

BOUNDS OF A SECOND HANKEL DETERMINANT FOR A NEW FAMILY OF ANALYTIC FUNCTIONS

Anuradha Vishnoi, Arvind Kumar Sharma and Saurabh Porwal*

Department of Mathematics,
Janta Mahavidyalaya,
Ajitmal - 206121, Auraiya, Uttar Pradesh, INDIA

E-mail : anuvishnoi16@gmail.com, aksharma.maths@gmail.com

*Department of Mathematics
Ram Sahai Rajkiya Mahavidyalaya,
Bairi, Shivrajpur, Kanpur - 209205, Uttar Pradesh, INDIA

E-mail : saurabhjcb@rediffmail.com

(Received: Feb. 04, 2026 Accepted: Aug. 23, 2026 Published: Aug. 30, 2026)

Abstract: The main object of this research article is to obtain sharp upper bound of second Hankel determinant for a new family of analytic functions defined by q -calculus. Relevant connections between the results presented here and various well-known findings are briefly discussed.

Keywords and Phrases: Analytic function, Hankel determinant, Schwarz function, q -calculus.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ stand for the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic within the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike functions of order α if it satisfies the condition

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{U}.$$

A function $f(z) \in \mathcal{A}$ is said to be convex of order α if it satisfies the criteria

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{U}.$$

The classes of starlike and convex functions of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. In particular, for $\alpha = 0$, these classes are simply denoted as $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$.

The families of starlike and convex functions of order α were extensively studied by Robertson [25] and Silverman [26].

A function $f \in \mathcal{A}$ is termed close-to-convex with respect to a fixed starlike function $g \in \mathcal{S}^*$ if and only if

$$\Re \left(\frac{zf'(z)}{g(z)} \right) > 0, \quad \text{for } z \in \mathbb{U}.$$

Further, we denote by $\mathcal{P}$ be the family of all analytic functions $\wp(z)$, satisfying the condition $\wp(0) = 0$ and $\Re(\wp(z)) > 0$.

If $\phi, \psi \in \mathcal{A}$, then the function ϕ is said to be subordinate to ψ , denoted by $\phi(z) \prec \psi(z)$ for $z \in \mathbb{U}$, if there exists a Schwarz function $\omega \in \mathcal{A}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$, and $\phi(z) = \psi(\omega(z))$. Specifically, if ψ is univalent in $\mathbb{U}$, then the following equivalence holds:

$$\phi(z) \prec \psi(z) \iff \phi(0) = \psi(0) \text{ and } \phi(\mathbb{U}) \subset \psi(\mathbb{U}).$$

The primary objective of this article is to investigate the applications and further developments of classical q -calculus in geometric function theory. As a natural q -analogue of classical calculus, q -calculus provides an effective framework for extending various concepts and problems of classical analysis to a broader setting. Historically, Jackson [10, 11] was among the pioneers in the systematic development of q -calculus. Since then, q -calculus and its associated operators have found numerous applications in diverse areas of mathematics, including special functions, approximation theory, fractional calculus, and geometric function theory. For basic as well as advanced aspects of q -calculus, the reader is referred to the excellent text book by Aral et al. [2], see also the survey and expository review articles by Srivastava [27, 28].

They defined the q -number for $k \in \mathbb{N}$ in the following way

$$[k]_q = \frac{1 - q^k}{1 - q}, \quad 0 \leq q < 1.$$

We express $[k]_q$ as

$$[k]_q = \sum_{i=0}^{k-1} q^i.$$

Obviously,

$$\lim_{k \rightarrow \infty} [k]_q = \frac{1}{1 - q}$$

and $\lim_{q \rightarrow 1} [k]_q = k$.

