

**A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS RELATED
TO q -DIFFERENTIAL OPERATOR AND INVOLUTION NUMBERS**

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Abstract: In this work, we employ the Salagean q -differential operator in conjunction with involution numbers to define a new subclass of bi-univalent functions in the open unit disk. For this class, we derive coefficient estimates, including bounds for the general Taylor–Maclaurin coefficients and the Fekete–Szegő functional coefficients.

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1. Introduction

We denote by $\mathcal{A}$ the class of functions, that are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and have the normalized form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

The subclass of $\mathcal{A}$ that consists of univalent (i.e., one-to-one) functions in the open unit disk $\mathbb{D}$ is denoted by S . The image of $\mathbb{D}$ under any function f in S contains a disk of radius $\frac{1}{4}$, under the Koebe one-quarter theorem [9]. Thus, for each function $f \in S$, there exists an inverse function f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D}),$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right).$$

The inverse function f^{-1} has the following series expansion:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Let Σ stand for the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). For more basic results one may refer Srivastava et al. [21] and references there in. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{D}$. We say that the function f is subordinate to g if there exists a Schwarz function w , which is analytic in $\mathbb{D}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{D}).$$

It is well known that, if the function g is univalent in $\mathbb{D}$, then (see [16])

$$f \prec g \quad (z \in \mathbb{D}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For $q \in (0, 1)$, the Jackson q -derivative of a function $f \in \mathcal{A}$ is given by (see [12, 13])

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0. \end{cases} \quad (1.3)$$

From (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (1.4)$$

where

$$[n]_q = \frac{1 - q^n}{1 - q},$$

is sometimes called the basic number n . In particular if $q \rightarrow 1^-$, $[n]_q \rightarrow n$.

For $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [11] defined the Salagean q -differential operator as follows :

$$\mathcal{S}_q^0 f(z) = f(z),$$

$$\mathcal{S}_q^1 f(z) = z D_q f(z) = z + \sum_{n=2}^{\infty} [n]_q a_n z^n,$$

$$\mathcal{S}_q^2 f(z) = z D_q (\mathcal{S}_q^1 f(z)) = z + \sum_{n=2}^{\infty} [n]_q^2 a_n z^n,$$

$$\mathcal{S}_q^m f(z) = z D_q (\mathcal{S}_q^{m-1} f(z)) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \quad (m \in \mathbb{N}_0, = \mathbb{N} \cup \{0\}, \quad z \in \mathbb{D}).$$

We note that $\lim_{q \rightarrow 1^-} \mathcal{S}_q^m f(z) = \mathcal{S}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad (m \in \mathbb{N}_0, \quad z \in \mathbb{D})$,

the familiar Salagean derivative [20]. The usual involution numbers, also known telephone numbers, are assumed by the recurrence relation

$$\mathcal{I}(n) = \mathcal{I}(n-1) + (n-1)\mathcal{I}(n-2) \quad \text{for } n \geq 2$$

with initial conditions $\mathcal{I}(0) = \mathcal{I}(1) = 1$. Heinrich August Rothe first noticed connections between these numbers and symmetric groups in 1800. He noted that $\mathcal{I}(n)$ is the number of involutions (self-inverse permutations) in the symmetric group $\mathcal{S}_n$ (see, for example, [6, 15]). Because involutions correspond to standard Young tableaux, it is clear that the n th involution number is also the number of Young tableaux on the set $1, 2, \dots, n$ (for details, see [3]). In 2017, Wloch and Wolowiec-Musial introduced generalized telephone numbers $\mathcal{I}(l, n)$ defined for integers $n \geq 0$ and $l \geq 1$ by the recurrence relation

$$\mathcal{I}(l, n) = l\mathcal{I}(l, n-1) + (n-1)\mathcal{I}(l, n-2)$$

with initial conditions

$$\mathcal{I}(l, 0) = 1 \quad \text{and} \quad \mathcal{I}(l, 1) = l.$$

In 2019 Bednarz and Wolowiec-Musial [2] considered a new generalization of telephone numbers by

$$\mathcal{I}_l(n) = \mathcal{I}_l(n-1) + l(n-1)\mathcal{I}_l(n-2),$$

where $n \geq 2$ and $l \geq 1$ with initial conditions

$$\mathcal{I}_l(0) = \mathcal{I}_l(1) = 1.$$

In 2021, Deniz [8], investigated the exponential-generating function for $\mathcal{I}_l(n)$ as follows:

$$e^{\left(x+l\frac{x^2}{2}\right)} = \sum_{n=0}^{\infty} \mathcal{I}_l(n) \frac{x^n}{n!} \quad (l \geq 1).$$

