

**RELATIVE ORDER AND RELATIVE LOWER ORDER OF
BICOMPLEX VALUED ENTIRE FUNCTIONS**

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(Received: Jan. 17, 2026 Accepted: Aug. 11, 2026 Published: Aug. 30, 2026)

Abstract: The main target of this paper is to define the relative order and relative lower order of a bicomplex valued entire function so that growth of any bicomplex

entire function with respect to another entire function can be measured easily. In this paper our objective is to explore some basic properties in connection with the sum and product of relative order and relative lower order of bicomplex entire functions. Also, some examples are provided corresponding to the relative order and relative lower order of bicomplex valued entire functions.

Keywords and Phrases: Bicomplex valued entire function, Relative order, Relative lower order, Idempotent representation.

2020 Mathematics Subject Classification: 30G35, 30D20, 32A30.

1. Introduction and Preliminaries

The study of bicomplex numbers has been an active area of research since the foundational contributions of Segre [10], Luna-Elizarrarás [5, 6], and Spampinato [12], who sought a richer algebraic framework beyond the complex field. In particular, bicomplex algebra provides a fully commutative four-dimensional extension of complex numbers, offering a compelling alternative [6] to Hamilton's and Riley's quaternion system [3, 9], which, while also four-dimensional, lacks commutativity. For further study on relative growth and other general algebras, one can see [7, 11].

1.1. The Bicomplex Number System [8]

A bicomplex number $\mathfrak{z}$ can be represented in the form

$$\mathfrak{z} = \mathfrak{x}_1 + i_1 \mathfrak{x}_2 + i_2 \mathfrak{x}_3 + (i_1 i_2) \mathfrak{x}_4 = (\mathfrak{x}_1 + i_1 \mathfrak{x}_2) + i_2 (\mathfrak{x}_3 + i_1 \mathfrak{x}_4) = \mathfrak{z}_1 + i_2 \mathfrak{z}_2,$$

where each $\mathfrak{x}_k$ ($k = 1, \dots, 4$) is a real number, $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We set $j = i_1 i_2$, so that $j^2 = +1$. Here $\mathfrak{z}_1 = \mathfrak{x}_1 + i_1 \mathfrak{x}_2$ and $\mathfrak{z}_2 = \mathfrak{x}_3 + i_1 \mathfrak{x}_4$ lie in the complex plane $\mathbb{C}$.

We denote the collection of real numbers by $\mathbb{C}_0$, the usual complex numbers by $\mathbb{C}_1$, and the bicomplex numbers by $\mathbb{C}_2$.

1.2. Algebraic Structure of $\mathbb{C}_2$ [8]

We equip the bicomplex field $\mathbb{C}_2$ with addition and scalar multiplication operations and a norm as follows.

Addition and Scalar Multiplication: For $\mathfrak{z} = (\mathfrak{x}_1 + i_1 \mathfrak{x}_2 + i_2 \mathfrak{x}_3 + i_1 i_2 \mathfrak{x}_4)$ and $\mathfrak{w} = (\mathfrak{y}_1 + i_1 \mathfrak{y}_2 + i_2 \mathfrak{y}_3 + i_1 i_2 \mathfrak{y}_4)$ in $\mathbb{C}_2$, define

$$\mathfrak{z} \oplus \mathfrak{w} = (\mathfrak{x}_1 + \mathfrak{y}_1) + i_1 (\mathfrak{x}_2 + \mathfrak{y}_2) + i_2 (\mathfrak{x}_3 + \mathfrak{y}_3) + i_1 i_2 (\mathfrak{x}_4 + \mathfrak{y}_4).$$

Given a scalar $\mathfrak{a} \in \mathbb{C}_1$, scalar multiplication is

$$\mathfrak{a} \odot \mathfrak{z} = \mathfrak{a} \mathfrak{x}_1 + i_1 \mathfrak{a} \mathfrak{x}_2 + i_2 \mathfrak{a} \mathfrak{x}_3 + i_1 i_2 \mathfrak{a} \mathfrak{x}_4.$$

With these operations, $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.