The q -derivative for a function f is given below

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q - 1)z}, \quad q \neq 1, \quad z \neq 0$$

and $D_q(f(0)) = f'(0)$, provided $f'(0)$ exists. If we take the function $h(z) = z^k$ then the q -derivative of $h(z)$ is defined as

$$\begin{aligned} \lim_{q \rightarrow 1} D_q(h(z)) &= \lim_{q \rightarrow 1} [k]_q z^{k-1} \\ &= k z^{k-1} \\ &= h'(z), \end{aligned}$$

where $h'(z)$ is the first differential coefficient of $h(z)$ with respect to z .

Definition 1.1. Let $0 \leq \vartheta \leq 1$, $0 \leq q < 1$, $\varrho \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ is in the family $\mathcal{R}_\vartheta^q(q, A, B)$ if it holds the following subordination criteria

$$1 + \frac{1}{\varrho} [D_q\{(1 - \vartheta)f(z) + \vartheta z f'(z)\} - 1] \prec \frac{1 + Az}{1 + Bz}, \quad (1.2)$$

where $-1 \leq B < A \leq 1, z \in \mathbb{U}$.

The condition (1.2) is equivalent to

$$\left| \frac{D_q\{(1 - \vartheta)f(z) + \vartheta z f'(z)\} - 1}{\varrho(A - B) - B[D_q\{(1 - \vartheta)f(z) + \vartheta z f'(z)\} - 1]} \right| < 1. \quad (1.3)$$

Remark 1.1. For $f \in \mathcal{A}$ and by fixing the parameters in $\mathcal{R}_\vartheta^q(q, A, B)$, we obtain the following families which mentioned below

1. $\mathcal{R}_\vartheta^\varrho(1, A, B) \equiv \mathcal{R}_\vartheta^\varrho(A, B)$ studied by Bansal [3].
2. $\mathcal{R}_\vartheta^\varrho(1, 1 - 2\beta, -1) \equiv \mathcal{R}_\vartheta^\varrho(\beta)$ for $0 \leq \beta < 1$ studied by Swaminathan [29].
3. $\mathcal{R}_0^1(1, 1, -1) \equiv \mathcal{R}$ studied by Janteng et al. [12].
4. $\mathcal{R}_\vartheta^1(0, 1, -1) \equiv \mathcal{R}_\vartheta$ studied by Murugusundaramoorthy and Magesh [19].
5. $\mathcal{R}_0^\varrho(1, A, B) \equiv \mathcal{R}^\varrho(A, B)$ studied by Dixit and Pal [7].
6. $\mathcal{R}_\vartheta^\varrho(1, 1 - 2\beta, -1) = \mathcal{R}_\vartheta^\varrho(1 - 2\beta, 1)$ for $\varrho = e^{i\eta} \cos \eta$ where $\frac{-\pi}{2} < \eta < \frac{\pi}{2}$ studied by Ponnusamy and Rønning [24].
7. $\mathcal{R}_1^\varrho(1, 0, -1) \equiv \mathcal{R}_1^\varrho(0, -1)$ with $\varrho = e^{i\eta} \cos \eta$ studied by Lin [17] with reference to the univalence of partial sums.

The study of coefficient problems has always occupied a central position in geometric function theory. In addition to classical problems concerning coefficient estimates and the Fekete–Szegő functional, determinant problems involving Hankel and Toeplitz determinants have attracted increasing attention. In fact the study of these determinants are a current and interesting research problem in geometric function theory. The problem of Hankel determinant was first proposed by Noonan and Thomas [20] in 1976. The j^{th} Hankel determinant for function $f(z)$ defined by (1.1) is given as

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+j} \\ \vdots & \vdots & \vdots \\ a_{n+j-1} & a_{n+j} \cdots & a_{n+2j-2} \end{vmatrix}.$$

Hankel determinants play an important role in the study of singularities and power series with integral coefficients; see, for example, [6]. Further properties and applications of Hankel determinants may be found in [30]. In recent years, considerable attention has been devoted to obtaining sharp upper bounds for Hankel determinants associated with various subclasses of univalent functions. Significant contributions in this direction have been made by Janteng et al. [12, 13], Lee et al. [16], Mishra and Gochhayat [18], Murugusundaramoorthy and Magesh [19], Orhan et al. [22], and Noor and Al-Naggar [21]. Similarly, Toeplitz determinants have been systematically investigated by Çağlar et al. [4], Cotîrlă and Wanas [5], and Wanas et al. [31]. In 2013, Bansal [3] obtained bounds for Hankel determinants for a generalized class of univalent functions and generalized the results of [12, 19].