Clearly, when $l = 1$, we have $\mathcal{I}_l(n) \equiv \mathcal{I}(n)$. $\mathcal{I}_l(n)$ for some values of n are as follows:

1. $\mathcal{I}_l(0) = \mathcal{I}_l = 1$,
2. $\mathcal{I}_l(2) = 1 + l$,
3. $\mathcal{I}_l(3) = 1 + 3l$
4. $\mathcal{I}_l(4) = 1 + 6l + 3l^2$
5. $\mathcal{I}_l(5) = 1 + 10l + 15l^2$
6. $\mathcal{I}_l(6) = 1 + 15l + 45l^2 + 15l^3$

with the domain of the open unit disc $\mathbb{D}$, we consider the function

$$F(z) = e^{\left(z+l\frac{z^2}{2}\right)} = 1+z+\frac{1+l}{2}z^2+\frac{1+3l}{6}z^3+\frac{3l^2+6l+1}{24}z^4+\frac{1+10l+15l^2}{120}z^5+\dots(z \in \mathbb{D}).$$

Motivated by earlier studies on bi-univalent functions [10, 24, 25] and recent investigations involving special number sequences and q -calculus in geometric function theory, particularly generalized Involution numbers [1, 14, 17-19, 22, 23, 26], several researchers have studied analytic and bi-univalent function classes, Hankel determinants, and related coefficient problems using generalized telephone numbers and Salagean-type q -difference operators. These developments demonstrate the growing interest in combining special number sequences with q -calculus in the

study of analytic and bi-univalent functions. Motivated by these investigations, we introduce a new subclass $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$ of the function class Σ , defined by means of the Salagean q -differential operator associated with involution numbers. For the proposed class, we obtain general Taylor-Maclaurin coefficient estimates and corresponding bounds for the Fekete-Szegő functional.

2. Main Results

We now introduce the subclass of bi-univalent functions that is the main object of investigation in this paper.

Definition 2.1. For $\alpha \geq 0$, $0 \leq \beta \leq 1$, and $1 \leq l < 2$, a function $f \in \Sigma$ is said to be in the class $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$ if it satisfies the subordinations:

$$(1 - \alpha) \left[(1 - \beta) \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} + \beta \frac{\mathcal{S}_q^{m+2} f(z)}{\mathcal{S}_q^{m+1} f(z)} \right] + \alpha \left[\frac{(1 - \beta) \mathcal{S}_q^{m+1} f(z) + \beta \mathcal{S}_q^{m+2} f(z)}{(1 - \beta) \mathcal{S}_q^m f(z) + \beta \mathcal{S}_q^{m+1} f(z)} \right] \prec F(z) = e^{\left(z + l \frac{z^2}{2}\right)}$$

and

$$(1 - \alpha) \left[(1 - \beta) \frac{\mathcal{S}_q^{m+1} g(w)}{\mathcal{S}_q^m g(w)} + \beta \frac{\mathcal{S}_q^{m+2} g(w)}{\mathcal{S}_q^{m+1} g(w)} \right] + \alpha \left[\frac{(1 - \beta) \mathcal{S}_q^{m+1} g(w) + \beta \mathcal{S}_q^{m+2} g(w)}{(1 - \beta) \mathcal{S}_q^m g(w) + \beta \mathcal{S}_q^{m+1} g(w)} \right] \prec F(w) = e^{\left(w + l \frac{w^2}{2}\right)}$$

where the function $g = f^{-1}$ is given by (1.2).

Suitably fixing the parameter $\alpha = 0$ we state the following new subclass :

Example 1. For $0 \leq \beta \leq 1$ and $1 \leq l < 2$, a function $f \in \Sigma$ is said to be in the class $\mathcal{W}_\Sigma^m(\beta, F, q)$ if it satisfies the subordinations

$$(1 - \beta) \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} + \beta \frac{\mathcal{S}_q^{m+2} f(z)}{\mathcal{S}_q^{m+1} f(z)} \prec F(z) = e^{\left(z + l \frac{z^2}{2}\right)}$$

and

$$(1 - \beta) \frac{\mathcal{S}_q^{m+1} g(w)}{\mathcal{S}_q^m g(w)} + \beta \frac{\mathcal{S}_q^{m+2} g(w)}{\mathcal{S}_q^{m+1} g(w)} \prec F(w) = e^{\left(w + l \frac{w^2}{2}\right)}$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 2.2. It should be remarked that the class $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$ is a generalization of well-known classes considered earlier. These classes are:

