

**A NEW APPROACH FOR SOLVING LINEAR TIME FRACTIONAL
DIFFERENTIAL EQUATIONS USING SHEHU TRANSFORM
WITH ITERATIVE METHOD**

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(Received: Jul. 18, 2025 Accepted: Aug. 04, 2026 Published: Aug. 30, 2026)

Abstract: This paper presents an iterative method that combines the Shehu transform and fractional integration to obtain solutions of linear time-fractional partial differential equations involving the Caputo fractional derivative. By transforming the given differential equation into an equivalent Volterra-type integral equation, a recursive series solution is constructed. A convergence analysis is provided to establish the validity of the proposed iterative scheme and six examples are considered to demonstrate its applicability. Numerical results are obtained for different fractional orders and truncation levels, and the approximations are compared with available exact solutions or higher-order reference solutions. The corresponding absolute errors and three-dimensional representations further illustrate the convergence and solution behaviour. In addition, a comparison with the ADM and the VIM at an equal truncation order for a representative example demonstrates the consistency of the proposed formulation with these established approaches.

Keywords and Phrases: Fractional differential equation, Shehu transform, Caputo fractional derivative, Iterative method.

2020 Mathematics Subject Classification: 35R11, 44A05, 35C10.

1. Introduction

Fractional calculus has attracted considerable attention and has been successfully applied across a wide range of scientific and engineering disciplines [18] over the past few decades. Its applications have expanded significantly in fields such as electromagnetic theory, biological sciences [10], fluid mechanics, and different branches of applied physics [13, 17, 18]. Fractional differential equations have played an important role in real-life processes such as bioengineering, astrophysics, signal and image processing, wave propagation, heat transfer analysis, and thermostat control [20].

In particular, time-fractional partial differential equations are useful for describing processes with memory effects, where the present state of the system depends on its previous history [2]. The Caputo fractional derivative is commonly used in such problems because the initial conditions can be expressed in terms of the usual derivatives of integer order [3]. Therefore, effective analytical and semi-analytical methods are needed to obtain accurate solutions of time-fractional partial differential equations. This has motivated researchers to develop different transform-based and iterative methods to solve such problems [2, 3].

Furthermore, several analytical and semi-analytical methods have been developed for solving fractional differential equations, such as the Adomian Decomposition Method (ADM) [19, 27], Variational Iteration Method (VIM) [4, 24], Homotopy Analysis Method (HAM) [9, 14], Homotopy Perturbation Method (HPM) [15], Laplace Decomposition Method (LDM) [7, 8], and Sumudu Decomposition Method (SDM) [6]. These methods provide useful series solutions for fractional differential equations. However, the construction of successive terms and the analysis of convergence may vary depending on the structure of the problem.

Integral transforms provide a useful approach for solving fractional differential equations. Among them, the Shehu transform has been applied to fractional differential equations, particularly those involving the Caputo fractional derivative. Belgacem, Baleanu, and Bokhari [3] developed the Shehu transform formulas for the Riemann–Liouville and Caputo fractional derivatives and applied them to fractional differential equations. Later, several researchers extended the use of the Shehu transform to different types of fractional derivatives and fractional differential equations [11, 21]. The Shehu transform has also been combined with different analytical and iterative methods to obtain approximate solutions of fractional differential equations. For example, q-homotopy analysis Shehu transform methods have been applied to time-fractional partial differential equations [28]. Maitama and Zhao [12] developed a homotopy analysis transform method for multidimen-

sional fractional diffusion equations and discussed the convergence of the resulting series solutions. More recently, Bekela and Deresse [2] proposed a fractional Shehu transform iterative method for nonlinear time-fractional hyperbolic partial differential equations and studied its convergence and stability.

Motivated by these considerations, this paper develops a Shehu-transform-based iterative method (ISTM) for solving linear time-fractional partial differential equations involving the Caputo fractional derivative. We consider the general form [16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a_0(x)u + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} + \cdots + a_n(x)\frac{\partial^n u}{\partial x^n} = q(x, t), t > 0, x \in \mathbb{R}. \quad (1)$$

Subject to the following initial conditions

$$u(x, 0) = f(x), \quad 0 < \alpha \leq 1,$$

and

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = g(x), \quad 1 < \alpha \leq 2.$$

where α denotes the order of the time-fractional derivative and $a_i (i = 0, 1, \dots, n)$, $f(x)$, $g(x)$, and $q(x, t)$ are continuous functions.

The proposed algorithm transforms the given differential equation into an equivalent Volterra-type integral equation and constructs the solution recursively as a series of successive components. A convergence analysis is established under suitable assumptions on the spatial operator. The numerical performance of the method is investigated through six examples covering fractional diffusion, wave, transport-diffusion, and inhomogeneous problems. For problems with known exact solutions, the truncated approximations are compared with the exact solutions, and the corresponding absolute errors are evaluated. For the inhomogeneous problem without a simple closed-form solution, a higher-order partial sum is used as a reference to examine truncation error and convergence behaviour. Three-dimensional representations are also provided to visualize the approximate solutions and their errors for different fractional orders. Furthermore, consistency with the Adomian Decomposition Method and the Variational Iteration Method is examined for a representative linear diffusion problem at an equal truncation order.

The remaining sections of the paper are organized as follows. Section 2 presents the basic definitions and properties of the Caputo fractional derivative and the Shehu transform. Section 3 describes the proposed iterative method and derives the recursive solution. Section 4 presents the convergence analysis and numerical error analysis. Section 5 contains the numerical examples and graphical results.

Finally, Section 6 presents the conclusions of the work.

2. Preliminaries and Basic Definitions

We need some basic definitions and properties of the Shehu transform and fractional calculus to proceed further.

Definition 2.1. [5, 22] *Riemann-Liouville Fractional integral of order α of a function $u(t) \in C_\beta, \beta \geq -1$ is defined as,*

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, t > 0, \alpha \in \mathbb{R}$$

Definition 2.2. [5, 27] *Riemann-Liouville Fractional derivative of order α of a function $u(t) \in C_\beta, \beta \geq -1$ is defined as,*

$$D_{t,RL}^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} u(s) ds, m-1 < \alpha < m, m \in \mathbb{N}$$

Definition 2.3. [5, 25] *Caputo fractional order derivative of order α of a function $u(t) \in C_\beta^m, \beta \geq -1, \alpha \geq 0$ and $m \in \mathbb{N} \cup 0$ is defined as,*

$${}^C D_t^\alpha u(t) = I_t^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{d^m}{dt^m} u(s) ds, m-1 < \alpha < m, m \in \mathbb{N}$$

Definition 2.4. [23, 26] *The Mittag-Leffler function is the extension of the exponential function defined as,*

For one parameter,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad z \in \mathbb{C}, \quad \alpha > 0$$

For two parameter,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad z \in \mathbb{C}, \quad \alpha, \beta > 0$$

Definition 2.5. [1, 11] *The Shehu transform of $u(t)$ is defined as Maitama and Zhao*

$$\mathbb{S}[u(t)] = U(s, p) = \int_0^\infty \exp\left(\frac{-st}{p}\right) u(t) dt$$

Definition 2.6. [1, 11] *The inverse Shehu transform is given by Maitama and Zhao*

$$\mathbb{S}^{-1}[U(s, p)] = u(t), \text{ for } t \geq 0$$

Eventually,

$$u(t) = \mathbb{S}^{-1}[U(s, p)] = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{1}{p} \exp\left(\frac{st}{p}\right) U(s, p) ds$$

where s and p are Shehu transform variables, with β being a real constant.

Definition 2.7. [3] The Shehu transform $\mathbb{S}[u(t)]$, of the Riemann-Liouville fractional integral is given by,

$$\mathbb{S}[I_t^\alpha u(t)] = \left(\frac{s}{p}\right)^{-\alpha} U(s, p).$$

The Shehu transform $\mathbb{S}[u(t)]$, of the Caputo fractional derivative is given by,

$$\mathbb{S}[{}^C D_t^\alpha u(x, t)] = \left(\frac{s}{p}\right)^\alpha U(s, p) - \sum_{k=0}^{m-1} \left(\frac{s}{p}\right)^{\alpha-k-1} u^{(k)}(x, 0),$$

$$m - 1 < \alpha \leq m.$$

Property 1. [1, 11] Linearity property of Shehu transform

Let the function $\alpha u(t)$ and $\beta v(t)$ be in set X , then $(\alpha u(t) + \beta v(t)) \in X$

$$\mathbb{S}[\alpha u(t) + \beta v(t)] = \alpha \mathbb{S}[u(t)] + \beta \mathbb{S}[v(t)].$$

where α and β are non-zero constants.