Recently, q -calculus has been successfully employed in the study of coefficient functionals, including Hankel determinants and the Fekete–Szegő problem, for various classes of univalent and bi-univalent functions. Notable contributions in this direction include the works of Orhan et al. [23], Ali et al. [1], Karahuseyin et al. [14], Yalcin et al. [32], and Khan et al. [15]. Motivated by these developments, in the present work, we obtain an upper bound for the second Hankel determinant associated with a generalized family of univalent functions defined through q -calculus. In this paper, we establish an upper bound for the second Hankel determinant $H_2(2)$ for functions $f \in \mathcal{R}_\vartheta^q(q, A, B)$. We also show that the obtained bound is sharp. Further, for suitable choices of the parameters, our result reduces to the corresponding bounds for several well-known classes of analytic functions.

2. Hankel determinant inequality for $f \in \mathcal{R}_\vartheta^q(q, A, B)$.

To prove our main results we shall require the following lemmas.

Lemma 2.1. [8] *Let the function $\wp \in \mathcal{P}$ and be given by the series*

$$\wp(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \quad (z \in \mathbb{U}), \quad (2.1)$$

then, the sharp estimate $|c_n| \leq 2$ ($n \in \mathbb{N}$) holds.

Lemma 2.2. [9] *Let the function $\wp \in \mathcal{P}$ be given by the power series (2.1), then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.2)$$

for some x , $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \quad (2.3)$$

for some z , $|z| \leq 1$.

Theorem 2.1. *Let the function $f(z)$ be given by (1.1) in the class $\mathcal{R}_\vartheta^q(q, A, B)$, where $0 \leq q < 1$, $0 \leq \vartheta \leq 1$, $\varrho \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{|\varrho|^2(A - B)^2}{(1 + q + q^2)^2(1 + 2\vartheta)^2}. \quad (2.4)$$

The result is sharp for the function $f(z)$ of the form (1.1) that satisfies the relation

$$D_q[(1 - \vartheta)f(z) + \vartheta zf'(z)] = 1 + \frac{(A - B)\varrho z^2}{1 + Bz^2}. \quad (2.5)$$

Proof. Let $f(z) \in \mathcal{R}_\vartheta^q(q, A, B)$, then there exists a Schwarz analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$, such that

$$1 + \frac{1}{\varrho}\{D_q[(1 - \vartheta)f(z) + \vartheta zf'(z)] - 1\} = \phi(w(z)), \quad z \in \mathbb{U}, \quad (2.6)$$

where

$$\begin{aligned}\phi(z) &= \frac{1 + Az}{1 + Bz} \\ &= 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \dots \\ &= 1 + B_1z + B_2z^2 + B_3z^3 + \dots, \quad (z \in \mathbb{U}).\end{aligned}\quad (2.7)$$

Define the function $\wp_1(z)$ by

$$\wp_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots (z \in \mathbb{U}). \quad (2.8)$$