- (i) For $\alpha = 1$ the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$ reduces to the class $\mathcal{P}\Sigma_q^m(\beta, F)$ which was studied by Breaz et.al. [4].
- (ii) For $q \rightarrow 1$ and $m = 0$ the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$ reduces to the class $\mathcal{W}_{\sigma}(\alpha, \beta, F)$, which was introduced and studied by Breaz et.al. [5].
- (iii) For $\alpha = 0, \beta = 0, m = 0$ and $q \rightarrow 1$, the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$ reduces to the class $S_{\Sigma}^*(F)$ which was introduced by Cotirla and Wanas [7].
- (iv) For $\alpha = 0, \beta = 1, m = 0$ and $q \rightarrow 1$, the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$ reduces to the class $\mathcal{C}_{\Sigma}(F)$, which was studied by Cotirla and Wanas [7].

We need the following lemma to prove our results.

Lemma 2.3. [9] Let $\mathcal{P}$ be the class of all analytic functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

satisfying $\operatorname{Re}\{p(z)\} > 0, (z \in \mathbb{D})$. Then

$$|p_n| \leq 2 \quad (n = 1, 2, 3, \dots).$$

Lemma 2.4. [27] Let $r, s \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then

$$|(r+s)z_1 + (r-s)z_2| \leq \begin{cases} 2|r|R, & |r| \geq |s|, \\ 2|s|R, & |r| \leq |s|. \end{cases}$$

Next, we investigate the optimal bounds for the Taylor- Maclaurin coefficients $|a_2|$ and $|a_3|$. Furthermore we establish Fekete-Szegő inequality for the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$.

Theorem 2.5. For $\alpha \geq 0, 0 \leq \beta \leq 1$, and $0 < q < 1$, let $f \in \mathcal{A}$ be in the class $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$. Then

$$|a_2| \leq \sqrt{\frac{2}{2\left\{q[1+\beta(q+q^2)][2]_q[3]_q^m - q\left((1-\alpha)[1+\beta(q^2+2q)] + \alpha[1+\beta q]^2\right)[2]_q^{2m}\right\} + (1-l)q^2(1+\beta q)^2[2]_q^{2m}}}$$

and

$$|a_3| \leq \frac{1}{q} \left(\frac{1}{q(1+\beta q)^2[2]_q^{2m}} + \frac{1}{\{1+\beta(q+q^2)\}[2]_q[3]_q^m} \right).$$

Proof. Let $f \in \mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$. Then there are two analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \dots \quad (z \in \mathbb{D}) \quad (2.1)$$

and

$$v(w) = v_1w + v_2w^2 + v_3w^3 + \dots \quad (w \in \mathbb{D}), \quad (2.2)$$

with $u(0) = v(0) = 0$ and $\max\{|u(z)|, |v(w)|\} < 1$ ($z, w \in \mathbb{D}$), such that

$$(1 - \alpha) \left[(1 - \beta) \frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)} + \beta \frac{\mathcal{S}_q^{m+2}f(z)}{\mathcal{S}_q^{m+1}f(z)} \right] + \alpha \left[\frac{(1 - \beta)\mathcal{S}_q^{m+1}f(z) + \beta\mathcal{S}_q^{m+2}f(z)}{(1 - \beta)\mathcal{S}_q^m f(z) + \beta\mathcal{S}_q^{m+1}f(z)} \right] = F(u(z)) \quad (2.3)$$

and

$$(1 - \alpha) \left[(1 - \beta) \frac{\mathcal{S}_q^{m+1}g(w)}{\mathcal{S}_q^m g(w)} + \beta \frac{\mathcal{S}_q^{m+2}g(w)}{\mathcal{S}_q^{m+1}g(w)} \right] + \alpha \left[\frac{(1 - \beta)\mathcal{S}_q^{m+1}g(w) + \beta\mathcal{S}_q^{m+2}g(w)}{(1 - \beta)\mathcal{S}_q^m g(w) + \beta\mathcal{S}_q^{m+1}g(w)} \right] = F(v(w)). \quad (2.4)$$

Define the functions ϕ and ψ by

$$\phi(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1z + c_2z^2 + \dots$$

and

$$\psi(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + d_1z + d_2z^2 + \dots$$

