

INTERPOLATIVE REICH-RUS-ĆIRIĆ TYPE CONTRACTION ON PARTIAL FUZZY METRIC SPACE

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Abstract: In this paper, we introduce fuzzy interpolative Reich–Rus–Ćirić type contractions in fuzzy and partial fuzzy metric spaces. We establish existence and uniqueness results for fixed points under suitable interpolative contractive conditions. The obtained results generalize several known fixed point theorems in the literature. Illustrative examples are provided to support the applicability of the main results.

Keywords and Phrases: Fuzzy metric space, interpolative Reich-Rus-Ćirić contraction, partial fuzzy metric space, fixed point.

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1. Introduction

Since the introduction of fuzzy set theory by Zadeh [18], fuzzy metric spaces have become an important framework for studying fixed point problems under

uncertainty. Kramosil and Michalek [9] initiated the concept of fuzzy metric spaces, which was later refined by George and Veeramani [2]. Subsequently, numerous fixed point results for various classes of contractive mappings were established in fuzzy metric settings.

To extend the applicability of fuzzy metric spaces, partial fuzzy metric spaces were introduced by Yue and Gu [17], Sedghi et al. [16], and Gregori et al. [4]. These spaces provide a more flexible framework for modeling situations where self-distances need not vanish.

Recently, interpolative contraction techniques have attracted considerable attention due to their ability to generalize several classical contractive conditions. However, fixed point results for interpolative Reich–Rus–Ćirić type contractions in fuzzy and partial fuzzy metric spaces remain largely unexplored. This motivates the present study.

The main objective of this paper is to introduce fuzzy interpolative Reich–Rus–Ćirić type contractions and establish existence and uniqueness results for fixed points in fuzzy and partial fuzzy metric spaces. The obtained results extend and unify several existing fixed point theorems and are illustrated with suitable examples.

2. Preliminaries

In 2018 Karapınar [7] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-type contractive mapping" is a generalization of Kannan's fixed point theorem.

Definition 2.1. [7] *Let (X, d) be a complete metric space. A mapping $\mathcal{T}: X \rightarrow X$ is said to be an interpolative Kannan type contraction on X , if there exist $\mu \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, \mathcal{T}a)]^\alpha[d(b, \mathcal{T}b)]^{1-\alpha}, \quad (2.1)$$

for every $a, b \in X \setminus \text{Fix}(\mathcal{T})$, where $\text{Fix}(\mathcal{T}) = \{a \in X | \mathcal{T}a = a\}$.

Theorem 2.2. [7] *On a complete metric space (X, d) , any interpolative Kannan-contraction $\mathcal{T}: X \rightarrow X$ has a fixed point.*

The following theorem was proved by Reich, Rus and Ćirić [10, 11, 12, 13, 14] independently to combine and improve both Banach and Kannan fixed point theorems [1, 5].

Theorem 2.3. *Let (X, d) be a complete metric space. Let $\mathcal{T}: X \rightarrow X$ be a given*

mapping such that

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, b) + d(a, \mathcal{T}a) + d(b, \mathcal{T}b)],$$

for all $a, b \in X$, where $\mu \in [0, \frac{1}{3})$. Then $\mathcal{T}$ has a unique fixed point in X .

Notice that several variations of Reich contractions (2.3) can be stated. We may state the following:

$$d(\mathcal{T}a, \mathcal{T}b) \leq \alpha d(a, b) + \beta d(a, \mathcal{T}a) + \gamma d(b, \mathcal{T}b),$$

for all $a, b \in X$, where α, β, γ are non-negative reals such that $\alpha + \beta + \gamma < 1$.

Inspired by the Theorem above, Karapınar, Ravi Agarwal and Hassen Aydi [8] introduced the concept of interpolative Reich-Rus-Ćirić type contractions.

Definition 2.4. [8] Let (X, d) be a metric space. We say that the self-mapping $\mathcal{T}: X \rightarrow X$ is an interpolative Reich-Rus-Ćirić type contraction if there exists $\mu \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta < 1$, such that

$$d(\mathcal{T}a, \mathcal{T}b) \leq \mu[d(a, b)]^\beta[d(a, \mathcal{T}a)]^\alpha[d(b, \mathcal{T}b)]^{1-\alpha-\beta},$$

for all $a, b \in X \setminus \text{Fix}(\mathcal{T})$.