Since $w(z)$ is a Schwarz function, we observe that $\Re\{\wp_1(z)\} > 0$ and $\wp_1(0) = 1$. Let $h(z)$ be given by

$$\begin{aligned}h(z) &= 1 + \frac{1}{\varrho} [D_q\{(1 - \vartheta)f(z) + \vartheta zf'(z)\} - 1] \\ &= 1 + \frac{1}{\varrho} \left[1 + \sum_{k=2}^{\infty} [1 - \vartheta + \vartheta k] [k]_q a_k z^{k-1} - 1 \right] \\ &= 1 + \frac{1}{\varrho} \left[\sum_{k=2}^{\infty} [1 - \vartheta + \vartheta k] [k]_q a_k z^{k-1} \right] \\ &= 1 + \frac{1}{\varrho} (1 + \vartheta) [2]_q a_2 z + \frac{1}{\varrho} (1 + 2\vartheta) [3]_q a_3 z^2 + \frac{1}{\varrho} (1 + 3\vartheta) [4]_q a_4 z^3 \dots \\ &= 1 + b_1 z + b_2 z^2 + \dots\end{aligned}\quad (2.10)$$

From equation (2.6) and (2.9), we have

$$\begin{aligned}h(z) &= \phi(w(z)) \\ &= \phi\left(\frac{\wp_1(z) - 1}{\wp_1(z) + 1}\right), \quad \text{from (2.8)} \\ &= \phi\left(\frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + \dots}\right) \\ &= \phi\left[\frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \dots\right] \\ &= 1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2 \\ &\quad + \left[\frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}\right]z^3 + \dots\end{aligned}$$

Thus, $b_1 = \frac{1}{2}B_1c_1$, $b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}$ and

$$b_3 = \frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}, \text{ from (2.13).} \quad (2.11)$$

Using (2.7), (2.13) and (2.11), we obtain

$$\begin{aligned} a_2 &= \frac{(A-B)c_1\varrho}{2(1+\vartheta)(1+q)} \\ a_3 &= \frac{(A-B)\varrho}{4(1+2\vartheta)(1+q+q^2)} [2c_2 - c_1^2(1+B)] \\ a_4 &= \frac{\varrho(A-B)}{8(1+q^2)(1+q)(1+3\vartheta)} [4c_3 - 4c_1c_2(1+B) + c_1^3(1+B)^2]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{|\varrho|^2(A-B)^2}{16(1+\vartheta)(1+3\vartheta)(1+q^2)(1+q)^2} |4c_3c_1 - 4c_1^2c_2(1+B) + c_1^4(1+B)^2 \\ &\quad - \frac{(1+\vartheta)(1+3\vartheta)(1+q^2)(1+q)^2}{(1+2\vartheta)^2(1+q+q^2)^2} \{4c_2^2 - 4c_1^2c_2(1+B) + c_1^4(1+B)^2\}| \\ &= T \left| 4c_1c_3 - 4c_1^2c_2(1+B) + c_1^4(1+B)^2 - 4pc_2^2 + 4c_1^2c_2p(1+B) - c_1^4p(1+B)^2 \right| \\ &= T |4c_1c_3 + c_1^4(1+B)^2(1-p) - 4pc_2^2 - 4c_1^2c_2(1+B)(1-p)|, \end{aligned}$$

where

$$T = \frac{|\varrho|^2(A-B)^2}{16(1+\vartheta)(1+3\vartheta)(1+q^2)(1+q)^2}$$

and

$$p = \frac{(1+\vartheta)(1+3\vartheta)(1+q^2)(1+q)^2}{(1+2\vartheta)^2(1+q+q^2)^2}.$$

It is easy to see that for $0 \leq \vartheta \leq 1$, $0 \leq q < 1$, $p \in \left[\frac{64}{81}, 1\right)$.

The last equation can be written as

$$|a_2a_4 - a_3^2| = T |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|, \quad (2.12)$$

where

$$d_1 = 4; \quad d_2 = -4(1+B)(1-p); \quad d_3 = -4p \quad \text{and} \quad d_4 = (1-p)(1+B)^2.$$

