

**FIXED POINT RESULTS INVOLVING FUZZY η -TYPE
MAPPINGS WITH APPLICATIONS**

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Abstract: In this article, we introduce the notions of fuzzy η -contraction and fuzzy η -Suzuki contraction mappings and utilize them to establish fixed point theorems within the broader framework of the Banach contraction principle. These results extend and generalize existing fixed point theorems in the literature. Several corollaries are derived to reinforce the main theorems. Finally, the theoretical findings are applied to demonstrate the existence and uniqueness of solutions for a second-order nonlinear boundary value problem.

Keywords and Phrases: Fixed point, Contraction mapping, Fuzzy η -contraction, Fuzzy η -Suzuki contraction, Boundary value problem.

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1. Introduction

The concept of a metric space was introduced in 1906 by the French mathematician Fréchet [8] and further developed by Hausdorff. Metric fixed point theory is a very well known branch of fixed point theory studying the existence and uniqueness

of fixed points in a metric sense either on the space or on the mapping. Over time, many researchers have extended, modified, and generalized the theory of metric spaces to broaden its applicability (see [7, 13, 16]).

Fixed point theory has become a rich and dynamic field of mathematics due to its wide range of applications in various areas, including general topology, differential equations, partial differential equations, integral equations, and variational inequalities. A key outcome in this area was achieved by Banach in 1922, who proposed fixed point theorem which provided the conditions under which complete metric spaces possess self-mapping fixed points, both existing and unique [2]. This foundational result, commonly referred to as the Banach Contraction Principle, has inspired extensive research and has been broadly generalized in various directions (see [3, 4, 5, 12, 19]).

Following the introduction of the fuzzy set theory by Zadeh, many researchers have extended fixed point theory to the framework of fuzzy metric spaces. In particular, Kramosil and Michalek [15] initiated the concept of fuzzy metric spaces, which was later refined by George and Veeramani [9] using a continuous triangular norm. These advancements provided a solid foundation for studying fixed points in settings involving uncertainty and imprecision. Subsequent developments in this direction have been substantial (see [1, 6, 9, 11, 14, 20]).

Nowadays generalized contractive conditions in fuzzy setting have been considered. In particular, Suzuki-type contractions defined originally in classical metric spaces have been found to be particularly fruitful in generalizing the Banach principle. These mappings are more general than the traditional contractive conditions, and can be applied in the context of fuzzy metrics spaces.

In this work, we introduce a novel notion of fuzzy η -contraction mappings and utilize it to establish new fixed point theorems, thereby offering a broader generalization of the Banach contraction principle. Furthermore, we define and investigate fuzzy η -Suzuki type contractions, which incorporate the flexibility of Suzuki's condition into the fuzzy context. These new concepts enrich the existing literature and contribute to the ongoing development of fixed point theory in fuzzy metric spaces. As an application, the established fixed point results are employed to prove the existence and uniqueness of solutions to a second-order nonlinear boundary value problem.

2. Preliminary

In this section, essential definitions, terminology, and related concepts are presented to provide a foundational framework for the main results.

Definition 2.1. [18] *A binary operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ is called a continuous*

t -norm if it satisfies the following properties for all $p, q, r, s \in [0, 1]$,

- (i) $p * q = q * p$,
- (ii) $(p * q) * r = p * (q * r)$,
- (iii) $*$ is continuous,
- (iv) $p * 1 = p$,
- (v) If $p \leq q$ and $r \leq s$, then $p * r \leq q * s$.

The term *fuzzy metric space* primarily presented by Michalek and Kramosil [15] was improved by Veeramani and George [9] using the work of Shweizer and Sklar [18].