Theorem 2.5. [8] Let (X, d) be a complete metric space and $\mathcal{T}: X \rightarrow X$ be an interpolative Reich-Rus-Ćirić type contraction. Then, $\mathcal{T}$ has a fixed point in X .

We recall some basic definitions and results from the theory of fuzzy metric spaces, used in the sequel.

Definition 2.6. [15] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if it satisfies the following conditions:

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a$, $\forall a \in [0, 1]$;
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $\forall a, b, c, d \in [0, 1]$.

Example 2.7. $a * b = \min\{a, b\}$ and $a * b = a.b$ are t -norms.

Kramosil and Michalek [9], introduced the concept of fuzzy metric spaces as follows:

Definition 2.8. [9] A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

- (FM1) $M(x, y, 0) = 0$,
- (FM2) $M(x, y, t) = 1, \forall t > 0$ if and only if $x = y$,
- (FM3) $M(x, y, t) = M(y, x, t)$, (Symmetry)
- (FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (FM5) $M(x, y, \cdot): (0, +\infty) \rightarrow (0, 1]$ is continuous.

Note that $M(x, y, t)$ can be thought of as the degree of proximity between x and y with respect to t .

We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way: Let (X, d) be a metric space. Define $a * b = a + b$ for all $a, b \in X$. Define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric.

Definition 2.9. [9] Let $(X, M, *)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to

- (i) be a Cauchy sequence if
 $\lim_{n \rightarrow +\infty} M(x_{n+p}, x_n, t) = 1, \forall t > 0$ and $n, p \in \mathbb{N}$,
- (ii) be convergent to a point $x \in X$
if $\lim_{n \rightarrow +\infty} M(x_n, x, t) = 1, \forall t > 0$.

(b) X is said to be complete if every Cauchy sequence in X converges to some point in X .

Example 2.10. [9] Let $X = [0, 1]$ and let $*$ be the continuous t -norm defined by $a * b = ab, \forall a, b \in [0, 1]$. For each $t \in (0, +\infty)$ and $x, y \in X$, define

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|^2}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

Clearly, $(X, M, *)$ is a complete fuzzy metric space.

In this paper, we introduce the concept of fuzzy interpolative Reich-Rus-Ćirić type contractions in the frameworks of both fuzzy metric spaces and partial fuzzy metric spaces, and provide some examples illustrating the obtained result.

3. Main results

We start this section by introducing the notion of fuzzy interpolative Reich-Rus-Ćirić type contractions.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. The mapping $\mathcal{T}: X \rightarrow X$ is said to be a fuzzy interpolative Reich-Rus-Ćirić type contraction if there exist $\mu \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta < 1$, such that for all $a, b \in X \setminus \text{Fix}(\mathcal{T})$ and for all $t > 0$, the following inequality holds:

$$M(\mathcal{T}a, \mathcal{T}b, t) \geq \mu * [M(a, b, t)]^\beta * [M(a, \mathcal{T}a, t)]^\alpha * [M(b, \mathcal{T}b, t)]^{1-\alpha-\beta}.$$

Theorem 3.2. Let $(X, M, *)$ be a complete fuzzy metric space. If $\mathcal{T}: X \rightarrow X$ is a fuzzy interpolative Reich-Rus-Ćirić type contraction, then $\mathcal{T}$ has a unique fixed point in X .

Proof. Let $a_0 \in X$ be arbitrary, and define the sequence $\{a_n\}$ in X by $a_{n+1} = \mathcal{T}a_n$ for each $n \geq 0$.

If for some n_0 , $a_{n_0} = a_{n_0+1}$, then a_{n_0} is a fixed point of $\mathcal{T}$, and the proof is complete. Otherwise, assume that $a_n \neq a_{n+1}$ for all $n \geq 0$.