Or, equivalently,

$$u(z) = \frac{\phi(z) - 1}{\phi(z) + 1} = \frac{1}{2} \left[c_1z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \dots \right] \quad (2.5)$$

$$v(z) = \frac{\psi(z) - 1}{\psi(z) + 1} = \frac{1}{2} \left[d_1z + \left(d_2 - \frac{d_1^2}{2} \right) z^2 + \dots \right]. \quad (2.6)$$

Then $\phi(z)$ and $\psi(z)$ are analytic in $\mathbb{D}$, where $\phi(0) = 1 = \psi(0)$. Since $u, v : \mathbb{D} \rightarrow \mathbb{D}$, we say that $\phi(z)$ and $\psi(z)$ have a positive real part in $\mathbb{D}$, and

$$|\phi_i| \leq 2 \quad \text{and} \quad |\psi_i| \leq 2, \quad (i = 1, 2, 3, \dots).$$

Further we have

$$\begin{aligned} F(u(z)) &= e^{\left(u(z) + l \frac{u(z)^2}{2} \right)} = e^{\left(\frac{\phi(z)-1}{\phi(z)+1} + l \frac{\left(\frac{\phi(z)-1}{\phi(z)+1} \right)^2}{2} \right)} \\ &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} + \frac{(l-1)}{8}c_1^2 \right) z^2 \\ &\quad + \left(\frac{c_3}{2} + (l-1)\frac{c_1c_2}{4} + \frac{(1-3l)}{48}c_1^3 \right) z^3 + \dots \end{aligned} \quad (2.7)$$

$$\begin{aligned}
F(v(w)) &= e^{\left(v(w) + l \frac{v(w)^2}{2}\right)} = e^{\left(\frac{\psi(w)-1}{\psi(w)+1} + l \frac{\left(\frac{\psi(w)-1}{\psi(w)+1}\right)^2}{2}\right)} \\
&= 1 + \frac{d_1}{2}w + \left(\frac{d_2}{2} + \frac{(l-1)}{8}d_1^2\right)w^2 \\
&\quad + \left(\frac{d_3}{2} + (l-1)\frac{d_1d_2}{4} + \frac{(1-3l)}{48}d_1^3\right)w^3 + \dots \quad (2.8)
\end{aligned}$$

Using (2.7) and (2.8) in (2.3) and (2.4) respectively, we have

$$\begin{aligned}
(1-\alpha) \left[(1-\beta) \frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)} + \beta \frac{\mathcal{S}_q^{m+2}f(z)}{\mathcal{S}_q^{m+1}f(z)} \right] + \alpha \left[\frac{(1-\beta)\mathcal{S}_q^{m+1}f(z) + \beta\mathcal{S}_q^{m+2}f(z)}{(1-\beta)\mathcal{S}_q^m f(z) + \beta\mathcal{S}_q^{m+1}f(z)} \right] \\
= F(u(z)) = 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} + \frac{(l-1)}{8}c_1^2\right)z^2 + \dots \quad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
(1-\alpha) \left[(1-\beta) \frac{\mathcal{S}_q^{m+1}g(w)}{\mathcal{S}_q^m g(w)} + \beta \frac{\mathcal{S}_q^{m+2}g(w)}{\mathcal{S}_q^{m+1}g(w)} \right] + \alpha \left[\frac{(1-\beta)\mathcal{S}_q^{m+1}g(w) + \beta\mathcal{S}_q^{m+2}g(w)}{(1-\beta)\mathcal{S}_q^m g(w) + \beta\mathcal{S}_q^{m+1}g(w)} \right] \\
= F(v(w)) = 1 + \frac{d_1}{2}w + \left(\frac{d_2}{2} + \frac{(l-1)}{8}d_1^2\right)w^2 + \dots \quad (2.10)
\end{aligned}$$

Equating the coefficients in (2.9) and (2.10), we get

$$q(1+\beta q)[2]_q^m a_2 = \frac{c_1}{2}, \quad (2.11)$$

$$\begin{aligned}
q \{1 + \beta(q + q^2)\} [2]_q [3]_q^m a_3 - q \{ (1-\alpha)(1 + \beta(q^2 + 2q)) + \alpha(1 + \beta q)^2 \} [2]_q^{2m} a_2^2 \\
= \frac{c_2}{2} + \frac{(l-1)}{8}c_1^2, \quad (2.12)
\end{aligned}$$

$$-q(1+\beta q)[2]_q^m a_2 = \frac{d_1}{2} \quad (2.13)$$

and

$$\begin{aligned}
q \{1 + \beta(q + q^2)\} [2]_q [3]_q^m (2a_2^2 - a_3) - q \{ (1-\alpha)(1 + \beta(q^2 + 2q)) + \alpha(1 + \beta q)^2 \} [2]_q^{2m} \\
= \frac{d_2}{2} + \frac{(l-1)}{8}d_1^2. \quad (2.14)
\end{aligned}$$