Property 2. [1, 11] Linearity property of inverse Shehu transform

Let the function $\alpha u(t)$ and $\beta v(t)$ be in set X , then $(\alpha u(t) + \beta v(t)) \in X$

$$\mathbb{S}^{-1}[\alpha u(t) + \beta v(t)] = \alpha \mathbb{S}^{-1}[u(t)] + \beta \mathbb{S}^{-1}[v(t)].$$

where α and β are non-zero constants.

3. Construction of algorithm

In this section, an iterative method based on the Shehu transform is developed for solving linear time-fractional partial differential equations involving the Caputo fractional derivative. The resulting integral representation is then used to construct the approximate solution through an iterative decomposition.

3.1. Two elementary transform facts

Lemma 3.1. For every integer $k \geq 0$,

$$S[t^k] = k! \left(\frac{p}{s}\right)^{k+1} = k! \left(\frac{s}{p}\right)^{-(k+1)}, \quad \text{hence} \quad S^{-1}\left[\left(\frac{s}{p}\right)^{-(k+1)}\right] = \frac{t^k}{k!}.$$

Proof. With $\tau = st/p$, $S[t^k] = \int_0^\infty e^{-st/p} t^k dt = \left(\frac{p}{s}\right)^{k+1} \int_0^\infty e^{-\tau} \tau^k d\tau = \left(\frac{p}{s}\right)^{k+1} \Gamma(k+1) = k! \left(\frac{p}{s}\right)^{k+1}$.

Lemma 3.2. For any $\alpha > 0$ and any function $h(x, t)$ (of exponential order in t , for fixed x),

$$S^{-1} \left[\left(\frac{s}{p} \right)^{-\alpha} S[h(x, t)] \right] = I_t^\alpha h(x, t).$$

In particular, if $h(x, t) = t^\beta \phi(x)$ for a constant $\beta \geq 0$ and a function ϕ that does not depend on t , then

$$S^{-1} \left[\left(\frac{s}{p} \right)^{-\alpha} S[t^\beta \phi(x)] \right] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta + \alpha} \phi(x). \quad (2)$$

Proof. By Definition 2.7 $(s/p)^{-\alpha}$ is exactly the Shehu transform of I_t^α ; since Shehu transform obeys the same convolution structure as the Laplace transform (setting $p = 1$ recovers the Laplace transform, under which the analogous statement is the classical result $\mathcal{L}^{-1}[s^{-\alpha} \mathcal{L}[h]] = I_t^\alpha h$), the first identity follows. For (2), apply Lemma 3.1 with $\Gamma(\beta + 1)$ in place of $k!$ (valid for real $\beta \geq 0$ by the same substitution $\tau = st/p$), so that $(s/p)^{-\alpha} S[t^\beta] = \Gamma(\beta + 1)(s/p)^{-\alpha - \beta - 1} [(s/p)^{-1}]^{-1}$; inverting term by term with Lemma 3.1 (index $\beta + \alpha$) gives $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta + \alpha}$, and $\phi(x)$ passes through unchanged since the transform acts on the t -variable only.

3.2. Derivation

We consider the linear time-fractional partial differential equation involving the Caputo fractional derivative:

$${}^C D_t^\alpha u(x, t) + L_x[u(x, t)] = q(x, t),$$

where

$$L_x[u(x, t)] = a_0(x)u + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} + \cdots + a_n(x)\frac{\partial^n u}{\partial x^n}$$

Here, ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order α , $q(x, t)$ represents the source term, and L_x is the linear spatial operator collecting every term of the linear time fractional partial differential equation other than $D_t^\alpha u$.

Solving for $U(x; s, p)$, The corresponding initial conditions are given by

$$u(x, 0) = f(x), \quad 0 < \alpha \leq 1,$$

and

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = g(x), \quad 1 < \alpha \leq 2.$$

Applying the Shehu transform to the linear time fractional partial differential equation with initial and boundary conditions in the t -variable and using Definition 2.7, we obtain

$$\mathbb{S} \{ {}^C D_t^\alpha u(x, t) \} + \mathbb{S} \{ L_x[u(x, t)] \} = \mathbb{S} \{ q(x, t) \}.$$

Since,

$$\begin{aligned} \left(\frac{s}{p}\right)^\alpha U(x; s, p) - \sum_{k=0}^{m-1} \left(\frac{s}{p}\right)^{\alpha-k-1} u^{(k)}(x, 0) &= \mathbb{S}[q(x, t)] - \mathbb{S} \{ L_x[u(x, t)] \}, \\ U(x; s, p) &= \sum_{k=0}^{m-1} \left(\frac{s}{p}\right)^{-k-1} u^{(k)}(x, 0) + \left(\frac{s}{p}\right)^{-\alpha} \mathbb{S}[q(x, t)] - \left(\frac{s}{p}\right)^{-\alpha} \mathbb{S} \{ L_x[u(x, t)] \}. \end{aligned}$$

Applying $\mathbb{S}^{-1}[\cdot]$ term by term and using Lemma 3.1 on the first sum and Lemma 3.2 on the last two terms,

$$u(x, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(x, 0) + I_t^\alpha q(x, t) - I_t^\alpha \{ L_x[u(x, t)] \}. \quad (3)$$

Equation (3) is an exact Volterra-type integral equation.

Following the iterative decomposition, the solution $u(x, t)$ is expressed as an infinite series of components:

$$u(x, t) = \sum_{r=0}^{\infty} u_r(x, t),$$

where the initial component is defined by

$$u_0(x, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(x, 0) + I_t^\alpha q(x, t), \quad (4)$$

and the remaining components are generated recursively as

$$u_{r+1}(x, t) = -I_t^\alpha [L_x[u_r](x, t)], \quad r = 0, 1, 2, \dots \quad (5)$$

To evaluate the successive components, we use Lemma 3.2. In particular, if the action of the spatial differential operator on $u(x, t)$ can be represented in the form

$$L_x[u(x, t)] = t^\beta \phi(x), \quad \beta \geq 0,$$

where $\phi(x)$ is independent of t , then Lemma 3.2 gives

$$I_t^\alpha (L_x[u(x, t)]) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta + \alpha} \phi(x).$$

Consequently, the recursive relation takes the form

$$u_{r+1}(x, t) = -\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta + \alpha} \phi(x), \quad r = 0, 1, 2, \dots$$

Thus, once $u_0(x, t)$ is determined, the successive components $u_1(x, t)$, $u_2(x, t)$, $u_3(x, t)$, $\dots$ can be obtained recursively by applying the spatial differential operator L_x followed by the fractional integral operator I_t^α .

The approximate solution after N iterations is given by the partial sum

$$u_N(x, t) = \sum_{r=0}^N u_r(x, t).$$

As N increases, the partial sums provide successively improved approximations to the solution. Hence, the series solution of the considered problem is formally expressed as

$$u(x, t) = \lim_{N \rightarrow \infty} u_N(x, t) = \sum_{r=0}^{\infty} u_r(x, t). \quad (6)$$

4. Convergence analysis

Convergence analysis is an essential part of establishing the mathematical validity of transform-based iterative schemes for fractional differential equations. Similar convergence analyses have been reported for Shehu-transform-based iterative approaches to fractional differential equations and partial differential equations [3, 2, 12, 21]

Fix a bounded spatial domain $x \in [-R, R]$ and a time horizon $T > 0$, and let $X = C([-R, R] \times [0, T])$ with the sup norm $\|u\| = \sup_{|x| \leq R, 0 \leq t \leq T} |u(x, t)|$.

The following lemma provides an estimate for the fractional integral in the supremum norm.