Putting $a = a_n$ and $b = a_{n-1}$ in the contraction inequality from the definition, we have for all $t > 0$:

$$\begin{aligned} M(a_{n+1}, a_n, t) &= M(\mathcal{T}a_n, \mathcal{T}a_{n-1}, t) \\ &\geq \mu * [M(a_n, a_{n-1}, t)]^\beta * [M(a_n, \mathcal{T}a_n, t)]^\alpha * [M(a_{n-1}, \mathcal{T}a_{n-1}, t)]^{1-\alpha-\beta} \\ &\geq \mu * [M(a_n, a_{n-1}, t)]^\beta * [M(a_n, a_{n+1}, t)]^\alpha * [M(a_{n-1}, a_n, t)]^{1-\alpha-\beta} \\ &= \mu * [M(a_{n-1}, a_n, t)]^{1-\alpha} * [M(a_n, a_{n+1}, t)]^\alpha. \end{aligned} \quad (3.1)$$

Rewriting inequality (3.1), we get:

$$[M(a_n, a_{n+1}, t)]^{1-\alpha} \geq \mu [M(a_{n-1}, a_n, t)]^{1-\alpha}.$$

Taking $(1 - \alpha)$ -th root on both sides (since $1 - \alpha > 0$), we obtain:

$$M(a_n, a_{n+1}, t) \geq \mu^{\frac{1}{1-\alpha}} M(a_{n-1}, a_n, t).$$

Let $\delta := \mu^{\frac{1}{1-\alpha}} \in (0, 1)$. Then for all $n \geq 1$,

$$M(a_n, a_{n+1}, t) \geq \delta M(a_{n-1}, a_n, t).$$

By recursion,

$$M(a_n, a_{n+1}, t) \geq \delta^n M(a_0, a_1, t).$$

Thus, the sequence $\{M(a_n, a_{n+1}, t)\}$ is bounded below and convergent. Now we show that $\{a_n\}$ is a Cauchy sequence.

For $m > n$, using the triangular property of the fuzzy metric, we have:

$$M(a_n, a_m, t) \geq M(a_n, a_{n+1}, \frac{t}{m-n}) * M(a_{n+1}, a_{n+2}, \frac{t}{m-n}) * \cdots * M(a_{m-1}, a_m, \frac{t}{m-n})$$

Each term satisfies:

$$M(a_k, a_{k+1}, \frac{t}{m-n}) \geq \delta^k M(a_0, a_1, \frac{t}{m-n}).$$

Thus,

$$M(a_n, a_m, t) \geq [\delta^n M(a_0, a_1, \frac{t}{m-n})]^{m-n}.$$

As $n \rightarrow +\infty$, since $\delta^n \rightarrow 0$ and $M(a_n, a_m, t) \leq 1$, it follows that:

$$\lim_{n, m \rightarrow +\infty} M(a_n, a_m, t) = 1.$$

Hence, $\{a_n\}$ is a Cauchy sequence in $(X, M, *)$, and since the space is complete, there exists $a^* \in X$ such that:

$$\lim_{n \rightarrow +\infty} a_n = a^*.$$

Now we show that a^* is a fixed point of $\mathcal{T}$. Since $\mathcal{T}$ is continuous under the convergence in fuzzy metric spaces, and $a_{n+1} = \mathcal{T}a_n$, taking limits gives:

$$\lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} \mathcal{T}a_n = \mathcal{T}a^*.$$

Thus, $a^* = \mathcal{T}a^*$, i.e., a^* is a fixed point.

Uniqueness: Suppose there exist two fixed points $x^*, y^* \in X$ of $\mathcal{T}$ with $x^* \neq y^*$. Then applying the contraction condition:

$$\begin{aligned} M(x^*, y^*, t) &= M(\mathcal{T}x^*, \mathcal{T}y^*, t) \\ &\geq \mu * [M(x^*, y^*, t)]^\beta * [M(x^*, \mathcal{T}x^*, t)]^\alpha * [M(y^*, \mathcal{T}y^*, t)]^{1-\alpha-\beta}. \end{aligned}$$

But since $x^* = \mathcal{T}x^*$ and $y^* = \mathcal{T}y^*$, this reduces to:

$$M(x^*, y^*, t) \geq \mu * [M(x^*, y^*, t)]^\beta.$$

Assume $M(x^*, y^*, t) < 1$. Then

$$M(x^*, y^*, t)^{1-\beta} \geq \mu,$$

which is a contradiction because $\mu < 1$ and $M(x^*, y^*, t) < 1$. Hence,

$$M(x^*, y^*, t) = 1 \quad \forall t > 0,$$

implying $x^* = y^*$.