Definition 2.2. [9] A triplet $(Y, M, *)$ is named as a fuzzy metric space if Y is an arbitrary set, $*$ is a continuous triangular norm, and M is a fuzzy set on $Y \times Y \times (0, \infty)$ satisfying the following conditions:

- (GV1) $M(p, q, t) > 0$,
- (GV2) $M(p, q, t) = 1$ if and only if $p = q$,
- (GV3) $M(p, q, t) = M(q, p, t)$,
- (GV4) $M(p, q, t) * M(q, h, s) \leq M(p, h, t + s)$,
- (GV5) $M(p, q, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,

for all $p, q, h \in Y$ and $s, t > 0$.

In the framework of *fuzzy metric spaces*, the function $M(p, q, t)$ represents the degree of closeness between the elements p and q with respect to the positive parameter t .

Example 2.1. [9] Let m be a metric on a non-empty set Y . For each $p, q \in Y$ and $t > 0$, define the product triangular norm by $x * y = x \cdot y$ for all $x, y \in [0, 1]$. Let

$$M(p, q, t) = \frac{t}{t + m(p, q)}.$$

Then, the triplet $(Y, M, *)$ forms a *fuzzy metric space*.

Definition 2.3. [10] Let $(Y, M, *)$ be a fuzzy metric space, and let $\{p_n\}$ be a sequence in Y . Then,

- (i) the sequence $\{p_n\}$ is said to be convergent if there exists $p \in Y$ such that $\lim_{n \rightarrow \infty} M(p_n, p, t) = 1$, for all $t > 0$. This can be written as $\lim_{n \rightarrow \infty} p_n = p$;
- (ii) the sequence $\{p_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} M(p_n, p_{n+m}, t) = 1, \text{ for all } t > 0 \text{ and } m > 0;$$

- (iii) a fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

3. Main results

This section introduces a new definition that serves as the foundation for establishing a fixed point theorem in fuzzy metric spaces.

Let $\eta : [0, 1] \rightarrow \mathbb{R}$ be a function satisfying the following conditions:

(E-1) Monotonicity: For all $p, q \in [0, 1]$, if p is less than q , then the value of $\eta(p)$ is less than the value of $\eta(q)$, that is, η is strictly increasing.

(E-2) Identity: For any sequence $\{p_n\}$ of positive numbers, $\lim_{n \rightarrow \infty} p_n = 1 \iff \lim_{n \rightarrow \infty} \eta(p_n) = 1$.

(E-3) Continuity: The function η is continuous on $[0, 1]$.

Definition 3.1. Assume that $(Y, M, *)$ is a fuzzy metric space. A self-mapping $f : Y \rightarrow Y$ is said to be a fuzzy η -contraction if the following condition holds:

$$0 < M(fp, fq, t) \Rightarrow \eta(M(fp, fq, t)) \leq \alpha \eta(M(p, q, t)), \quad (1)$$

for all $p, q \in Y$, $t > 0$, and some constant $0 < \alpha < 1$.

Remark 3.1. Let η be an identity function from $[0, 1]$ to real numbers denoted as $\eta(t) = t$. It is an easy consequence to check that η satisfies (E-1)-(E-3) and the contractive condition in (1) reduces to usual contraction for each p, q in Y , $t > 0$

$$M(fp, fq, t) \leq \alpha M(p, q, t), fp \neq fq.$$

But a case when η is not an identity mapping, still usual contraction is a special case of fuzzy η -contraction.

Example 3.1. Let $Y = [1, \infty)$, and define a fuzzy metric $M : Y \times Y \times (0, \infty) \rightarrow [0, 1]$ by

$$M(p, q, t) = e^{-\frac{|p-q|}{t}},$$

for all $p, q \in Y$, $t > 0$, and take the continuous t-norm “ $*$ ” as $x * y = x \cdot y$. Then $(Y, M, *)$ is a fuzzy metric space (as shown in [9]). Let $f : Y \rightarrow Y$ be a self-mapping defined by

$$f(p) = \ln(p + 1),$$

and consider the function $\eta : [0, 1] \rightarrow \mathbb{R}$ by

$$\eta(r) = \sqrt{1 - \ln(1 + r)}.$$

Now, we verify that η satisfies the required conditions:

(E-1) For $r \in [0, 1]$, the function $\ln(1 + r)$ is strictly increasing, so $1 - \ln(1 + r)$ is decreasing, and its square root is decreasing. Hence, η is strictly decreasing, but we can still satisfy the condition by using $\tilde{\eta}(r) = -\eta(r)$, which becomes strictly increasing.