Lemma 4.1. *Let $h \in X$ and $\alpha > 0$. Then, for every $0 \leq t \leq T$,*

$$|I_t^\alpha h(x, t)| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|h\|_\infty.$$

Consequently,

$$\|I_t^\alpha h\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|h\|_\infty.$$

Proof. Using the definition of the Riemann–Liouville fractional integral, we obtain

$$|I_t^\alpha h(x, t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} h(x, \tau) d\tau \right|.$$

By the triangle inequality and the definition of the supremum norm,

$$\begin{aligned} |I_t^\alpha h(x, t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} |h(x, \tau)| d\tau \\ &\leq \frac{\|h\|_\infty}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau. \end{aligned}$$

The integral on the right-hand side is

$$\int_0^t (t - \tau)^{\alpha-1} d\tau = \frac{t^\alpha}{\alpha}.$$

Using the identity

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha),$$

we consequently obtain

$$|I_t^\alpha h(x, t)| \leq \frac{t^\alpha}{\Gamma(\alpha + 1)} \|h\|_\infty.$$

Since $0 \leq t \leq T$, it follows that

$$|I_t^\alpha h(x, t)| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|h\|_\infty.$$

Taking the maximum over $(x, t) \in \Omega$ gives

$$\|I_t^\alpha h\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|h\|_\infty.$$

This completes the proof.

Theorem 4.2. (Sufficient condition for geometric convergence). *Let X be a Banach space on which the spatial operator L_x is bounded, i.e., there exists $M > 0$ such that*

$$\|L_x[v]\|_X \leq M\|v\|_X, \quad v \in X.$$

Define

$$\lambda = \frac{MT^\alpha}{\Gamma(\alpha + 1)}.$$

If $\lambda < 1$, then the operator

$$\Phi[v] = u_0 - I_t^\alpha [L_x[v]]$$

is a contraction on X . Consequently, the partial sums

$$S_N = \sum_{r=0}^N u_r$$

converge in X to the unique solution u^* of the Volterra integral equation (3). Moreover,

$$\|u^* - S_N\|_X \leq \frac{\lambda^{N+1}}{1 - \lambda} \|u_0\|_X.$$

Proof. For $u, v \in X$, $\|\Phi(u) - \Phi(v)\| = \|I_t^\alpha L[u - v]\| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|L[u - v]\| \leq \frac{MT^\alpha}{\Gamma(\alpha + 1)} \|u - v\| = \lambda \|u - v\|$ by Lemma 4.1, so Φ is a λ -contraction; by the Banach fixed-point theorem it has a unique fixed point $u^* \in X$, which solves (3). A direct computation using (5) and linearity gives, for $N \geq 0$,

$$\Phi(S_{N-1}) = u_0 - I_t^\alpha L \left[\sum_{r=0}^{N-1} u_r \right] = u_0 + \sum_{r=0}^{N-1} u_{r+1} = \sum_{r=0}^N u_r = S_N,$$

so $\{S_N\}$ is exactly the Picard iteration sequence for Φ starting from $S_{-1} = 0$, and by the standard a priori error estimate for Picard iteration,

$$\|u^* - S_N\|_X \leq \frac{\lambda^{N+1}}{1 - \lambda} \|S_0 - S_{-1}\|_X = \frac{\lambda^{N+1}}{1 - \lambda} \|u_0\|_X.$$

Theorem 4.2 guarantees geometric convergence whenever the time horizon is short enough that $\lambda < 1$. All six examples below, however, converge for every $t > 0$.

Theorem 4.3. (Unconditional convergence for polynomial coefficients). *Suppose that the spatial operator L_x leaves invariant the finite-dimensional polynomial space*

$$P_d = \text{span}\{1, x, \dots, x^d\}.$$

Then there exist constants $C, \rho > 0$ such that

$$|u_r(x, t)| \leq C \rho^r \frac{t^{r\alpha}}{\Gamma(r\alpha + 1)}, \quad r = 0, 1, 2, \dots,$$

for x in a bounded spatial domain and $t \geq 0$. Consequently, the series

$$\sum_{r=0}^{\infty} u_r(x, t)$$

converges absolutely for every $t \geq 0$ and uniformly on compact time intervals.

Proof. Since L maps the $(d + 1)$ -dimensional space of polynomials of degree $\leq d$ in x into itself, its restriction is represented, in the monomial basis, by a $(d + 1) \times (d + 1)$ matrix with entries bounded by some ρ_0 depending only on d and $\max_i \|a_i\|_{\infty}$; hence for a coefficient vector $c_r(t) \in \mathbb{R}^{d+1}$ (writing $u_r(x, t) = \sum_n c_{r,n}(t)x^n$), $\|c_{r+1}(t)\|_{\infty} \leq \rho_0 \|c_r(t)\|_{\infty}$ componentwise up to the fractional-integral factor. Combining this matrix bound with (2) applied at each recursion step (each application of I_t^{α} multiplies the current power t^{β} by $\Gamma(\beta + 1)/\Gamma(\beta + \alpha + 1)$ and increases β by α) gives, by induction on r , a bound of the stated form with $\rho = (d + 1)\rho_0$ and $C = \|c_0\|_{\infty}$. The ratio of consecutive bounding terms is

$$\frac{\rho^{r+1}t^{(r+1)\alpha}/\Gamma((r+1)\alpha+1)}{\rho^rt^{r\alpha}/\Gamma(r\alpha+1)} = \rho t^{\alpha} \frac{\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)} \xrightarrow{r \rightarrow \infty} 0,$$

because $\Gamma(z)/\Gamma(z + \alpha) \sim z^{-\alpha} \rightarrow 0$ as $z \rightarrow \infty$ for any fixed $\alpha > 0$ (a standard consequence of Stirling's formula). By the ratio test, $\sum_r C\rho^rt^{r\alpha}/\Gamma(r\alpha+1)$ converges for every fixed $t \geq 0$, and dominates $\sum_r |u_r(x, t)|$; uniform convergence on compact t -intervals then follows from the Weierstrass M -test.

4.1. Numerical Error Analysis

To examine the numerical accuracy of the proposed iterative scheme, we compare the finite-term approximation with the corresponding exact solution whenever it is available. Let

$$S_N(x, t) = \sum_{n=0}^N u_n(x, t)$$

denote the N -th partial sum of the iterative series. The absolute error is defined by

$$E_N(x, t) = |u_{\text{exact}}(x, t) - S_N(x, t)|.$$

The value of $E_N(x, t)$ gives a direct measure of the accuracy of the truncated series solution. In general, a decrease in E_N as N increases shows that the approximate solution approaches the exact solution.

For problems where an exact solution is not available, the convergence of the iterative approximations is examined using the successive-approximation difference

$$D_N(x, t) = |S_N(x, t) - S_{N-1}(x, t)|.$$

The quantity D_N is used as a numerical indicator of convergence and does not represent the exact error. In addition, the residual

$$R_N(x, t) = {}^C D_t^\alpha S_N(x, t) + L_x[S_N](x, t) - q(x, t)$$

can be evaluated to check how accurately the truncated approximation satisfies the given fractional differential equation.

5. Numerical Examples and Applications

Here, some examples are given to illustrate the efficiency of the proposed method.

Example 1. Consider the linear time-fractional diffusion equation [16],

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \quad u(x, 0) = \sin x.$$

Using the iterative formula (3), we obtain successively the following iteration, Here, $a_2(x) = -1$, $q = 0$, so $L[u] = -\partial^2 u / \partial x^2$ and $m = 1$:

$$\begin{aligned} u_0(x, t) &= \sin x, \\ u_1(x, t) &= -I_t^\alpha [-D^2 u_0] = -I_t^\alpha [\sin x] = -\frac{t^\alpha}{\Gamma(\alpha + 1)} \sin x, \\ u_2(x, t) &= -I_t^\alpha [-D^2 u_1] = -I_t^\alpha \left[\frac{t^\alpha}{\Gamma(\alpha + 1)} \sin x \right] = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \sin x, \\ u_3(x, t) &= -\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \sin x, \quad u_4(x, t) = \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)} \sin x, \quad \dots \end{aligned}$$

so that the series solution is given by:

$$u(x, t) = \sin x \sum_{k=0}^{\infty} (-1)^k \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} = \sin x E_\alpha(-t^\alpha). \quad (7)$$

Table 1 shows a numerical comparison of the truncated series solutions S_5 , S_{10} , and S_{15} with the corresponding exact solution for selected values of α , x , and t . It can be seen that the accuracy of the approximation improves as the number of terms increases. In particular, S_{15} is very close to the exact solution, and the corresponding absolute error is very small for the considered values. These results provide numerical evidence for the accuracy and effectiveness of the proposed Shehu-transform-based iterative scheme.

Table 1: Numerical comparison of the truncated approximations S_5 , S_{10} , and S_{15} with the exact solution for Example 1.