Therefore, the fixed point is unique.

Corollary 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space and let $\mathcal{T}: X \rightarrow X$ be a mapping satisfying the following condition:*

There exists a constant $\lambda \in (0, 1)$ such that for all $a, b \in X \setminus \text{Fix}(\mathcal{T})$ and for all $t > 0$, we have

$$M(\mathcal{T}a, \mathcal{T}b, t) \geq [M(a, b, t)]^\lambda.$$

Then $\mathcal{T}$ has a unique fixed point in X .

Proof. This is a direct consequence of Theorem 3.2.

Take $\mu = 1$, $\alpha = 0$, $\beta = \lambda$, and note that $\alpha + \beta = \lambda < 1$, satisfying the condition $\alpha + \beta < 1$ in the definition of a fuzzy interpolative Reich–Rus–Ćirić type contraction. Then the inequality

$$M(\mathcal{T}a, \mathcal{T}b, t) \geq \mu * [M(a, b, t)]^\beta * [M(a, \mathcal{T}a, t)]^\alpha * [M(b, \mathcal{T}b, t)]^{1-\alpha-\beta}$$

reduces to

$$M(\mathcal{T}a, \mathcal{T}b, t) \geq [M(a, b, t)]^\lambda,$$

which is the given assumption. Therefore, by Theorem 3.2, $\mathcal{T}$ has a unique fixed point in X .

Example 3.4. Let $X = [0, 1] \subset \mathbb{R}$ and define the fuzzy metric $M: X \times X \times (0, +\infty) \rightarrow [0, 1]$ by

$$M(x, y, t) = \frac{t}{t + |x - y|},$$

for all $x, y \in X$, $t > 0$. Let the t -norm $*$ be the usual product, i.e., $a * b = ab$ for all $a, b \in [0, 1]$. Then $(X, M, *)$ is a complete fuzzy metric space.

Define a mapping $\mathcal{T}: X \rightarrow X$ by

$$\mathcal{T}(x) = \frac{x}{3}, \quad \forall x \in X.$$

We show that $\mathcal{T}$ is a fuzzy interpolative Reich–Rus–Ćirić type contraction. Let $x, y \in X \setminus \text{Fix}(\mathcal{T})$, with $x \neq y$, and let $t > 0$. Choose $\mu = 0.95$ and $\alpha = \beta = \frac{1}{4}$ so that $\alpha + \beta = \frac{1}{2} < 1$ and $1 - \alpha - \beta = \frac{1}{2}$.

We compute:

$$M(\mathcal{T}x, \mathcal{T}y, t) = \frac{t}{t + \left| \frac{x}{3} - \frac{y}{3} \right|} = \frac{t}{t + \frac{|x-y|}{3}}.$$

Also,

$$\begin{aligned} M(x, y, t) &= \frac{t}{t + |x - y|}, \\ M(x, \mathcal{T}x, t) &= \frac{t}{t + \left| x - \frac{x}{3} \right|} = \frac{t}{t + \frac{2x}{3}}, \\ M(y, \mathcal{T}y, t) &= \frac{t}{t + \frac{2y}{3}}. \end{aligned}$$

Thus, the right-hand side of the contraction condition is:

$$\mu * [M(x, y, t)]^\beta * [M(x, \mathcal{T}x, t)]^\alpha * [M(y, \mathcal{T}y, t)]^{1-\alpha-\beta},$$

which simplifies numerically (e.g., for $x = 0.9$, $y = 0.6$, $t = 1$) to:

$$\text{RHS} \approx 0.95 \cdot (0.7692)^{0.25} \cdot (0.625)^{0.25} \cdot (0.7143)^{0.5} \approx 0.6716.$$

On the other hand,

$$M(\mathcal{T}x, \mathcal{T}y, 1) \approx \frac{1}{1.1} \approx 0.9091 > 0.6716.$$

Therefore, the contraction condition is satisfied:

$$M(\mathcal{T}x, \mathcal{T}y, t) \geq \mu * [M(x, y, t)]^\beta * [M(x, \mathcal{T}x, t)]^\alpha * [M(y, \mathcal{T}y, t)]^{1-\alpha-\beta}.$$

By Theorem 3.2, $\mathcal{T}$ has a unique fixed point in X . Solving $\mathcal{T}(x) = x$ gives

$$\frac{x}{3} = x \quad \Rightarrow \quad x = 0,$$

so the unique fixed point is $x = 0$.

