soc. Parana. Mat.*, 43(3) (2025), 1-14.
- [26] Kalita, N. & Dutta, A. J., Spectral analysis of second order quantum difference operator over the sequence space ℓ_p ($1 < p < \infty$), *Bol. Kara. Univ. Math. Ser.*, 118(2) (2025), 122-136.
- [27] Kayaduman, K. & Furkan, H., The fine spectra of the difference operator Δ over the sequence space ℓ_1 and bv , *Int. Math. Forum*, 1(24) (2006), 1153-1160.
- [28] Maddox, I. J., Elements of Functional Analysis, Cambridge University Press, London, 1970.
- [29] Mursaleen, M., Quadratic function preserving wavelet type Baskakov operators for enhanced function approximation, *Comp. Appl. Math.*, 44 (8) (2025), 395.
- [30] Mursaleen, M. & Serra-Capizzano, S., Statistical Convergence via q-Calculus and a Korovkin's Type Approximation Theorem, *Axioms*, 11(2) (2022).
- [31] Mursaleen, M., Alshaban, E. & Nasiruzzaman, Md., Shape preserving approximation properties to family of α -Bernstein shifted knot operators, *J. Inequal. Appl.*, 2025 (2025), 107.
- [32] Mursaleen, M. & Başar, F., Sequence Spaces: Topics in Modern Summability Theory, CRC Press/Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.
- [33] Panigrahi, B. L. & Srivastava, P. D., Spectrum and fine spectrum of generalised second order difference operator Δ_{uv}^2 on sequence space c_0 , *Thai J. Math.*, 9(1)(2012), 57-74.

- [34] Panigrahi, B. L. & Srivastava, P. D., Spectrum and fine spectrum of generalised second order forward difference operator Δ_{uvw}^2 on sequence space ℓ_1 , *Demonstr. Math.*, 45(3) (2012), 593-609.
- [35] Patra, A. & Srivastava, P. D., Spectrum and fine spectrum of generalised lower triangular triple band matrices over the sequence space ℓ_p , *Linear Multilinear Algebra*, 68(9) (2020), 1797-1816.
- [36] Srivastava, P. D. & Kumar, S., On the fine spectrum of the generalised difference operator Δ_v over the sequence space c_0 , *Commun. Math. Anal.*, 6(1) (2009), 8-21.
- [37] Srivastava, P. D. & Kumar, S., Fine spectrum of the generalised difference operator Δ_v on the sequence space ℓ_1 , *Thai J. Math.*, 8(2) (2010), 221-223.
- [38] Taylor, R. B., *Introduction to Functional Analysis*, Wiley, New York, 1980.
- [39] Turkey, M. E. & Durna, N., The spectrum of q -Cesàro matrices on c and its various spectral decomposition for $0 < q < 1$, *Oper. and Matrices*, 15(3) (2021), 795-813.
- [40] Wanas, A. K. & Khachi, S. C., Coefficient bounds and Fekete-Szegő inequalities for new families of bi-starlike and bi-convex functions associated with The q -Bernoulli polynomial, *Appl. Math. E-Notes*, 25 (2025), 105-117.
- [41] Wanas, A. K. & Jabir, N. Y., Coefficient estimates and Fekete-Szegő inequalities for a families of bi-functions related to q -Srivastava-Attiya operator, *Asian-Eur. J. Math.*, 18(10) (2025), 1-22.
- [42] Wilansky, A., *Summability Through Functional Analysis*, North Holland, Amsterdam, 1984.
- [43] Yaying, T., Hazarika, B., Tripathy, B. C. & Mursaleen, M., The spectrum of second order quantum difference operator, *Symmetry*, 14(3) (2022), 557.
- [44] Yaying, T., Hazarika, B., Baliarsingh, P. & Mursaleen, M., Cesàro q -difference sequence spaces and spectrum of weighted q -difference operator, *Bull. Iranian Math. Soc.*, 50(2) (2024), 23.
- [45] Yaying, T., Hazarika, B. & Mursaleen, M., On some new BK-Spaces as the domain of (p, q) -Cesàro matrix and point spectrum, *Iran. J. Sci.*, 47(3) (2023), 1565-1574.

- [46] Yeşilkayagil, M. & Başar, F., On the fine spectrum of the operator defined by a lambda matrix over the sequence space of null and convergent sequences, *Abstr. Appl. Anal.*, 2013, Article ID 687393 (2013), 13 pages.
- [47] Yeşilkayagil, M. & Başar, F., On the fine spectrum of the operator defined by a lambda matrix over the sequence space c_0 and c , *AIP Conf. Proc.*, 18 (2012), 199-202.
- [48] Yeşilkayagil, M. & Başar, F., A survey for the spectrum of triangles over sequence spaces, *Numer. Funct. Anal. Optim.*, 40(16) (2019), 1898-1917.
- [49] Yildirim, M. E., The spectrum and fine spectrum of q -Cesàro matrices with $0 < q < 1$ on c_0 , *Numer. Funct. Anal. Optim.*, 41(3) (2020), 361-377.

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