From (2.11) and (2.12), we have

$$c_1 = -d_1 \quad (2.15)$$

and

$$8q^2(1+\beta q)^2[2]_q^{2m}a_2^2 = (c_1^2 + d_1^2). \quad (2.16)$$

From (2.12), (2.14) and (2.16), we get

$$a_2^2 = \frac{(c_2 + d_2)}{2 \left\{ 2 \left(q[1 + \beta(q + q^2)] [2]_q [3]_q^m - q \left((1 - \alpha) [1 + \beta(q^2 + 2q)] + \alpha [1 + \beta q]^2 \right) [2]_q^{2m} \right) + (1 - l) q^2 (1 + \beta q)^2 [2]_q^{2m} \right\}}. \quad (2.17)$$

Applying Lemma 2.3 for the coefficients c_2 and d_2 in (2.17), we obtain

$$|a_2| \leq \sqrt{\frac{2}{\left| 2 \left\{ q[1 + \beta(q + q^2)] [2]_q [3]_q^m - q \left((1 - \alpha) [1 + \beta(q^2 + 2q)] + \alpha [1 + \beta q]^2 \right) [2]_q^{2m} \right\} + (1 - l) q^2 (1 + \beta q)^2 [2]_q^{2m} \right|}}.$$

By subtracting (2.14) from (2.12) and using (2.15) and, we have

$$2q\{1 + \beta(q + q^2)\} [2]_q [3]_q^m (a_3 - a_2^2) = \frac{1}{2} (c_2 - d_2). \quad (2.18)$$

If we use (2.16) in (2.18), we will obtain

$$a_3 = \frac{c_1^2 + d_1^2}{8q^2(1 + \beta q)^2 [2]_q^{2m}} + \frac{c_2 - d_2}{4q\{1 + \beta(q + q^2)\} [2]_q [3]_q^m}. \quad (2.19)$$

If we apply Lemma 2.3 for the coefficients c_1 , c_2 , d_1 and d_2 in (2.19), we obtain

$$|a_3| \leq \frac{1}{q} \left(\frac{1}{q[1 + \beta q]^2 [2]_q^{2m}} + \frac{1}{[1 + \beta(q + q^2)] [2]_q [3]_q^m} \right).$$

Theorem 2.6. For $\alpha \geq 0$, $0 \leq \beta \leq 1$, $0 < q < 1$ and $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{q\{1 + \beta(q + q^2)\} [2]_q [3]_q^m}, \\ \text{for } |\mu - 1| \leq \left| 1 - \frac{[(1 - \alpha)[1 + \beta(q^2 + 2q)] + \alpha(1 + \beta q)^2] [2]_q^{2m-1}}{[1 + \beta(q + q^2)] [3]_q^m} + \frac{q(1 - l)[1 + \beta q]^2 [2]_q^{2m-1}}{2[1 + \beta(q + q^2)] [3]_q} \right| \\ \frac{|2 \{ q[1 + \beta(q + q^2)] [2]_q [3]_q^m - q \left[(1 - \alpha) [1 + \beta(q^2 + 2q)] + \alpha [1 + \beta q]^2 \right] \}}{2|\mu - 1|} \\ \quad + (1 - l) q^2 [1 + \beta q]^2 [2]_q^{2m} \\ \text{for } |\mu - 1| \geq \left| 1 - \frac{[(1 - \alpha)[1 + \beta(q^2 + 2q)] + \alpha(1 + \beta q)^2] [2]_q^{2m-1}}{[1 + \beta(q + q^2)] [3]_q^m} + \frac{q(1 - l)[1 + \beta q]^2 [2]_q^{2m-1}}{2[1 + \beta(q + q^2)] [3]_q} \right|. \end{cases}$$