α	x	t	S_5	S_{10}	S_{15}	u_{exact}	$ u_{\text{exact}} - S_{15} $
0.50	0.5	0.2	0.3081302834	0.3086487411	0.3086485391	0.3086485391	2.643×10^{-11}
0.50	0.5	0.4	0.2615720471	0.2654215711	0.2654129700	0.2654129764	6.415×10^{-9}
0.50	0.5	0.6	0.2264988610	0.2388421167	0.2387655389	0.2387656970	1.581×10^{-7}
0.50	1.0	0.2	0.5408195270	0.5417295059	0.5417291513	0.5417291514	4.639×10^{-11}
0.50	1.0	0.4	0.4591021344	0.4658586847	0.4658435883	0.4658435996	1.126×10^{-8}
0.50	1.0	0.6	0.3975429013	0.4192073534	0.4190729467	0.4190732242	2.775×10^{-7}
0.75	0.5	0.2	0.3512137765	0.3512198153	0.3512198152	0.3512198152	1.110×10^{-16}
0.75	0.5	0.4	0.2885426052	0.2886722228	0.2886722195	0.2886722195	1.565×10^{-14}
0.75	0.5	0.6	0.2448338131	0.2456026311	0.2456025404	0.2456025404	1.979×10^{-12}
0.75	1.0	0.2	0.6164381716	0.6164487705	0.6164487705	0.6164487705	2.220×10^{-16}
0.75	1.0	0.4	0.5064399174	0.5066674177	0.5066674120	0.5066674120	2.753×10^{-14}
0.75	1.0	0.6	0.4297237698	0.4310731723	0.4310730132	0.4310730132	3.474×10^{-12}
1.00	0.5	0.2	0.3925203908	0.3925204323	0.3925204323	0.3925204323	$< 10^{-15}$
1.00	0.5	0.4	0.3213659701	0.3213685491	0.3213685491	0.3213685491	1.110×10^{-16}
1.00	0.5	0.6	0.2630857232	0.2631143143	0.2631143142	0.2631143142	$< 10^{-15}$
1.00	1.0	0.2	0.6889381004	0.6889381731	0.6889381731	0.6889381731	$< 10^{-15}$
1.00	1.0	0.4	0.5640503427	0.5640548693	0.5640548693	0.5640548693	2.220×10^{-16}
1.00	1.0	0.6	0.4617588859	0.4618090680	0.4618090679	0.4618090679	$< 10^{-15}$

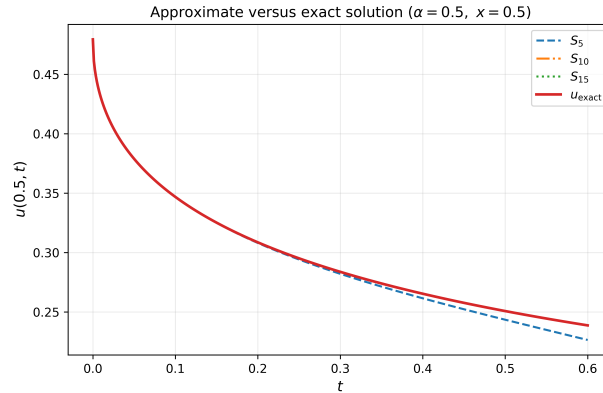
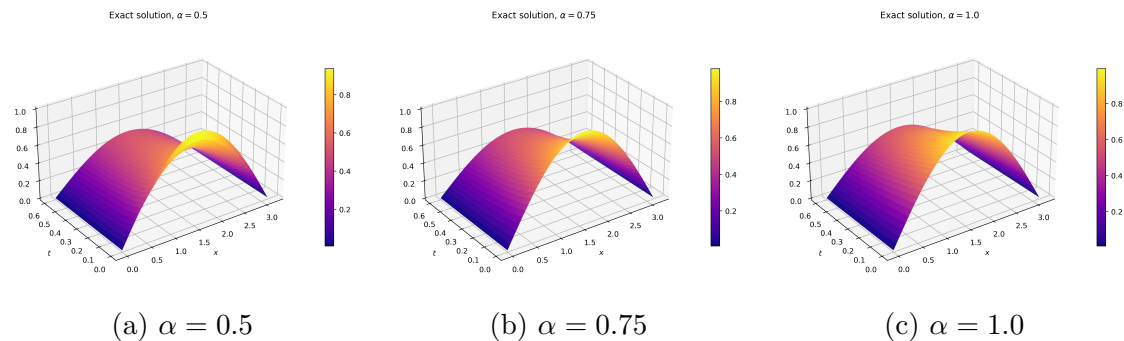
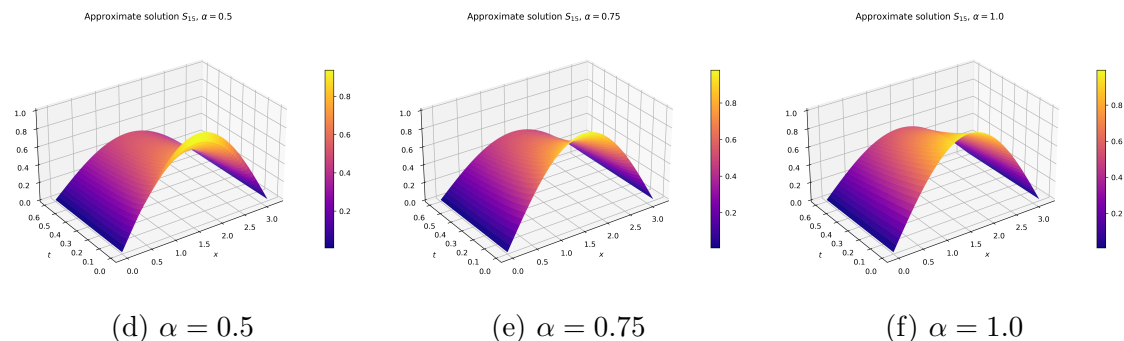


Figure 1: Comparison of the truncated series solutions S_5 , S_{10} , and S_{15} with the exact solution u_{exact} for $\alpha = 0.5$ and $x = 0.5$.

Figure 1 shows a comparison of the truncated series solutions S_5 , S_{10} , and S_{15} with the exact solution for $\alpha = 0.5$ and $x = 0.5$. It can be seen that the approximation improves as the number of terms increases. In particular, S_{15} is very close to the exact solution over the considered time interval. This shows the accuracy of the proposed iterative method.

Exact Solution

Approximate Solution $S_{15}(x, t)$ 

Absolute Error

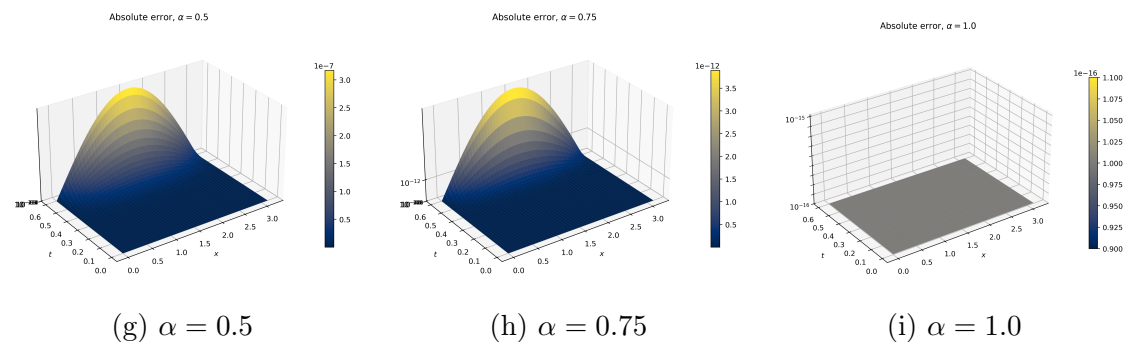


Figure 2: Three-dimensional graphical comparison of the exact solution, the truncated approximate solution $S_{15}(x, t)$, and the corresponding absolute error for Example 1 at $\alpha = 0.5, 0.75$, and 1.0 .

Figure 2 shows the three-dimensional surface of the exact solution, the approximate solution $S_{15}(x, t)$, and the corresponding absolute error for $\alpha = 0.5, 0.75$, and 1.0 . The approximate and exact solutions are in close agreement for the considered fractional orders. For $\alpha = 1$, the fractional solution reduces to the classical diffusion solution $u(x, t) = \sin(x)e^{-t}$, which confirms the consistency of the proposed method.