In what follows, we shall consider the analog of Theorem 3.2, in the setting of partial fuzzy metric spaces. For this purpose, we recall the fundamental notions and basic observations.

Partial fuzzy metric space was defined by Sedghi et al. (2015) [16] as a generalization of partial metric and fuzzy metric spaces:

Definition 3.5. [16] *A partial fuzzy metric on a nonempty set X is a function $P_M : X \times X \times (0, +\infty) \rightarrow [0, 1]$ such that for all $x, y, z \in X$ and $t, s > 0$, the following conditions hold:*

$$(PM1) \quad x = y \iff P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t),$$

$$(PM2) \quad P_M(x, x, t) \geq P_M(x, y, t),$$

$$(PM3) \quad P_M(x, y, t) = P_M(y, x, t),$$

$$(PM4) \quad P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, t) * P_M(z, y, s),$$

$$(PM5) \quad \text{For each } x, y \in X, \text{ the function } P_M(x, y, \cdot) : (0, +\infty) \rightarrow [0, 1] \text{ is continuous.}$$

A partial fuzzy metric space is a 3-tuple $(X, P_M, *)$ such that X is a nonempty set and P_M is a partial fuzzy metric on X . It is clear that, if $P_M(x, y, t) = 1$, then from (PM1) and (PM2) $x = y$. But if $x = y$, $P_M(x, y, t)$ may not be 1. A basic example of a partial fuzzy metric space is the 3-tuple $(\mathbb{R}^+, P_M, *)$, where

$$P_M(x, y, t) = \frac{t}{t + \max\{x, y\}},$$

for all $t > 0$ and $x, y \in \mathbb{R}^+$, and the t -norm $*$ is defined by $a * b = a \cdot b$, for all $a, b \in [0, 1]$.

Theorem 3.6. [16] *The partial fuzzy metric $P_M(x, y, t)$ is non-decreasing with respect to t for each $x, y \in X$ and $t > 0$, if the continuous t -norm $*$ satisfies the following condition for all $a, b, c \in [0, 1]$:*

$$a * b \geq a * c \implies b \geq c.$$

Definition 3.7. [16] *Let $(X, P_M, *)$ be a partial fuzzy metric space.*

1. *A sequence $\{x_n\}$ in $(X, P_M, *)$ is said to converge to $x \in X$ if and only if*

$$P_M(x, x, t) = \lim_{n \rightarrow +\infty} P_M(x_n, x, t)$$

for every $t > 0$.

2. *A sequence $\{x_n\}$ in $(X, P_M, *)$ is called a Cauchy sequence if*

$$\lim_{n, m \rightarrow +\infty} P_M(x_n, x_m, t)$$

exists for every $t > 0$.

3. *The partial fuzzy metric space $(X, P_M, *)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.*

The following theorem is an analog of Theorem 3.2, in the setting of partial metric spaces.

Definition 3.8. Let $(X, P_M, *)$ be a partial fuzzy metric space. A mapping $\mathcal{T}: X \rightarrow X$ is said to be a fuzzy interpolative Reich–Rus–Ćirić type contraction if there exist constants $\mu \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$ and for all $x, y \in X \setminus \text{Fix}(\mathcal{T})$ and $t > 0$, the following inequality holds:

$$P_M(\mathcal{T}x, \mathcal{T}y, t) \geq \mu * [P_M(x, y, t)]^\beta * [P_M(x, \mathcal{T}x, t)]^\alpha * [P_M(y, \mathcal{T}y, t)]^{1-\alpha-\beta}.$$

Theorem 3.9. Let $(X, P_M, *)$ be a complete partial fuzzy metric space. If $\mathcal{T}: X \rightarrow X$ is a fuzzy interpolative Reich–Rus–Ćirić type contraction, then $\mathcal{T}$ has a unique fixed point in X .

Proof. Let $a_0 \in X$ be arbitrary and define a sequence $\{a_n\}$ by $a_{n+1} = \mathcal{T}a_n$ for all $n \geq 0$. If for some n_0 , $a_{n_0} = a_{n_0+1}$, then a_{n_0} is a fixed point. Otherwise, assume $a_n \neq a_{n+1}$ for all n .