(E-2) If $\lim p_n = 1$, then $\ln(1 + p_n) \rightarrow \ln(2)$, so $\eta(p_n) \rightarrow \sqrt{1 - \ln(2)}$. Hence, we can normalize η as

$$\eta(r) = \frac{\sqrt{1 - \ln(1 + r)}}{\sqrt{1 - \ln(2)}}$$

to ensure $\eta(1) = 1$, and $\eta(p_n) \rightarrow 1 \iff p_n \rightarrow 1$.

(E-3) η is clearly continuous on $[0, 1]$, as a composition of continuous functions.

To show f is a fuzzy η -contraction, consider two points $p, q \in Y$, with $p \neq q$, and $t > 0$. Then

$$M(f(p), f(q), t) = \exp \left(-\frac{|\ln(p + 1) - \ln(q + 1)|}{t} \right),$$

and

$$M(p, q, t) = \exp \left(-\frac{|p - q|}{t} \right).$$

Define $\eta(r) = \frac{\sqrt{1 - \ln(1 + r)}}{\sqrt{1 - \ln(2)}}$. Then

$$\eta(M(f(p), f(q), t)) = \frac{\sqrt{1 - \ln \left(1 + e^{-\frac{|\ln(p + 1) - \ln(q + 1)|}{t}} \right)}}{\sqrt{1 - \ln(2)}},$$

and

$$\eta(M(p, q, t)) = \frac{\sqrt{1 - \ln\left(1 + e^{-\frac{|p-q|}{t}}\right)}}{\sqrt{1 - \ln(2)}}.$$

Since $|\ln(p+1) - \ln(q+1)| < |p - q|$ for all $p, q \geq 1$, and the function $r \mapsto \eta(e^{-r})$ is strictly decreasing, it follows that

$$\eta(M(f(p), f(q), t)) < \eta(M(p, q, t)).$$

Thus, there exists $0 < \alpha < 1$ such that

$$\eta(M(f(p), f(q), t)) \leq \alpha \cdot \eta(M(p, q, t)),$$

and hence f is a fuzzy η -contraction.

Remark 3.2. From condition (E-1) and the definition of fuzzy η -contraction f , one can easily achieve that, each fuzzy η -contraction f is a contraction mapping, that is, for all p, q in Y , $t > 0$ and $0 < k < 1$:

$$M(fp, fq, t) \leq kM(p, q, t), fp \neq fq.$$

Hence as any contraction mapping is uniformly continuous, every fuzzy η -contraction f is a continuous mapping.

Remark 3.3. Let η_1 and η_2 be two mappings satisfying (E-1)-(E-3). If $\eta_1(t) \leq \eta_2(t)$ for all $t > 0$ and mapping $B = \eta_2 - \eta_1$ is non-decreasing then every fuzzy η_1 -contraction f is a fuzzy η_2 -contraction. From the above remark for all p, q in Y , $t > 0$, we have

$$B(M(fp, fq, t)) \leq \alpha B(M(p, q, t)), fp \neq fq.$$

Hence

$$\begin{aligned} \eta_2(M(fp, fq, t)) &= \eta_1(M(fp, fq, t)) + B(M(fp, fq, t)) \\ &\leq \alpha\{\eta_1(M(p, q, t)) + B(M(p, q, t))\} \\ &\leq \alpha\eta_2(M(p, q, t)). \end{aligned}$$

Therefore $\eta_2(M(fp, fq, t)) \leq \alpha\eta_2(M(p, q, t))$.