Example 2. Consider the linear time-fractional wave equation [16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \alpha \leq 2, \quad u(x, 0) = x, \quad u_t(x, 0) = x^2.$$

Using the iterative formula (3), we obtain successively the following iteration,

Here $a_2(x) = -x^2/2$, $q = 0$, $m = 2$:

$$\begin{aligned} u_0(x, t) &= x + tx^2, \\ u_1(x, t) &= -I_t^\alpha \left[-\frac{x^2}{2} D^2 u_0 \right] = x^2 I_t^\alpha [t] = \frac{x^2 t^{\alpha+1}}{\Gamma(\alpha+2)}, \\ u_2(x, t) &= \frac{x^2 t^{2\alpha+1}}{\Gamma(2\alpha+2)}, \quad u_3(x, t) = \frac{x^2 t^{3\alpha+1}}{\Gamma(3\alpha+2)}, \quad \dots \end{aligned}$$

so that, in closed form,

$$u(x, t) = x + tx^2 + x^2 \sum_{k=1}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)} = x + x^2 t E_{\alpha,2}(t^\alpha). \quad (8)$$

Table 2: Numerical comparison of S_5 , S_{10} , and S_{15} with the exact solution for Example 2 at $x = 0.5$.

α	t	S_5	S_{10}	S_{15}	u_{exact}	$ u_{\text{exact}} - S_{15} $
1.50	0.2	0.551362462258	0.551362462258	0.551362462258	0.551362462258	5.41×10^{-44}
1.50	0.4	0.607884635768	0.607884635775	0.607884635775	0.607884635775	1.82×10^{-36}
1.50	0.6	0.672380651815	0.672380652237	0.672380652237	0.672380652237	4.60×10^{-32}
1.75	0.2	0.550679598094	0.550679598094	0.550679598094	0.550679598094	1.52×10^{-52}
1.75	0.4	0.604626759624	0.604626759624	0.604626759624	0.604626759624	8.15×10^{-44}
1.75	0.6	0.664360496770	0.664360496775	0.664360496775	0.664360496775	1.04×10^{-38}
2.00	0.2	0.550334000635	0.550334000635	0.550334000635	0.550334000635	2.47×10^{-61}
2.00	0.4	0.602688081451	0.602688081451	0.602688081451	0.602688081451	2.12×10^{-51}
2.00	0.6	0.659163395537	0.659163395537	0.659163395537	0.659163395537	1.38×10^{-45}

Table 2 shows that the truncated approximations S_5 , S_{10} , and S_{15} rapidly approach the exact solution. The approximation S_{15} agrees with the exact solution to a high

degree of accuracy for all considered values of α and t , confirming the high accuracy of the proposed ISTM for the considered parameter values.

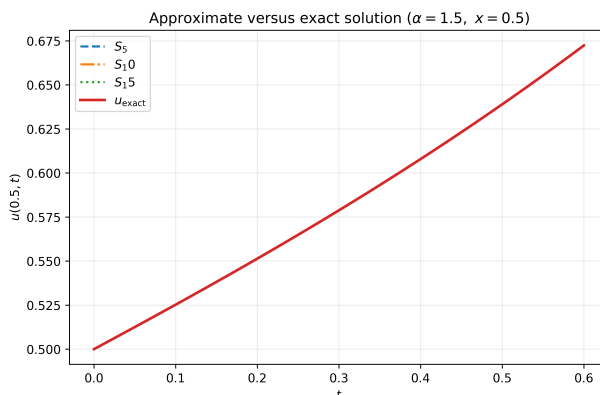


Figure 3: Comparison of the truncated solutions S_5 , S_{10} , and S_{15} with the exact solution for $\alpha = 1.5$ and $x = 0.5$.

Figure 3 illustrates the comparison between the truncated solutions S_5 , S_{10} , and S_{15} and the exact solution for $\alpha = 1.5$. The curves are in close agreement, and the higher-order approximations become indistinguishable from the exact solution over the considered time interval.

Figure 4 shows the three-dimensional plots of the exact solution, the approximate solution $S_{15}(x, t)$, and the corresponding absolute error for $\alpha = 1.5, 1.75$, and 2.0 . The approximate solution is close to the exact solution for all the considered values of α . The absolute error is also very small over the computational domain, which confirms the accuracy of the proposed iterative method.

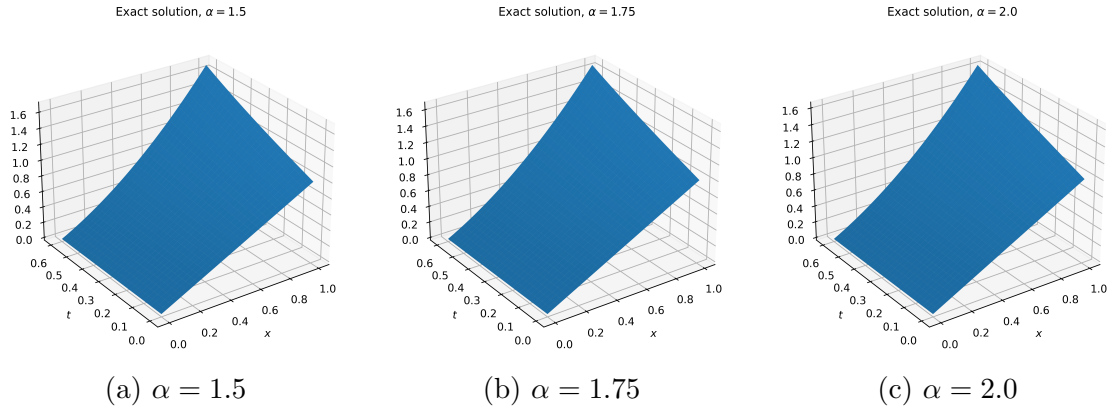
Example 3. Consider the one-dimensional linear inhomogeneous time-fractional equation [16]

$${}^C D_t^\alpha u + x \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t^2 + 2x^3 + 5, \quad 0 < \alpha \leq 1.$$

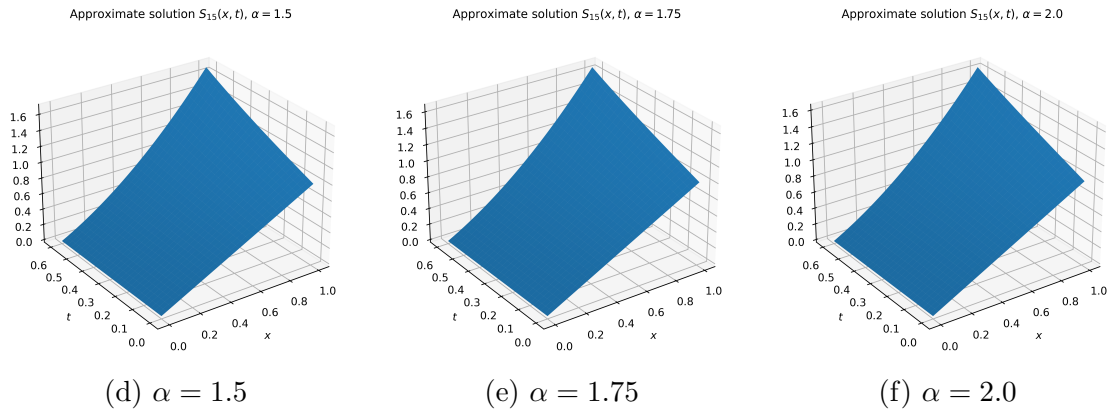
Subject to the initial condition, $u(x, 0) = x^2$.