Norm and Completeness: Introduce the norm as

$$\|\mathfrak{x}_1 + i_1 \mathfrak{x}_2 + i_2 \mathfrak{x}_3 + i_1 i_2 \mathfrak{x}_4\| = \sqrt{\mathfrak{x}_1^2 + \mathfrak{x}_2^2 + \mathfrak{x}_3^2 + \mathfrak{x}_4^2}.$$

Observe that $\mathbb{C}_2$ can be identified with $\mathbb{C}_0^4$ via

$$\mathfrak{x}_1 + i_1 \mathfrak{x}_2 + i_2 \mathfrak{x}_3 + i_1 i_2 \mathfrak{x}_4 \longleftrightarrow (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4).$$

Equipping $\mathbb{C}_0^4$ with the standard Euclidean norm makes it a complete normed linear space or a Banach space. Because the norm on $\mathbb{C}_2$ coincides with this Euclidean norm under the above correspondence, $(\mathbb{C}_2, \oplus, \odot, \|\cdot\|)$ is itself a Banach space.

Bicomplex Multiplication: Define the bilinear product $\otimes : \mathbb{C}_2 \times \mathbb{C}_2 \rightarrow \mathbb{C}_2$ by

$$\begin{aligned} \mathfrak{z} \otimes \mathfrak{w} = & (\mathfrak{x}_1 \mathfrak{y}_1 - \mathfrak{x}_2 \mathfrak{y}_2 - \mathfrak{x}_3 \mathfrak{y}_3 + \mathfrak{x}_4 \mathfrak{y}_4) \\ & + i_1 (\mathfrak{x}_1 \mathfrak{y}_2 + \mathfrak{x}_2 \mathfrak{y}_1 - \mathfrak{x}_3 \mathfrak{y}_4 - \mathfrak{x}_4 \mathfrak{y}_3) \\ & + i_2 (\mathfrak{x}_1 \mathfrak{y}_3 + \mathfrak{x}_3 \mathfrak{y}_1 - \mathfrak{x}_2 \mathfrak{y}_4 - \mathfrak{x}_4 \mathfrak{y}_2) \\ & + i_1 i_2 (\mathfrak{x}_1 \mathfrak{y}_4 + \mathfrak{x}_4 \mathfrak{y}_1 + \mathfrak{x}_2 \mathfrak{y}_3 + \mathfrak{x}_3 \mathfrak{y}_2). \end{aligned}$$

Banach Algebra Property: One can verify that, for all $\mathfrak{a} \in \mathbb{C}_1$, $\mathfrak{z}, \mathfrak{w} \in \mathbb{C}_2$:

$$\|\mathfrak{a} \odot \mathfrak{z}\| = |\mathfrak{a}| \|\mathfrak{z}\|, \quad \|\mathfrak{z} \otimes \mathfrak{w}\| \leq \sqrt{2} \|\mathfrak{z}\| \|\mathfrak{w}\|.$$

Therefore, $(\mathbb{C}_2, \oplus, \odot, \otimes, \|\cdot\|)$ is a commutative Banach algebra.

1.3. Idempotent Representation of Bicomplex Numbers [8]

The bicomplex number system $\mathbb{C}_2$ admits four idempotent elements:

$$0, \quad 1, \quad \frac{1 + i_1 i_2}{2}, \quad \frac{1 - i_1 i_2}{2}.$$

Among these, the two nontrivial idempotents are denoted by

$$e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2},$$

which satisfy the following relations

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = e_2 e_1 = 0, \quad e_1 + e_2 = 1.$$

The elements e_1 and e_2 are called orthogonal idempotents due to these properties.

Every bicomplex number $\mathfrak{z} = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 \in \mathbb{C}_2$, where $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_1$, can be uniquely expressed as

$$\mathfrak{z} = (\mathfrak{z}_1 - i_1 \mathfrak{z}_2)e_1 + (\mathfrak{z}_1 + i_1 \mathfrak{z}_2)e_2 = \xi_1 e_1 + \xi_2 e_2,$$

where $\xi_1, \xi_2 \in \mathbb{C}_1$. This decomposition is known as the idempotent representation of $\mathfrak{z}$.

A bicomplex number $\mathfrak{z} = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 \in \mathbb{C}_2$ is said to be non-singular if $|\mathfrak{z}_1^2 + \mathfrak{z}_2^2| \neq 0$, and singular otherwise. We denote the set of all singular elements in $\mathbb{C}_2$ by θ_2 .

1.4. Discus and Neighborhood [8]

Let

$$\mathfrak{a} = \mathfrak{a}_1 + i_1 \mathfrak{a}_2 + i_2 \mathfrak{a}_3 + i_1 i_2 \mathfrak{a}_4 = \alpha + i_2 \beta$$

be a fixed bicomplex point, where $\alpha = \mathfrak{a}_1 + i_1 \mathfrak{a}_2$ and $\beta = \mathfrak{a}_3 + i_1 \mathfrak{a}_4$ lie in $\mathbb{C}_1$. Choose positive real numbers $\mathfrak{r}, \mathfrak{r}_1, \mathfrak{r}_2 > 0$.