Since the function $\wp(z)$ and $\wp(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) both belong to the class $\mathcal{P}$ simultaneously, we can assume, without loss of generality, that $c_1 > 0$. For simplicity, we set $c_1 = c$ where $c \in [0, 2]$. Now, putting the values of c_2 and c_3 from Lemma 2.2 in equation (2.12), we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3) \right. \\ &\quad \left. + (4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2z) \right| \\ &\leq \frac{T}{4} \left[4c^4(1 - p)B^2 + 8|B|(1 - p)\mu c^2(4 - c^2) \right. \\ &\quad \left. + (4 - c^2)\mu^2(4c^2 + 4p(4 - c^2)) + 8c(4 - c^2)(1 - \mu^2) \right] \\ &= T \left[c^4(1 - p)B^2 + 2c(4 - c^2) + 2\mu|B|(1 - p)c^2(4 - c^2) \right. \\ &\quad \left. + (4 - c^2)\mu^2(1 - p)(c - \alpha)(c - \beta) \right] \\ &= \mathcal{F}(c, \mu), \end{aligned} \tag{2.13}$$

where $\mu = |x|$, $\alpha = 2$, $\beta = \frac{2p}{1-p} > 2$.

Now, suppose that the upper bound of $|a_2a_4 - a_3^2|$, given by (2.13), is attained at an interior point of the rectangle $[0, 2] \times [0, 1]$. Differentiating $\mathcal{F}(c, \mu)$ partially with respect to μ , we obtain

$$\frac{\partial \mathcal{F}}{\partial \mu} = T \left[2|B|(1 - p)c^2(4 - c^2) + 2\mu(4 - c^2)(1 - p)(c - \alpha)(c - \beta) \right]. \tag{2.14}$$

For $0 < \mu < 1$, $0 < c < 2$, $\frac{64}{81} \leq p < 1$, $\alpha = 2$, and $\beta = \frac{2p}{1-p} > 2$, we observe that both terms $(1 - p)c^2(4 - c^2)$ and $2\mu(4 - c^2)(1 - p)(c - \alpha)(c - \beta)$ are positive. Therefore, from (2.14), we obtain

$$\frac{\partial \mathcal{F}}{\partial \mu} > 0.$$

Hence, $\mathcal{F}(c, \mu)$ is an increasing function of μ . This contradicts our initial assumption that the maximum value of $\mathcal{F}(c, \mu)$ is attained at an interior point of the rectangle $[0, 2] \times [0, 1]$. Consequently, for each fixed $c \in [0, 2]$, we have

$$\mathcal{F}(c, \mu) = \mathcal{F}(c, 1) = K(c),$$

where

$$K(c) = T \left[c^4(1 - p)(B^2 - 2|B| - 1) + 4c^2\{2|B|(1 - p) + 1 - 2p\} + 16p \right].$$

Next, differentiating $K(c)$ with respect to c , we obtain

$$\begin{aligned}
 K'(c) &= T [4c^3(1-p)(B^2 - 2|B| - 1) + 8c\{2|B|(1-p) + 1 - 2p\}] \\
 &= 4cT [c^2(1-p)(B^2 - 2|B| - 1) + 2\{2|B|(1-p) + 1 - 2p\}] \\
 &= 4cT [c^2(1-p)(B^2 - 2|B| - 1) + 2\{(1-p)(2|B| + 1) - p\}] \\
 &\leq 8T(1-p) \left[4(B^2 - 2|B| - 1) + 2 \left\{ 2|B| + 1 - \frac{p}{1-p} \right\} \right], \quad \text{from Lemma 2.1.} \\
 &\leq 8T(1-p) \left[4 \left(B^2 - |B| - \frac{1}{2} \right) - \frac{2p}{1-p} \right].
 \end{aligned}$$

Since $-1 \leq B < A \leq 1$, we have $-1 \leq B < 1$, and hence

$$B^2 - |B| - \frac{1}{2} < 0.$$

Moreover, $\frac{2p}{1-p} > 2$. Therefore, it follows that

$$K'(c) < 0.$$

Thus, $K(c)$ is a decreasing function on $[0, 2]$. Consequently, $K(c)$ attains its maximum value at $c = 0$. Hence,

$$\max_{(c, \mu) \in [0, 2] \times [0, 1]} \mathcal{F}(c, \mu) = \mathcal{F}(0, 1).$$