Using the iterative formula (3), we obtain successively the following iteration,

Exact Solution



Approximate Solution $S_{15}(x, t)$



Absolute Error

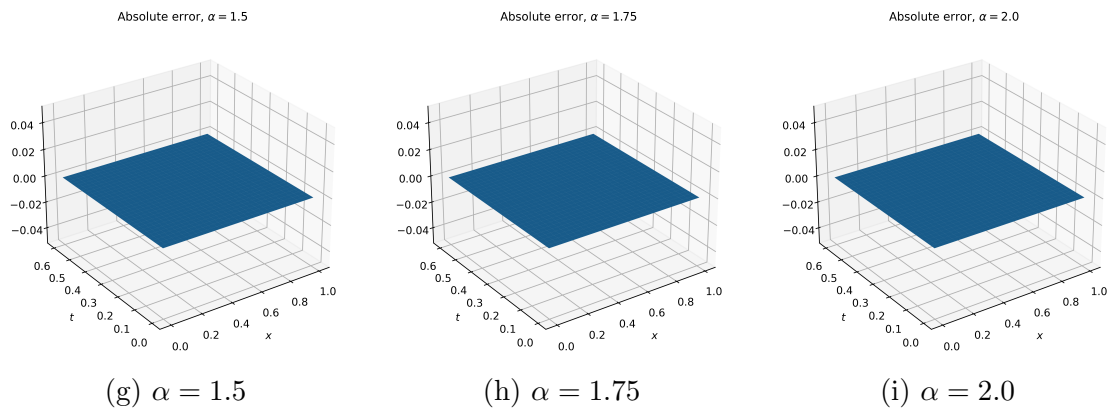


Figure 4: Three-dimensional graphical comparison of the exact solution, approximate solution $S_{15}(x, t)$, and corresponding absolute error for Example 2 at $\alpha = 1.5$, 1.75, and 2.0.

Here, $a_1(x) = x$, $a_2(x) = 1$, $q(x, t) = 2t^2 + 2x^3 + 5$, $m = 1$, and $L[u] = xDu + D^2u$.

$$\begin{aligned} u_0(x, t) &= x^2 + \frac{4t^{\alpha+2}}{\Gamma(\alpha+3)} + (2x^3 + 5)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_1(x, t) &= -(2x^2 + 2)\frac{t^\alpha}{\Gamma(\alpha+1)} - (6x^3 + 12x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_2(x, t) &= (4x^2 + 4)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (18x^3 + 48x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ u_3(x, t) &= -(8x^2 + 8)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - (54x^3 + 156x)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}. \end{aligned}$$

The first few components are therefore summarized by the partial sum $S_3(x, t)$ given by,

$$\begin{aligned} S_3(x, t) &= x^2 + \frac{4t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{(2x^3 + 5)t^\alpha}{\Gamma(\alpha+1)} \\ &\quad - \frac{(2x^2 + 2)t^\alpha}{\Gamma(\alpha+1)} - \frac{(6x^3 + 12x)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \frac{(4x^2 + 4)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(18x^3 + 48x)t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad - \frac{(8x^2 + 8)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{(54x^3 + 156x)t^{4\alpha}}{\Gamma(4\alpha+1)}. \end{aligned} \quad (9)$$

No simple closed form is available (the coefficients of x^3 and x mix under L), so the higher-order partial sum

$$S_{60}(x, t) = \sum_{n=0}^{60} u_n(x, t)$$

is used as a reference solution. The truncated approximations S_5 , S_{10} , and S_{15} are compared with S_{60} , with the absolute error defined by

$$E_N(x, t) = |S_{60}(x, t) - S_N(x, t)|.$$

Figure 5 compares the partial sums S_5 , S_{10} , and S_{15} with the higher-order reference solution S_{60} . The results show that the convergence rate depends on α and t , with higher truncation orders being required for smaller α and larger t .

Table 3: Numerical comparison of S_5 , S_{10} , and S_{15} with the reference solution S_{60} for selected values of α and t at $x = 0.5$.

α	t	S_5	S_{10}	S_{15}	S_{60}	$ S_{60} - S_{15} $
0.50	0.2	1.244147	1.788037	1.752561	1.753494	9.33×10^{-4}
0.50	0.4	-2.138782	4.519792	2.301957	2.601064	2.99×10^{-1}
0.50	0.6	-13.876836	23.293537	-5.196516	3.431922	8.63
0.75	0.2	1.143138	1.146292	1.146291	1.146291	4.73×10^{-11}
0.75	0.4	1.799261	1.893352	1.893009	1.893010	3.01×10^{-7}
0.75	0.6	2.035159	2.725024	2.712893	2.712942	4.97×10^{-5}
1.00	0.2	0.788201	0.788213	0.788213	0.788213	5.24×10^{-19}
1.00	0.4	1.371252	1.372328	1.372328	1.372328	6.49×10^{-14}
1.00	0.6	2.056091	2.071190	2.071187	2.071187	6.15×10^{-11}

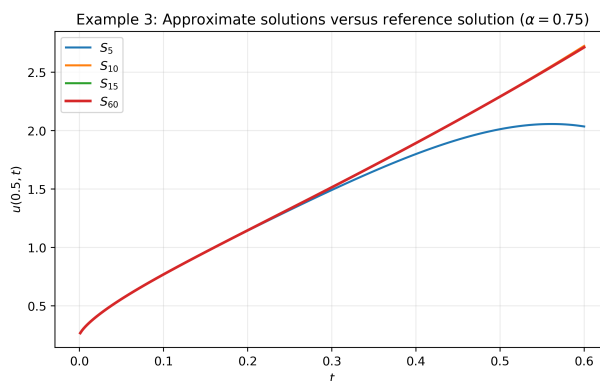


Figure 5: Comparison of the truncated approximations S_5 , S_{10} , and S_{15} with the higher-order reference solution S_{60} at $\alpha = 0.75$ and $x = 0.5$.

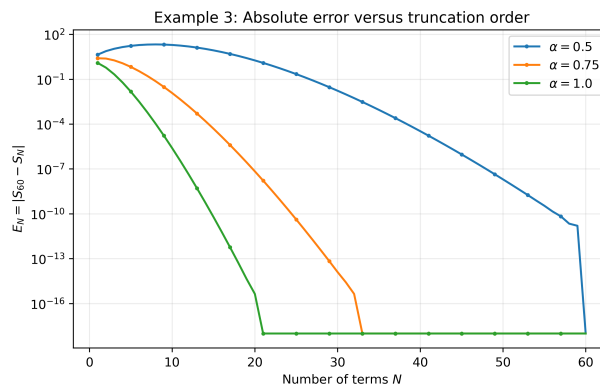


Figure 6: Absolute truncation error $E_N = |S_{60} - S_N|$ versus the number of terms N for different values of α .

Figure 6 shows that the absolute truncation error decreases as the number of terms N increases. The error is relatively larger for $\alpha = 0.5$ and decreases faster for larger values of α , indicating improved convergence of the proposed iterative method.

Figure 7 presents the three-dimensional surfaces of the reference solution S_{60} , the approximation S_{15} , and the corresponding absolute error for $\alpha = 0.5, 0.75$, and 1.0 . The surfaces demonstrate the dependence of the approximation accuracy on the fractional order and the truncation level.

Example 4.

Consider the diffusion equation with exponential data

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < \alpha \leq 1, \quad u(x, 0) = e^x.$$

As in Example 1, $a_2 = -1$, and since $D^2 e^x = e^x$ the iterates do not alternate in sign:

$$u_0 = e^x, \quad u_1 = \frac{t^\alpha}{\Gamma(\alpha + 1)} e^x, \quad u_2 = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} e^x, \quad \dots, \quad u(x, t) = e^x E_\alpha(t^\alpha).$$

At $\alpha = 1$, $E_1(t) = e^t$, so $u(x, t) = e^{x+t}$, which indeed solves $u_t = u_{xx}$ exactly. Table 4 compares S_5 and S_{15} .