Define the two complex subsets

$$\mathfrak{A}_1 = \{\mathfrak{z}_1 - i_1 \mathfrak{z}_2 : \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_1\}, \quad \mathfrak{A}_2 = \{\mathfrak{z}_1 + i_1 \mathfrak{z}_2 : \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_1\},$$

and let $\mathfrak{w}_1 \in \mathfrak{A}_1, \mathfrak{w}_2 \in \mathfrak{A}_2$. Note that $\mathfrak{w}_1, \mathfrak{w}_2$ are ordinary complex numbers.

The open ball $\mathfrak{B}(\mathfrak{a}, \mathfrak{r})$ and the closed ball $\overline{\mathfrak{B}}(\mathfrak{a}, \mathfrak{r})$ centered at $\mathfrak{a}$ with radius $\mathfrak{r}$ in the bicomplex plane are defined as

$$\mathfrak{B}(\mathfrak{a}, \mathfrak{r}) = \{\mathfrak{z} = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 \in \mathbb{C}_2 : \|\mathfrak{z} - \mathfrak{a}\| < \mathfrak{r}\},$$

$$\overline{\mathfrak{B}}(\mathfrak{a}, \mathfrak{r}) = \{\mathfrak{z} = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 \in \mathbb{C}_2 : \|\mathfrak{z} - \mathfrak{a}\| \leq \mathfrak{r}\}.$$

Using the idempotent decomposition $\mathfrak{z} = \mathfrak{w}_1 e_1 + \mathfrak{w}_2 e_2 \in \mathbb{C}_2$, define

$$\mathfrak{D}(\mathfrak{a}; \mathfrak{r}_1, \mathfrak{r}_2) = \{\mathfrak{z} \in \mathbb{C}_2 : \mathfrak{z} = \mathfrak{w}_1 e_1 + \mathfrak{w}_2 e_2, |\mathfrak{w}_1 - (\alpha - i_1 \beta)| < \mathfrak{r}_1, |\mathfrak{w}_2 - (\alpha + i_1 \beta)| < \mathfrak{r}_2\},$$

$$\overline{\mathfrak{D}}(\mathfrak{a}; \mathfrak{r}_1, \mathfrak{r}_2) = \{\mathfrak{z} \in \mathbb{C}_2 : \mathfrak{z} = \mathfrak{w}_1 e_1 + \mathfrak{w}_2 e_2, |\mathfrak{w}_1 - (\alpha - i_1 \beta)| \leq \mathfrak{r}_1, |\mathfrak{w}_2 - (\alpha + i_1 \beta)| \leq \mathfrak{r}_2\}.$$

Here $\mathfrak{D}(\mathfrak{a}; \mathfrak{r}_1, \mathfrak{r}_2)$ and $\overline{\mathfrak{D}}(\mathfrak{a}; \mathfrak{r}_1, \mathfrak{r}_2)$ are called the open discus and closed discus with center at $\mathfrak{a}$ and radii $\mathfrak{r}_1$ and $\mathfrak{r}_2$, respectively.

1.5. Bicomplex Holomorphic Functions [8]

Let

$$f : \Omega \subset \mathbb{C}_2 \longrightarrow \mathbb{C}_2$$

be a bicomplex valued function. We say f is differentiable at $\omega_0 \in \Omega$ if the limit

$$f'(\omega_0) = \lim_{\substack{\mathfrak{h} \rightarrow 0 \\ \mathfrak{h} \notin \theta_2}} \frac{f(\omega_0 + \mathfrak{h}) - f(\omega_0)}{\mathfrak{h}}$$

exists, where

$$\mathfrak{h} = \mathfrak{h}_0 + i_1 \mathfrak{h}_1 + i_2 \mathfrak{h}_2 + i_1 i_2 \mathfrak{h}_3$$

is required to be invertible in $\mathbb{C}_2$ (i.e. $\mathfrak{h} \notin \theta_2$).