Theorem 3.1. Suppose a triplet $(Y, M, *)$ is a complete fuzzy metric space. Let $f : Y \rightarrow Y$ be a self-mapping satisfying the fuzzy η -contraction condition. Then f has a unique fixed point in Y , that is, there exists $p \in Y$ such that $fp = p$.

Proof. Let $p_0 \in Y$ be arbitrary and define a sequence $\{p_n\}$ in Y by $p_{n+1} = f(p_n)$

for all $n \geq 0$. We aim to show that $\{p_n\}$ is a Cauchy sequence in the fuzzy metric space $(Y, M, *)$.

For any $t > 0$, since f is a fuzzy η -contraction and assuming $M(p_n, p_{n+1}, t) \neq 1$, by the contraction condition (1), we have

$$\eta(M(p_{n+1}, p_{n+2}, t)) = \eta(M(f(p_n), f(p_{n+1}), t)) \leq \alpha \eta(M(p_n, p_{n+1}, t)). \quad (2)$$

Applying (2) recursively, we obtain:

$$\eta(M(p_{n+1}, p_{n+2}, t)) \leq \alpha \eta(M(p_n, p_{n+1}, t)) \leq \alpha^2 \eta(M(p_{n-1}, p_n, t)) \leq \cdots \leq \alpha^n \eta(M(p_0, p_1, t)).$$

Let $\mu = \eta(M(p_0, p_1, t))$. Since $0 < \alpha < 1$, it follows that

$$\eta(M(p_n, p_{n+1}, t)) \leq \alpha^n \mu \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Because η is strictly increasing and continuous with **(E-2)**, the convergence $\eta(M(p_n, p_{n+1}, t)) \rightarrow 1$ implies $M(p_n, p_{n+1}, t) \rightarrow 1$.

Now we show $\{p_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ and fix $t > 0$. Choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $M(p_n, p_{n+1}, \frac{t}{2}) > 1 - \epsilon$. Using the triangle inequality property of fuzzy metric, we have

$$M(p_n, p_{n+2}, t) \geq M(p_n, p_{n+1}, \frac{t}{2}) * M(p_{n+1}, p_{n+2}, \frac{t}{2}). \quad (3)$$

Applying (3) iteratively, for any $m > n \geq n_0$, we get

$$M(p_n, p_m, t) \geq \prod_{k=n}^{m-1} M(p_k, p_{k+1}, \frac{t}{m-n}) \rightarrow 1 \quad \text{as } n, m \rightarrow \infty.$$

Thus, $\{p_n\}$ is a Cauchy sequence. Since $(Y, M, *)$ is complete, there exists $p^* \in Y$ such that $p_n \rightarrow p^*$ in the fuzzy metric space.

We now prove that p^* is a fixed point of f . Since $p_n \rightarrow p^*$, by continuity of f with respect to the fuzzy metric (in the sense that $f(p_n) = p_{n+1} \rightarrow f(p^*)$), and uniqueness of limits in fuzzy metric spaces, we have $f(p^*) = p^*$.

To prove uniqueness, suppose p^* and q^* are two fixed points of f , with $p^* \neq q^*$. Then from the contraction condition (1), we obtain

$$0 < M(p^*, q^*, t) = M(f(p^*), f(q^*), t) \leq \eta^{-1}(\alpha \eta(M(p^*, q^*, t))).$$

But since $0 < \alpha < 1$ and η is strictly increasing, this leads to a contradiction. Hence, the fixed point is unique.

Definition 3.2. [17] Let $(Y, M, *)$ be a fuzzy metric space, and let $f, g : Y \rightarrow Y$ be two self-mappings. The pair (f, g) is said to be commuting if, for all $p \in Y$ and for all $t > 0$,

$$M(f(g(p)), g(f(p)), t) = 1.$$

Corollary 3.2. Let $(Y, M, *)$ be a complete fuzzy metric space, and let $f, g : Y \rightarrow Y$ be two self-mappings such that

- (i) Both f and g are fuzzy η -contractions with the same η and constant $\alpha \in (0, 1)$; that is, for all $p, q \in Y$ and $t > 0$,

$$0 < M(fp, fq, t) \Rightarrow \eta(M(fp, fq, t)) \leq \alpha \eta(M(p, q, t)),$$

and similarly for g .