Example 5. Consider the wave equation with polynomial data

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \alpha \leq 2, \quad u(x, 0) = x^2, \quad u_t(x, 0) = 0.$$

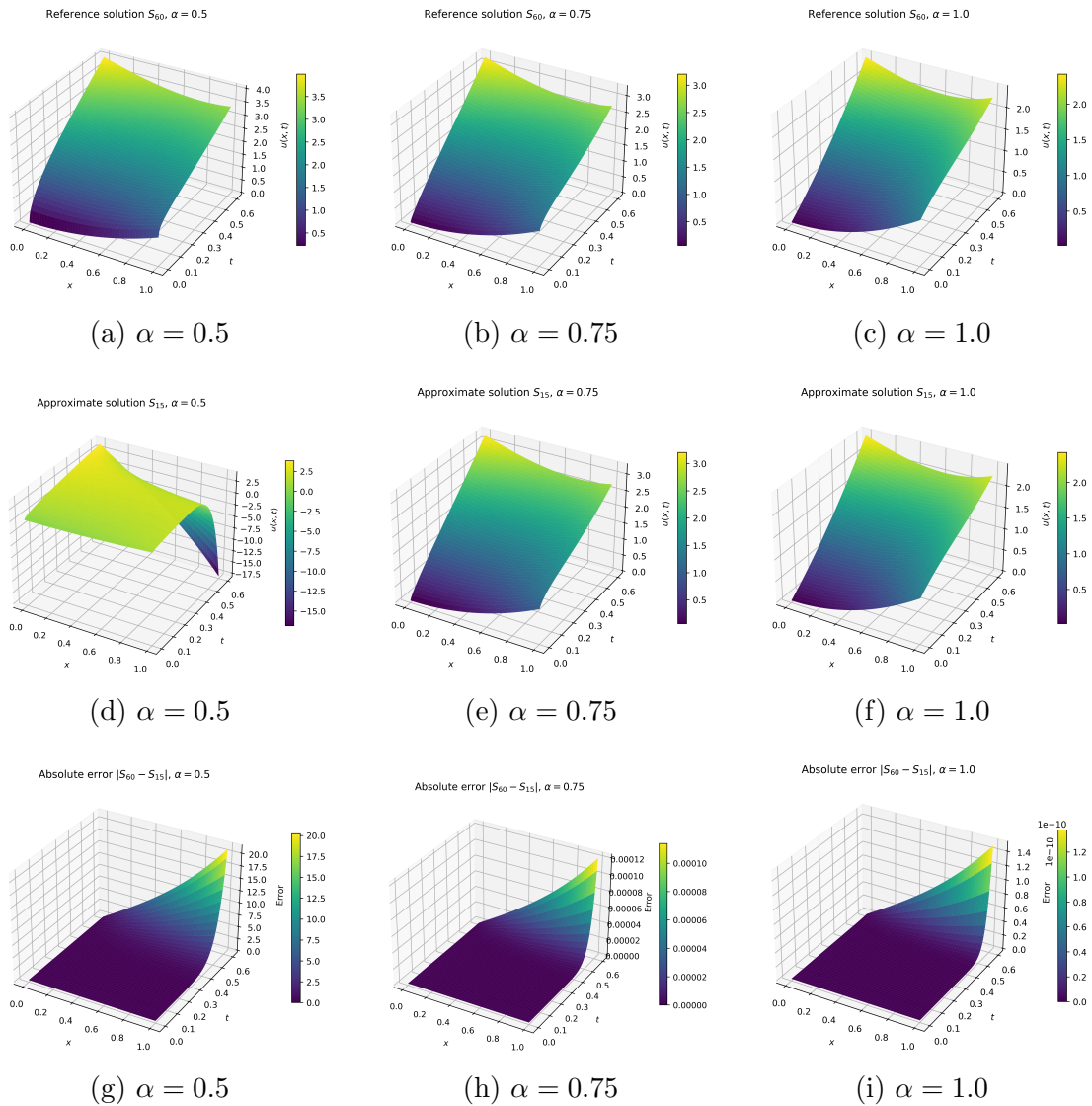


Figure 7: Three-dimensional graphical representation of the reference solution $S_{60}(x, t)$, approximate solution $S_{15}(x, t)$, and corresponding absolute error for Example 3 at $\alpha = 0.5, 0.75$, and 1.0 .

Table 4: Example 4: S_5 vs. S_{15} for $u(x, t) = e^x E_\alpha(t^\alpha)$.

x	$t = 0.2$		$t = 0.4$		$t = 0.6$	
	S_5	S_{15}	S_5	S_{15}	S_5	S_{15}
$\alpha = 0.5$						
0.25	2.307770	2.309984	3.100158	3.120237	3.964508	4.039817
0.5	2.963235	2.966078	3.980681	4.006464	5.090529	5.187227
0.75	3.804869	3.808520	5.111296	5.144401	6.536369	6.660532
1.0	4.885548	4.890236	6.563034	6.605542	8.392864	8.552292
$\alpha = 0.75$						
0.25	1.803621	1.803640	2.311369	2.311831	2.902603	2.905640
0.5	2.315896	2.315920	2.967856	2.968449	3.727016	3.730915
0.75	2.973669	2.973700	3.810803	3.811564	4.785583	4.790590
1.0	3.818266	3.818307	4.893168	4.894146	6.144810	6.151239
$\alpha = 1.0$ (exact: e^{x+t})						
0.25	1.568312	1.568312	1.915533	1.915541	2.339556	2.339647
0.5	2.013753	2.013753	2.459593	2.459603	3.004049	3.004166
0.75	2.585709	2.585710	3.158180	3.158193	3.857276	3.857426
1.0	3.320117	3.320117	4.055184	4.055200	4.952840	4.953032

Again $a_2 = -1$. Since $D^2 x^2 = 2$ is independent of x , $L[u_1] = 0$ and the recursion terminates, giving an exact closed form after two terms:

$$u_0 = x^2, \quad u_1 = \frac{2t^\alpha}{\Gamma(\alpha+1)}, \quad u_r = 0 \ (r \geq 2), \quad u(x, t) = x^2 + \frac{2t^\alpha}{\Gamma(\alpha+1)}.$$

This is verified to satisfy the PDE exactly for every $\alpha \in (1, 2]$ (using $D_t^\alpha t^\alpha = \Gamma(\alpha+1)$): $D_t^\alpha u = 2 = u_{xx}$. At $\alpha = 2$, $u = x^2 + t^2$, the classical d'Alembert solution of $u_{tt} = u_{xx}$ with these data. Table 5 lists exact values.

Example 6. Consider the inhomogeneous transport-diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = x + t, \quad 0 < \alpha \leq 1, \quad u(x, 0) = 0.$$

Here $a_1(x) = 1$ (no sign flip needed), $q = x + t$:

$$u_0 = \frac{x t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad u_1 = -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad u_r = 0 \ (r \geq 2),$$

again an exact, finite closed form:

$$u(x, t) = \frac{x t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Table 5: Example 5 (exact): $u(x, t) = x^2 + 2t^\alpha/\Gamma(\alpha + 1)$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$
$\alpha = 1.5$			
0.25	0.197067	0.443113	0.761731
0.5	0.384567	0.630613	0.949231
0.75	0.697067	0.943113	1.261731
1.0	1.134567	1.380613	1.699231
$\alpha = 1.75$			
0.25	0.136879	0.312680	0.571141
0.5	0.324379	0.500180	0.758641
0.75	0.636879	0.812680	1.071141
1.0	1.074379	1.250180	1.508641
$\alpha = 2.0$ (exact: $x^2 + t^2$)			
0.25	0.102500	0.222500	0.422500
0.5	0.290000	0.410000	0.610000
0.75	0.602500	0.722500	0.922500
1.0	1.040000	1.160000	1.360000

A direct check with $D_t^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}$ confirms $D_t^\alpha u + u_x = x + t$ exactly, for every $\alpha \in (0, 1]$. At $\alpha = 1$, $u(x, t) = xt$, matching the exact solution obtained by the method of characteristics for $u_t + u_x = x + t$, $u(x, 0) = 0$. Table 6 lists exact values.

5.1. Comparison with ADM and VIM

To further validate the proposed ISTM, its numerical results are compared with the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) for Example 1. For the present linear fractional diffusion equation, the three approaches lead to the same recursive fractional series at an equal truncation order. In particular, the recursive relation can be written as

$$u_{n+1}(x, t) = I_t^\alpha \left[\frac{\partial^2 u_n(x, t)}{\partial x^2} \right], \quad n = 0, 1, 2, \dots,$$

with

$$u_0(x, t) = \sin x.$$

Consequently, the corresponding N -term approximations satisfy

$$S_N^{\text{ISTM}} = S_N^{\text{ADM}} = S_N^{\text{VIM}}.$$

Table 6: Example 6 (exact): $u(x, t) = xt^\alpha/\Gamma(\alpha+1) + t^{\alpha+1}/\Gamma(\alpha+2) - t^{2\alpha}/\Gamma(2\alpha+1)$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$
$\alpha = 0.5$			
0.25	-0.006560	-0.031281	-0.031875
0.5	0.119597	0.147131	0.186635
0.75	0.245753	0.325544	0.405145
1.0	0.371910	0.503956	0.623654
$\alpha = 0.75$			
0.25	0.051258	0.071600	0.090147
0.5	0.132610	0.208417	0.275589
0.75	0.213961	0.345234	0.461031
1.0	0.295313	0.482051	0.646474
$\alpha = 1.0$ (exact: xt)			
0.25	0.050000	0.100000	0.150000
0.5	0.100000	0.200000	0.300000
0.75	0.150000	0.300000	0.450000
1.0	0.200000	0.400000	0.600000

Thus, the comparison is performed at the same truncation order $N = 5$ to ensure a fair assessment. The results are presented in Table 7 and Figure 8. The overlapping numerical approximations confirm the consistency of the proposed ISTM with the established ADM and VIM approaches for this benchmark problem.