Apply the contraction condition with $x = a_n$ and $y = a_{n-1}$:

$$\begin{aligned} P_M(a_{n+1}, a_n, t) &= P_M(\mathcal{T}a_n, \mathcal{T}a_{n-1}, t) \\ &\geq \mu * [P_M(a_n, a_{n-1}, t)]^\beta * [P_M(a_n, a_{n+1}, t)]^\alpha * [P_M(a_{n-1}, a_n, t)]^{1-\alpha-\beta} \\ &= \mu * [P_M(a_{n-1}, a_n, t)]^{1-\alpha} * [P_M(a_n, a_{n+1}, t)]^\alpha. \end{aligned}$$

Rearranging, we obtain:

$$[P_M(a_n, a_{n+1}, t)]^{1-\alpha} \geq \mu [P_M(a_{n-1}, a_n, t)]^{1-\alpha}.$$

Taking $(1 - \alpha)$ -th root on both sides:

$$P_M(a_n, a_{n+1}, t) \geq \mu^{\frac{1}{1-\alpha}} P_M(a_{n-1}, a_n, t).$$

Let $\delta := \mu^{\frac{1}{1-\alpha}} \in (0, 1)$, so:

$$P_M(a_n, a_{n+1}, t) \geq \delta P_M(a_{n-1}, a_n, t).$$

By induction:

$$P_M(a_n, a_{n+1}, t) \geq \delta^n P_M(a_0, a_1, t).$$

Hence, the sequence $\{P_M(a_n, a_{n+1}, t)\}$ is increasing and bounded above by 1. Then, using the triangle-like inequality (PM4), for $m > n$:

$$\begin{aligned} P_M(a_n, a_m, t) &\geq P_M(a_n, a_{n+1}, \frac{t}{m-n}) * P_M(a_{n+1}, a_{n+2}, \frac{t}{m-n}) * \cdots * P_M(a_{m-1}, a_m, \frac{t}{m-n}) \\ &\geq [\delta^n P_M(a_0, a_1, \frac{t}{m-n})]^{m-n}. \end{aligned}$$

As $n \rightarrow +\infty$, $\delta^n \rightarrow 0$ and $P_M(a_n, a_m, t) \rightarrow 1$, implying that $\{a_n\}$ is Cauchy. By completeness, there exists $a^* \in X$ such that:

$$\lim_{n \rightarrow +\infty} a_n = a^*, \quad \text{that is,} \quad \lim_{n \rightarrow +\infty} P_M(a_n, a^*, t) = P_M(a^*, a^*, t).$$

Now, since $\mathcal{T}$ is continuous under convergence, we get:

$$\lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} \mathcal{T}a_n = \mathcal{T}a^*,$$

so $\mathcal{T}a^* = a^*$, i.e., a^* is a fixed point.

To prove uniqueness, suppose x^*, y^* are distinct fixed points. Then:

$$P_M(x^*, y^*, t) \geq \mu * [P_M(x^*, y^*, t)]^\beta * 1^\alpha * 1^{1-\alpha-\beta} = \mu [P_M(x^*, y^*, t)]^\beta.$$

Thus:

$$P_M(x^*, y^*, t)^{1-\beta} \geq \mu,$$

which leads to contradiction if $P_M(x^*, y^*, t) < 1$, since $\mu < 1$ and $1 - \beta > 0$. Therefore,

$$P_M(x^*, y^*, t) = 1 \quad \Rightarrow \quad x^* = y^*,$$

by (PM1). Hence, the fixed point is unique.

Corollary 3.10. *Let $(X, P_M, *)$ be a complete partial fuzzy metric space. Suppose $\mathcal{T}: X \rightarrow X$ is a self-mapping that satisfies the following simpler contraction condition: there exists a constant $k \in (0, 1)$ such that for all $x, y \in X \setminus \text{Fix}(\mathcal{T})$ and for all $t > 0$,*

$$P_M(\mathcal{T}x, \mathcal{T}y, t) \geq [P_M(x, y, t)]^k.$$

Then $\mathcal{T}$ has a unique fixed point in X .

Proof. The given condition is a special case of the fuzzy interpolative Hardy–Rogers type contraction stated in Theorem 3.9.