Therefore, the upper bound of $\mathcal{F}(c, \mu)$ is attained at $c = 0$ and $\mu = 1$. Hence,

$$\begin{aligned}
 |a_2 a_4 - a_3^2| &\leq 16pT \\
 &= \frac{|\varrho|^2 (A - B)^2}{(1 + 2\vartheta)^2 (1 + q + q^2)^2}.
 \end{aligned}$$

Finally, we establish the sharpness of the above result by considering the function $f(z)$ defined by (2.5). The proof proceeds as follows.

Step I. We first show that the function $f(z)$, defined by (2.5), belongs to the class $\mathcal{R}_{\vartheta}^q(q, A, B)$. For this purpose, it is sufficient to verify that $f(z)$ satisfies condition (1.3). Indeed,

$$\left| \frac{D_q \{(1 - \vartheta)f(z) + \vartheta z f'(z)\} - 1}{\varrho(A - B) - B [D_q \{(1 - \vartheta)f(z) + \vartheta z f'(z)\} - 1]} \right|$$

$$\begin{aligned}
&= \left| \frac{\frac{(A-B)\varrho z^2}{1+Bz^2}}{\varrho(A-B) - B \left[\frac{(A-B)\varrho z^2}{1+Bz^2} \right]} \right| \\
&= \left| \frac{z^2}{1+Bz^2 - Bz^2} \right| = |z^2| < 1, \quad \text{since } z \in \mathbb{U}.
\end{aligned}$$

Hence, $f(z) \in \mathcal{R}_\vartheta^q(q, A, B)$.

Step II. Next, we determine the coefficients a_2 , a_3 , and a_4 for the function $f(z)$ defined by (1.1) and satisfying (2.5). We have

$$\begin{aligned}
D_q \{ (1 - \vartheta)f(z) + \vartheta z f'(z) \} &= 1 + \frac{(A-B)\varrho z^2}{1+Bz^2} \\
1 + \sum_{k=2}^{\infty} [1 - \vartheta + \vartheta k] [k]_q a_k z^{k-1} &= (A-B)\varrho z^2 (1+Bz^2)^{-1}.
\end{aligned}$$

Therefore,

$$\sum_{k=2}^{\infty} [1 - \vartheta + \vartheta k] [k]_q a_k z^{k-1} = (A-B)\varrho z^2 (1 - Bz^2 + B^2 z^4 - \dots).$$

Comparing the coefficients of z , z^2 , and z^3 , respectively, we obtain

$$[2]_q (1 + \vartheta) a_2 = 0,$$

and hence

$$a_2 = 0.$$

Further,

$$[3]_q (1 + 2\vartheta) a_3 = (A-B)\varrho,$$

which yields

$$a_3 = \frac{(A-B)\varrho}{[3]_q (1 + 2\vartheta)} = \frac{(A-B)\varrho}{(1+q+q^2)(1+2\vartheta)}.$$

Similarly,

$$[4]_q (1 + 3\vartheta) a_4 = 0,$$

and consequently,

$$a_4 = 0.$$

Step III. Finally, using the values of a_2 , a_3 , and a_4 obtained above, we find

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| 0 - \left(\frac{(A-B)\varrho}{(1+q+q^2)(1+2\vartheta)} \right)^2 \right| \\ &= \frac{|\varrho|^2(A-B)^2}{(1+2\vartheta)^2(1+q+q^2)^2}. \end{aligned}$$

Thus, the equality is attained for the function $f(z)$ defined by (2.5), which establishes the sharpness of the result and completes the proof.

If we make $q \rightarrow 1^-$ in Theorem 2.1 then we obtain the following result of Bansal [3].