Table 7: Comparison of ISTM, ADM, and VIM at equal truncation order $N = 5$ for Example 1.

α	x	t	u_{exact}	ISTM ₅	ADM ₅	VIM ₅	E_5
0.5	0.25	0.2	0.1592757674	0.1590083255	0.1590083255	0.1590083255	2.67×10^{-4}
0.5	0.50	0.2	0.3086485391	0.3081302834	0.3081302834	0.3081302834	5.18×10^{-4}
0.5	0.75	0.4	0.3773594804	0.3718985149	0.3718985149	0.3718985149	5.46×10^{-3}
0.5	1.00	0.6	0.4190732242	0.3975429013	0.3975429013	0.3975429013	2.15×10^{-2}

$$E_5 = |u_{\text{exact}} - S_5|.$$

Figure 8 shows that the ISTM, ADM, and VIM approximations coincide at the same truncation order $N = 5$. This agreement confirms the consistency of the proposed ISTM formulation with the established methods for the considered linear fractional diffusion problem.

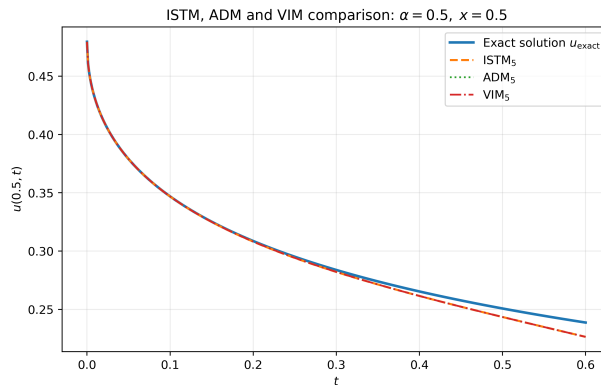


Figure 8: Comparison of the exact solution with the fifth-order approximations obtained using ISTM, ADM, and VIM for Example 1 with $\alpha = 0.5$ and $x = 0.5$.

6. Conclusion

In this work, an iterative method based on the Shehu transform is developed to solve linear time-fractional partial differential equations involving the Caputo fractional derivative. The proposed method uses the Shehu transform and an iterative procedure to obtain an equivalent Volterra-type integral equation and construct a recursive series solution. A convergence analysis is also provided for the resulting iterative sequence. The method was applied to six examples covering fractional diffusion, wave, transport-diffusion, and inhomogeneous equations. The numerical results show that the truncated series solutions get closer to the exact or higher-order reference solutions as the number of terms increases. The numerical tables and three-dimensional plots also show the effect of the fractional order and the number of terms on the accuracy of the solutions.

A comparison with ADM and VIM for a representative linear problem shows that the proposed method produces the same recursive series at the same truncation order. The results obtained by the three approaches are in good agreement, confirming the consistency of the proposed formulation with these established methods. Overall, the numerical and analytical results confirm the accuracy, convergence, and applicability of the proposed iterative approach.

References

- [1] Aggarwal S., Sharma SD., Gupta AR., Application of Shehu transform for handling growth and decay problems, *Global J Eng Sci Res.*, 6(4) (2019), 190-198.

- [2] Bekela A. S., Deresse A. T., An efficient numerical method for nonlinear time fractional hyperbolic partial differential equations based on fractional Shehu transform iterative method, *Journal of Applied Mathematics*, (2025), 7007124.
- [3] Belgacem R., Baleanu D., Bokhari A., Shehu transform and applications to Caputo fractional differential equations, *Int. J. Anal. Appl.*, 17 (2019), 917-927.
- [4] Bhalekar, S., Daftardar-Gejji, V., Solving evolution equations using a new iterative method, *Numer. Methods PDEs*, 26(4) (2010), 906-916.
- [5] Chakraverty, S., Jena, R. M., Jena, S. K., *Computational Fractional Dynamical Systems: Fractional Differential Equations and Applications*, John Wiley & Sons, 2022.
- [6] Eltayeb H, Kilicman A., Application of Sumudu decomposition method to solve nonlinear system of partial differential equations, *Abstr. Appl. Anal.*, 2012 (2012), 1-13.
- [7] Jafari H., Khalique CM., Nazari M., Application of Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations, *Appl. Math. Lett.*, 24(11) (2011), 1799-1805.
- [8] Khan M., Hussain M., Application of Laplace decomposition method on semi-infinite domain, *Numer. Algorithms*, 56 (2011), 211-218.
- [9] Liao SJ., Homotopy analysis method and its applications in mathematics, *J. Basic Sci. Eng.*, 5(2) (1997), 111-125.
- [10] Magin, R. L., Fractional calculus models of complex dynamics in biological tissues, *Comput. Math. Appl.*, 59(5) (2010), 1586-1593.
- [11] Maitama S., Zhao W. New integral transform: Shehu transform, a generalization of Sumudu and Laplace transform for solving differential equations, *Int. J. Anal. Appl.*, 17 (2019), 167-190.
- [12] Maitama S., Zhao W., New homotopy analysis transform method for solving multidimensional fractional diffusion equations, *Arab Journal of Basic and Applied Sciences*, 27 (2020), 27-44.

- [13] Miller, K. S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, 1993.
- [14] Mohammad OH, Khlaif AL., Homotopy analysis method for solving delay differential equations of fractional order, Math. Theory Model, 4 (2004), 12-24.
- [15] Momani, S., Abdulaziz, O., Hashim, I., Application of homotopy-perturbation method to fractional IVPs, J. Comput. Math. Appl., 216 (2008), 574-584.
- [16] Momani, S., Odibat, Z., Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A, 355(4-5) (2006), 271-279.
- [17] Oldham, K., Spanier, J., The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Mathematics in Science and Engineering, Vol. 111 (1974).
- [18] Podlubny I., Fractional Differential Equations, Academic Press, San Diego, 1999.
- [19] Raut, S. R., Sontakke, B. R., Bhadgaonkar, V. N., Analytical solution of linear fractional partial differential equation of order $0 < \alpha \leq 1$ by improved Adomian decomposition method, J. Adv. Math. Comput. Sci., 37(5) (2022), 32-43.
- [20] Renardy M, Rogers RC., An Introduction to Partial Differential Equations, Springer, 2006.
- [21] Rezapour S., Liaqat M. I., Etemad S., Owolabi K. M., An effective new iterative method to solve conformable Cauchy reaction-diffusion equation via the Shehu transform, Journal of Mathematics, 4172218 (2022), 1-12.
- [22] Sonawane, J., Sontakke, B., Takale, K., Strong and mild solutions of the system of fractional ordinary differential equation and its applications, Malaya J. Mat., 8(4) (2020), 1443-1453.
- [23] Sontakke, B., Shaikh, A., Properties of Caputo operator and its applications to linear fractional differential equations, Int. J. Eng. Res. Appl., 5(5) (2015), 22-27.

- [24] Sontakke, B. R., Shelke, A. S., Shaikh, A. S., Solution of nonlinear fractional differential equations by variational iteration method and applications, Far East J. Math. Sci., 110(1) (2019), 113-129.
- [25] Thange, T. G., Gade, A., A new approach for solving a system of fractional partial differential equation, Adv. Math., 9(4) (2020), 1933-1944.
- [26] Thange T. G., Gade A. R., Analytical solution for time fractional diffusion equation by Laplace-Carson decomposition method, Our Heritage, 68(6) (2020), 45-50.
- [27] Thange T. G., Gade A. R., On the Aboodh transform of fractional differential operator, Malaya J. Mat., 8(1) (2020), 225-229.
- [28] Yadav L. K., Agarwal G., Suthar D. L., and Purohit S. D., Time-fractional partial differential equations: A novel technique for analytical and numerical solutions, Arab Journal of Basic and Applied Sciences, 29(1) (2022), 86–98.