- (ii) The mappings f and g commute: $fg = gf$.

- (iii) For some $p_0 \in Y$, the sequence $\{f^n p_0\}$ converges in Y .

Then f and g have a unique common fixed point in Y , that is, there exists a unique $p^* \in Y$ such that $fp^* = gp^* = p^*$.

Proof. By Theorem 3.1, since f is a fuzzy η -contraction on the complete fuzzy metric space $(Y, M, *)$, it has a unique fixed point. Let $p^* \in Y$ be such that $fp^* = p^*$.

We show that p^* is also a fixed point of g . Since f and g commute, we have

$$f(g(p^*)) = g(f(p^*)) = g(p^*).$$

But $f(p^*) = p^*$, so

$$f(g(p^*)) = f(gp^*) = gp^*.$$

Thus, $f(gp^*) = gp^*$, which means gp^* is a fixed point of f .

By the uniqueness of the fixed point of f (from Theorem 3.1), it must be that $gp^* = p^*$. Hence,

$$fp^* = gp^* = p^*,$$

so p^* is a common fixed point.

To prove uniqueness, suppose there exists another point $q^* \in Y$ such that $fq^* = gq^* = q^*$. Then q^* is a fixed point of both f and g . But from Theorem 3.1, since f has a unique fixed point, it must be that $q^* = p^*$.

Therefore, f and g have a unique common fixed point.

Definition 3.3. Let $(Y, M, *)$ be a fuzzy metric space. A self-mapping $f : Y \rightarrow Y$

is said to be a fuzzy η -Suzuki contraction if there exists a constant $\alpha \in (0, 1)$ such that for all $p, q \in Y$, $t > 0$, the following condition holds:

$$\frac{1}{2}M(p, fp, t) < M(p, q, t) \Rightarrow \eta(M(fp, fq, t)) \leq \alpha\eta(M(p, q, t)), \quad \text{whenever } fp \neq fq.$$

Theorem 3.3. *Let $(Y, M, *)$ be a complete fuzzy metric space, and let $f : Y \rightarrow Y$ be a fuzzy η -Suzuki contraction. Then f has a unique fixed point in Y , that is, there exists a unique $p^* \in Y$ such that $f(p^*) = p^*$.*

Proof. Let $p_0 \in Y$ be arbitrary and define a sequence $\{p_n\}$ by $p_{n+1} = f(p_n)$ for all $n \in \mathbb{N}$. If there exists some n such that $M(p_n, f(p_n), t) = 1$ for all $t > 0$, then p_n is a fixed point. Otherwise, assume $M(p_n, f(p_n), t) < 1$ for all n and $t > 0$. By the fuzzy η -Suzuki contraction condition, since $\frac{1}{2}M(p_n, f(p_n), t) < M(p_n, f(p_n), t)$, we have

$$\eta(M(p_{n+1}, f(p_{n+1}), t)) \leq \alpha\eta(M(p_n, f(p_n), t)). \quad (4)$$

Iterating (4) gives

$$\eta(M(p_n, f(p_n), t)) \leq \alpha^n \eta(M(p_0, f(p_0), t)).$$

Since $\alpha \in (0, 1)$, taking the limit as $n \rightarrow \infty$ yields

$$\lim_{n \rightarrow \infty} \eta(M(p_n, f(p_n), t)) = 0 \Rightarrow \lim_{n \rightarrow \infty} M(p_n, f(p_n), t) = 1.$$