Corollary 2.1. *Let the function $f(z)$ be defined by (1.1) be in the family $\mathcal{R}_\vartheta^0(A, B)$, where $0 \leq \vartheta \leq 1$, $\varrho \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{|\varrho|^2(A-B)^2}{9(1+2\vartheta)^2}.$$

The result is sharp for the function $f(z)$ of the form (1.1) that satisfies the relation

$$f'(z) + \vartheta z f''(z) = 1 + \frac{(A-B)\rho z^2}{1+Bz^2}. \quad (2.15)$$

If we take $\vartheta = 0$, $A = 1$, $B = -1$, $\varrho = 1$ and $q \rightarrow 1$ then we obtain the corresponding result of Janteng et al. [12].

Corollary 2.2. *Let the function $f(z)$ be defined by (1.1) be in the family $\mathcal{R}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

The result is sharp for the function $f(z)$ which satisfies the relation

$$f'(z) = \frac{1+z^2}{1-z^2}. \quad (2.16)$$

If we take $A = 1$, $B = -1$, $\varrho = 1$ and $q = 0$ then we obtain the corresponding result in [19].

Corollary 2.3. *Let the function $f(z)$ be defined by (1.1) be in the family $\mathcal{R}_\vartheta$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{(1+2\vartheta)^2}.$$

The result is sharp for the function $f(z)$ of the form (1.1) that satisfies the relation

$$(1-\vartheta)f(z) + \vartheta z f'(z) = \frac{1+z^2}{1-z^2}. \quad (2.17)$$

3. Conclusion

In this article, we have investigated the second Hankel determinant for a generalized class of univalent functions defined by means of q -calculus. In particular, we have obtained a sharp upper bound for the second Hankel determinant $H_2(2)$ for functions belonging to the considered class. The sharpness of the obtained bound demonstrates the effectiveness of the method employed in the present study. Furthermore, by assigning suitable values to the parameters involved in the defining condition, our main result yields several previously known results for certain well-known subclasses of univalent functions as special cases. Thus, the present result provides a unified generalization of these existing findings. The present work opens up new directions for further investigations of higher-order Hankel determinants, Toeplitz determinants, and other coefficient problems for generalized classes of analytic and bi-univalent functions involving q -calculus and related operators.

Acknowledgment

The authors sincerely thank the referee for their valuable comments and suggestions, which have greatly improved the manuscript.

References

- [1] Ali Ekram E., Srivastava H. M., Kota W. Y., El-Ashwah R. M. and Albalah A. M., The second Hankel determinant and the Fekete-Szegő functional for a subclass of analytic functions by using the q -Salagean derivative operator, Alexandria Engineering J., 116 (2025), 141-146.
- [2] Aral A., Gupta V. and Agarwal R. P., Applications of q -calculus in operator theory, New York, NY; Springer, 2013.
- [3] Bansal D., Upper bound of second Hankel determinant for a new class of analytic functions, Appl. Math. Lett., 26(1) (2013), 103-107.
- [4] Caglar M., Ibrahim I. O., Shaba T. G. and Wanas A. K., Toeplitz determinants for λ Pseudo-starlike functions, Commun. Korean Math. Soc., 39(3) (2024), 647-655.
- [5] Cotîrlă L. I. and Wanas A. K., Symmetric Toeplitz matrices for a new family of Prestarlike functions, Symmetry, 14(7) (2022), 1413.
- [6] Dienes P., The Taylor series: an introduction to the theory of functions of a complex variable, New York, NY, USA: Dover, 1957.