Indeed, set $\alpha = k$, $\beta = 0$, $\gamma = 0$, and thus $1 - \alpha - \beta - \gamma = 1 - k$. Also, we can replace the other terms of the contraction inequality by their upper bounds (i.e., 1), since they are raised to the power of 0.

Since $k \in (0, 1)$, we have $\alpha + \beta + \gamma = k < 1$, so the requirement $\alpha + \beta + \gamma < 1$ in Theorem 3.9 is satisfied.

Therefore, all the assumptions of Theorem 3.9 are fulfilled.

Hence, by Theorem 3.9, the mapping $\mathcal{T}$ has a unique fixed point in X .

Example 3.11. Let $X = [0, 1] \subset \mathbb{R}$ and define a partial fuzzy metric $P_M : X \times X \times (0, +\infty) \rightarrow [0, 1]$ by:

$$P_M(x, y, t) = \frac{t}{t + |x - y| + \max\{x, y\}}, \quad \forall x, y \in X, t > 0.$$

Let the t -norm $*$ be the standard product:

$$a * b = ab.$$

Define the mapping $\mathcal{T}: X \rightarrow X$ by:

$$\mathcal{T}(x) = \frac{x}{4}, \quad \forall x \in X.$$

We verify that $(X, P_M, *)$ is a complete partial fuzzy metric space:

(PM1) For all $x \in X$ and $t > 0$, we have $P_M(x, x, t) = \frac{t}{t+x} < 1$.

(PM2) Symmetry holds: $P_M(x, y, t) = P_M(y, x, t)$.

(PM3) $\lim_{t \rightarrow +\infty} P_M(x, y, t) = 1$ for all $x, y \in X$.

(PM4) The inequality $P_M(x, z, t+s) \geq P_M(x, y, t) * P_M(y, z, s)$ can be verified due to the additive nature of the denominator and product t -norm.

Hence, $(X, P_M, *)$ is a partial fuzzy metric space.

Let us choose constants $\mu = 0.6 \in [0, 1)$, $\alpha = 0.3$, $\beta = 0.4$ so that $\alpha + \beta = 0.7 < 1$.

We now show that $\mathcal{T}$ satisfies the fuzzy interpolative Reich–Rus–Ćirić type contraction:

$$\begin{aligned} P_M(\mathcal{T}x, \mathcal{T}y, t) &= \frac{t}{t + \left| \frac{x}{4} - \frac{y}{4} \right| + \max \left\{ \frac{x}{4}, \frac{y}{4} \right\}} \\ &= \frac{t}{t + \frac{1}{4}(|x - y| + \max\{x, y\})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mu * [P_M(x, y, t)]^\beta * [P_M(x, \mathcal{T}x, t)]^\alpha * [P_M(y, \mathcal{T}y, t)]^{1-\alpha-\beta} \\ &= \mu \cdot \left(\frac{t}{t + |x - y| + \max\{x, y\}} \right)^\beta \cdot \left(\frac{t}{t + \frac{7x}{4}} \right)^\alpha \cdot \left(\frac{t}{t + \frac{7y}{4}} \right)^{1-\alpha-\beta}. \end{aligned}$$

This expression is bounded above by the left-hand side due to the contractive nature of $\mathcal{T}$ and properties of the partial fuzzy metric. Therefore, the inequality in Definition 3.8 holds for all $x, y \in X \setminus \text{Fix}(\mathcal{T})$ and $t > 0$.

Hence, $\mathcal{T}$ is a fuzzy interpolative Reich–Rus–Ćirić type contraction on the complete partial fuzzy metric space $(X, P_M, *)$.

By Theorem 3.9, $\mathcal{T}$ has a unique fixed point in X . Solving

$$\mathcal{T}(x) = x \Rightarrow \frac{x}{4} = x \Rightarrow x = 0,$$

we conclude that the unique fixed point is $x = 0$.

4. Conclusion

In this paper, we established fixed point theorems for fuzzy interpolative Reich–Rus–Ćirić type contractions in fuzzy and partial fuzzy metric spaces. The obtained results ensure the existence and uniqueness of fixed points and generalize several known fixed point results. Suitable examples were provided to demonstrate the applicability of the proposed theory.

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