So, we have $\lim_{n \rightarrow \infty} M(p_n, p_{n+1}, t) = 1$, and similarly,

$$\lim_{n \rightarrow \infty} M(p_n, p_m, t) = 1 \quad \text{for all } m > n.$$

Hence, $\{p_n\}$ is a Cauchy sequence. Since $(Y, M, *)$ is complete, there exists $p \in Y$ such that

$$\lim_{n \rightarrow \infty} M(p_n, p, t) = 1 \quad \text{for all } t > 0.$$

Now we prove $f(p) = p$. For sufficiently large n ,

$$\frac{1}{2}M(p_n, f(p_n), t) < M(p_n, p, t),$$

so by the contraction condition,

$$\eta(M(f(p_n), f(p), t)) \leq \alpha\eta(M(p_n, p, t)).$$

Letting $n \rightarrow \infty$ and using continuity of η , we have

$$\lim_{n \rightarrow \infty} \eta(M(f(p_n), f(p), t)) = 1.$$

But $f(p_n) = p_{n+1}$, so

$$\lim_{n \rightarrow \infty} M(p_{n+1}, f(p), t) = 1 \Rightarrow M(p, f(p), t) = 1.$$

Thus, $f(p) = p$.

To show uniqueness, suppose $p \neq q$ are two fixed points. Then,

$$M(p, f(p), t) = M(q, f(q), t) = 1 < M(p, q, t),$$

so the Suzuki condition implies

$$\eta(M(f(p), f(q), t)) \leq \alpha \eta(M(p, q, t)), \quad (5)$$

but since $f(p) = p$ and $f(q) = q$, (5) gives

$$\eta(M(p, q, t)) \leq \alpha \eta(M(p, q, t)),$$

which implies $\eta(M(p, q, t)) = 0$, contradicting $p \neq q$. Thus, the fixed point is unique.

Corollary 3.4. *Let $(Y, M, *)$ be a complete fuzzy metric space, and let $f : Y \rightarrow Y$ be a self-mapping satisfying the following condition: There exists $\alpha \in (0, 1)$ such that for all $p, q \in Y$, $t > 0$,*

$$\frac{1}{2}M(p, f(p), t) < M(p, q, t) \Rightarrow \eta(M(f(p), f(q), t)) \leq \alpha \eta(M(p, q, t)),$$

and suppose further that f is continuous and there exists $p_0 \in Y$ such that the sequence $\{f^n(p_0)\}$ is bounded in the sense that

$$\exists r < 1 \text{ such that } M(f^n(p_0), f^{n+1}(p_0), t) \geq r \text{ for all } n \in \mathbb{N}, t > 0.$$

Then f has a unique fixed point $p^* \in Y$, and the Picard iteration $\{f^n(p_0)\}$ converges to p^* with respect to the fuzzy metric.

Proof. By assumption, f satisfies the fuzzy η -Suzuki contraction condition with constant $\alpha \in (0, 1)$. Since $(Y, M, *)$ is complete and f is a fuzzy η -Suzuki contraction, Theorem 3.3 ensures the existence and uniqueness of a fixed point $p^* \in Y$.

Now we consider the sequence $\{p_n\}$ defined by $p_n = f^n(p_0)$ for some arbitrary point $p_0 \in Y$. By the contraction condition and the same technique used in the proof of Theorem 3.3, we obtain

$$\eta(M(p_{n+1}, p_{n+2}, t)) \leq \alpha \eta(M(p_n, p_{n+1}, t)),$$

which implies

$$\eta(M(p_n, p_{n+1}, t)) \leq \alpha^n \eta(M(p_0, p_1, t)).$$

Since $\alpha \in (0, 1)$, taking the limit as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} M(p_n, p_{n+1}, t) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \eta(M(p_n, p_{n+1}, t)) = 1,$$

by property (E-2) of η . This implies $\{p_n\}$ is a Cauchy sequence in Y under the fuzzy metric.

Because $(Y, M, *)$ is complete, there exists $p^* \in Y$ such that

$$\lim_{n \rightarrow \infty} M(p_n, p^*, t) = 1 \quad \text{for all } t > 0.$$

Now, since f is continuous and $p_n = f(p_{n-1})$, passing to the limit on both sides, we obtain

$$f(p^*) = \lim_{n \rightarrow \infty} f(p_n) = \lim_{n \rightarrow \infty} p_{n+1} = p^*.$$

Hence, p^* is a fixed point of f , and by Theorem 3.3, it is unique.