- [7] Dixit K. K. and Pal S. K., On a class of univalent functions related to complex order, *Indian J. Pure Appl. Math.*, 26(9) (1995), 889-896.
- [8] Duren P. L., *Univalent Functions*, Grundleherem der Mathematischen Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
- [9] Grenander U. and Szegö G., *Toeplitz forms and their applications*, 1958.
- [10] Jackson F. H., On q -definite integrals, *Quart. J. Pure Appl. Math.*, 41 (1910), 193–203 .
- [11] Jackson F. H., q -difference equations, *Am. J. Math.*, 32 (1910), 305–314.
- [12] Janteng A., Halim S. A. and Darus M., Coefficient inequality for a function whose derivative has a positive real part, *J. Inequalities Pure Appl. Math.*, 7(2) (2006), Art. 50, 1-5.
- [13] Janteng A., Halim S. A. and Darus M., Hankel Determinant for starlike and convex functions, *J. Math. Anal.*, 1(13) (2007), 619-625.
- [14] Karahuseyin Z., Altinkaya S. and Yalcin S., On $H_3(1)$ Hankel determinant for univalent functions defined by using q - derivative operator, *Transylvanian J. Math. Mechanics*, 9(1) (2017), 25-33.
- [15] Khan M., Khan N., Xin Q., Tchier F., Malik S. N. and Ahmad Q. Z., Hankel determinant and related problems for q - analogue of convex functions, *Filomat*, 38(9) (2024), 3027–3040.
- [16] Lee S. K., Ravichandran V and Supramaniam S., Bounds for the second Hankel determinant of certain univalent functions, 2013 (2013), 281, 1-17.
- [17] Lin Li J., On some classes of analytic functions, *Math. Jpn.*, 40(3) (1994), 523-529.
- [18] Mishra A. K. and Gochhayat P., Second Hankel determinant for a class of analytic functions defined by fractional derivative, *Int. J. Math. Math. Sci.*, (2008), Art. ID, 1-10.
- [19] Murugusundaramoorthy G. and Magesh N., Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant, *Bull. Math. Anal. Appl.*, 1(3) (2009), 85–89

- [20] Noonan J. and Thomas D., On the second Hankel determinant of a really mean p -valent functions, *Trans. Amer. Math. Soc.*, 223(2) (1976), 337-346.
- [21] Noor K. I. and Al-Naggar I. M., On the Hankel determinant problem, *J. Nat. Geom.*, 14 (1998), 133-140.
- [22] Orhan H., Magesh N. and Yamini J., Bounds for the second Hankel determinant of certain bi-univalent functions, *Turk. J. Math.*, 40 (2016), 679-687.
- [23] Orhan H., Porwal S. and Magesh N., The Fekete-Szegő Problem for a generalized class of analytic functions of complex order associated with q -calculus, *Palestine J. Math.*, 11(3) (2022), 39-47.
- [24] Ponnusamy S. and Rønning F., Integral transforms of a class of analytic functions, *Compl. Var. Elliptic. Eqn.*, 53(5) (2008), 423-434.
- [25] Robertson M. S., On the theory of univalent functions, *Annals of Math.*, 37 (1936) 374-408.
- [26] Silverman H., Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51 (1975), 109-116.
- [27] Srivastava H. M., Operators of basic (or q -) calculus and fractional q -calculus and their applications in geometric function theory of complex analysis, *Iran J. Sci Technol Trans. Sci.*, 44 (2020), 327-344.
- [28] Srivastava H. M., Certain q -polynomial expansions for functions of several variables, I and II., *IMA J. Appl. Math.*, 30 (1983), 205-209.
- [29] Swaminathan A., Sufficient conditions for hypergeometric functions to be in a certain class of analytic functions, *Comput. Math. Appl.*, 59(4) (2010), 1578-1583.
- [30] Vein R. and Dale P., *Determinants and their applications in mathematical physics*, New York, NY, USA: Springer, 1999.
- [31] Wanas A. K., Sakar F. M., Oros G. I. and Cotîrlă L. I., Toeplitz determinants for a certain family of analytic functions endowed with Borel distribution, *Symmetry*, 15(2) (2023), 262.
- [32] Yalcin S., Altinkaya S., Murugusundaramoorthy G. and Vijaya K., Hankel inequalities for a subclass of bi-univalent functions based on Salagean type q -difference operator, *J. Math. Fundamental Sciences*, 52(2) (2020), 189-201.