Proof. It follows from (2.17) and (2.18) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{c_2 - d_2}{4q[1 + \beta(q + q^2)] [2]_q [3]_q^m} + (1 - \mu) a_2^2 \\ &= \frac{c_2 - d_2}{4q[1 + \beta(q + q^2)] [2]_q [3]_q^m} + \left[\frac{(c_2 + d_2)(1 - \mu)}{4 \{ q[1 + \beta(q + q^2)] [2]_q [3]_q^m - q \left[(1 - \alpha) [1 + \beta(q^2 + 2q)] + \alpha [1 + \beta q]^2 \right] \}} \right. \\ &\quad \left. + 2(1 - l) q^2 [1 + \beta q]^2 [2]_q^{2m} \right] \end{aligned}$$

$$= \frac{1}{2} \left[\left(\eta(\mu, l) + \frac{1}{2q[1 + \beta(q + q^2)][2]_q[3]_q^m} \right) c_2 + \left(\eta(\mu, l) - \frac{1}{2q[1 + \beta(q + q^2)][2]_q[3]_q^m} \right) d_2 \right],$$

where

$$\eta(\mu, l) = \left[\frac{1 - \mu}{2 \{ q[1 + \beta(q + q^2)][2]_q[3]_q^m - q [(1 - \alpha)[1 + \beta(q^2 + 2q)] + \alpha[1 + \beta q]^2 \} + (1 - l)q^2[1 + \beta q]^2[2]_q^{2m}} \right].$$

Then, applying the above Lemma 2.3 and Lemma 2.4, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{q[1 + \beta(q + q^2)][2]_q[3]_q^m}, & \text{if } |\eta(\mu, l)| \leq \frac{1}{q[1 + \beta(q + q^2)][2]_q[3]_q^m} \\ 2|\eta(\mu, l)|, & \text{if } |\eta(\mu, l)| \geq \frac{1}{q[1 + \beta(q + q^2)][2]_q[3]_q^m}. \end{cases}$$

After some computations, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{q[1 + \beta(q + q^2)][2]_q[3]_q^m}, \\ \text{for } |\mu - 1| \leq \left| 1 - \frac{[(1 - \alpha)[1 + \beta(q^2 + 2q)] + \alpha(1 + \beta q)^2][2]_q^{2m-1}}{[1 + \beta(q + q^2)][3]_q^m} + \frac{q(1 - l)[1 + \beta q]^2[2]_q^{2m-1}}{2[1 + \beta(q + q^2)][3]_q} \right| \\ \frac{q(1 - l)[1 + \beta q]^2[2]_q^{2m-1}}{2[1 + \beta(q + q^2)][3]_q} \\ \text{for } |\mu - 1| \geq \left| 1 - \frac{[(1 - \alpha)[1 + \beta(q^2 + 2q)] + \alpha(1 + \beta q)^2][2]_q^{2m-1}}{[1 + \beta(q + q^2)][3]_q^m} + \frac{q(1 - l)[1 + \beta q]^2[2]_q^{2m-1}}{2[1 + \beta(q + q^2)][3]_q} \right|. \end{cases}$$

Taking $\mu = 1$ in the Theorem 2.6, we led to the following corollary.

Corollary 2.7. For $\alpha \geq 0$, $0 \leq \beta \leq 1$, $0 < q < 1$ and let $f \in \mathcal{A}$ be in the class $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$. Then

$$|a_3 - a_2^2| \leq \frac{1}{q[1 + \beta(q + q^2)][2]_q[3]_q^m}.$$

Remark 2.8. We can state the estimates for f in the class $\mathcal{W}_\Sigma^m(\beta, F, q)$ defined in Example 1 by taking $\alpha = 0$ in Theorems 2.5 and 2.6.

Remark 2.9. Theorems 2.5 and 2.6 provide the established results for the classes mentioned in Remark 2.2.

3. Applications of the Obtained Results

The coefficient estimates obtained in the preceding sections have several direct consequences for functions belonging to the class $\mathcal{W}_\Sigma^m(\alpha, \beta, F, q)$. In particular, the general Taylor–Maclaurin coefficient estimates can be used to obtain bounds for

the deviation of the functions from the identity mapping. Moreover, the Fekete-Szegő estimate yields bounds for several related coefficient functionals by assigning suitable values to the parameter involved.

4. Conclusion

In this paper, we introduced a new subclass $\mathcal{W}_{\Sigma}^m(\alpha, \beta, F, q)$ of bi-univalent functions using the Salagean q -differential operator and involution numbers. We obtained estimates for the general Taylor–Maclaurin coefficients and the Fekete-Szegő functional. These results contribute to the study of bi-univalent functions associated with q -calculus and special number sequences. Future research may consider other q -differential operators, special number sequences, and related coefficient problems such as Hankel determinants and other functionals.

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