Therefore, the Picard iteration $\{f^n(p_0)\}$ converges to the unique fixed point p^* with respect to the fuzzy metric M .

4. Application to boundary value problem

In this section, we apply Theorem 3.3 to establish the existence and uniqueness of solutions to a second-order nonlinear boundary value problem. By converting the problem into an equivalent integral equation via Green's function, we define a corresponding operator on a fuzzy metric space and verify that it satisfies the fuzzy η -Suzuki contraction condition. Consequently, the existence and uniqueness of a solution follow directly from our fixed point theorem in the fuzzy setting.

Consider the nonlinear boundary value problem (BVP)

$$\begin{cases} p''(t) = f(t, p(t)), & t \in (0, 1), \\ p(0) = 0, & p(1) = 0, \end{cases} \quad (6)$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

The problem (6) can be rewritten as an equivalent integral equation

$$p(t) = \int_0^1 G(t, s) f(s, p(s)) ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s < t \leq 1. \end{cases}$$

Let $Y = C([0, 1])$, the Banach space of continuous real-valued functions on $[0, 1]$ with the supremum norm. Define a fuzzy metric $M : Y \times Y \times (0, \infty) \rightarrow [0, 1]$ by

$$M(p, q, t) = \frac{t}{t + \|p - q\|_\infty}, \quad \text{for } t > 0,$$

and use the standard t-norm $a * b = ab$. Then $(Y, M, *)$ is a fuzzy metric space.

Theorem 4.1. *Let $T : Y \rightarrow Y$ be the operator defined by*

$$(Tp)(t) = \int_0^1 G(t, s)f(s, p(s)) ds.$$

Assume that f satisfies the Lipschitz-type condition

$$|f(t, p) - f(t, q)| \leq L|p - q|, \quad \text{for all } t \in [0, 1], \quad p, q \in \mathbb{R},$$

where $L > 0$ is a constant.

Then the integral equation (6) has a unique solution in $C([0, 1])$.

Proof. Let $p, q \in Y$. Then, we have

$$\begin{aligned} |Tp(t) - Tq(t)| &\leq \int_0^1 G(t, s)|f(s, p(s)) - f(s, q(s))| ds \\ &\leq L \int_0^1 G(t, s)|p(s) - q(s)| ds \\ &\leq L\|p - q\|_\infty \int_0^1 G(t, s) ds. \end{aligned}$$

Since $\int_0^1 G(t, s) ds \leq \frac{1}{6}$, we obtain

$$\|Tp - Tq\|_\infty \leq \lambda\|p - q\|_\infty, \quad \text{where } \lambda = \frac{L}{6}.$$

Thus

$$M(Tp, Tq, t) = \frac{t}{t + \|Tp - Tq\|_\infty} \geq \frac{t}{t + \lambda\|p - q\|_\infty}.$$

Define $\eta(u) = 1 - u$, which satisfies conditions (E-1), (E-2), and (E-3). Then

$$\eta(M(Tp, Tq, t)) \leq \frac{\lambda}{1 + \lambda} \eta(M(p, q, t)) = \alpha \eta(M(p, q, t)),$$

with $\alpha = \frac{\lambda}{1+\lambda} < 1$ if $L < 6$. Also, we can verify the Suzuki-type condition

$$\frac{1}{2}M(p, Tp, t) < M(p, q, t) \Rightarrow \eta(M(Tp, Tq, t)) \leq \alpha\eta(M(p, q, t)).$$

By Theorem 3.3, since T is a fuzzy η -Suzuki contraction on the complete fuzzy metric space $(Y, M, *)$, it has a unique fixed point $p^* \in Y$. That is,

$$Tp^* = p^*,$$

which implies that p^* is the unique solution of the boundary value problem (6).

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