If f is differentiable at every point of Ω , it is called bicomplex valued holomorphic function. Writing

$$\omega = \mathfrak{z}_1 + i_2 \mathfrak{z}_2, \quad f(\omega) = g_1(\mathfrak{z}_1, \mathfrak{z}_2) + i_2 g_2(\mathfrak{z}_1, \mathfrak{z}_2),$$

with $z_1, z_2 \in \mathbb{C}_1$, one can show that f is holomorphic when both g_1 and g_2 are holomorphic in $\mathbb{C}_1$ and satisfy the Cauchy–Riemann equations (cf. G.B. Price [8], 1991, Chapter 3. Derivatives and Holomorphic Functions, pp. 159-176.)

$$\frac{\partial g_1}{\partial \mathfrak{z}_1} = \frac{\partial g_2}{\partial \mathfrak{z}_2}, \quad \frac{\partial g_1}{\partial \mathfrak{z}_2} = -\frac{\partial g_2}{\partial \mathfrak{z}_1}.$$

In this case, the derivative takes the form

$$f'(\omega) = \frac{\partial g_1}{\partial \mathfrak{z}_2} + i_2 \frac{\partial g_2}{\partial \mathfrak{z}_1}.$$

1.6. Bicomplex Entire Functions and Growth Orders [8]

A function $f : \mathbb{C}_2 \rightarrow \mathbb{C}_2$ is called bicomplex valued entire function if it is holomorphic on the entire bicomplex plane $\mathbb{C}_2$.

Definition 1.1. (Order of a Bicomplex Valued Entire Function [4]) *Let f_1 and f_2 be the idempotent components of*

$$f(\mathfrak{w}) = f_1(\mathfrak{z}_1 - i_1 \mathfrak{z}_2) e_1 + f_2(\mathfrak{z}_1 + i_1 \mathfrak{z}_2) e_2,$$

with e_1, e_2 the standard idempotent elements of $\mathbb{C}_2$. Then $M_{f_i}(\mathfrak{r})$ denotes the maximal modulus of f_i on the circle $|\mathfrak{z}| = \mathfrak{r}$ and defined by

$$M_{f_i}(\mathfrak{r}) = \max_{|\mathfrak{z}|=\mathfrak{r}} |f_i(\mathfrak{z})|$$

The order of f_i in the classical sense is defined by

$$\rho_{f_i} = \limsup_{\mathfrak{r} \rightarrow \infty} \frac{\log^{[2]} M_{f_i}(\mathfrak{r})}{\log \mathfrak{r}}, \quad i = 1, 2.$$

Then the order of the bicomplex valued entire function f is defined by

$$\rho(f) = \max\{\rho_{f_1}, \rho_{f_2}\}.$$

We now introduce the notions of relative order and relative lower order for bicomplex valued entire functions.

Bernal [1, 2] defined the relative order of a complex valued entire function f with respect to another complex valued entire function g as

$$\rho_g(f) = \inf \{ \mu > 0 : M_g(\mathfrak{r}) < M_f(\mathfrak{r}^\mu) \quad \forall \mathfrak{r} > \mathfrak{r}_0(\mu) > 0 \},$$

where $M_f(\mathfrak{r}) = \max\{|f(\mathfrak{z})| : |\mathfrak{z}| = \mathfrak{r}\}$ denotes the maximum modulus of f on the circle of radius $\mathfrak{r}$. Equivalently, it can be expressed as

$$\rho_g(f) = \limsup_{\mathfrak{r} \rightarrow \infty} \frac{\log^{[2]} M_f^{-1} M_g(\mathfrak{r})}{\log \mathfrak{r}}.$$

By replacing the $\limsup$ with $\liminf$, one obtains the definition of the relative lower order, denoted by $\lambda_g(f)$, which captures the lower growth behavior of f relative to g .

Now, consider two bicomplex valued entire functions f and g , represented in their idempotent forms as

$$f(\mathfrak{z}_1 + i_2 \mathfrak{z}_2) = f_1(\mathfrak{z}_1 - i_1 \mathfrak{z}_2)e_1 + f_2(\mathfrak{z}_1 + i_1 \mathfrak{z}_2)e_2,$$

$$g(\mathfrak{z}_1 + i_2 \mathfrak{z}_2) = g_1(\mathfrak{z}_1 - i_1 \mathfrak{z}_2)e_1 + g_2(\mathfrak{z}_1 + i_1 \mathfrak{z}_2)e_2,$$

where f_1, f_2, g_1, g_2 are complex valued entire functions in $\mathbb{C}_1$.

The relative order of the bicomplex valued entire function f with respect to g is defined as

$$\rho_g(f) = \max \{ \rho_{g_1}(f_1), \rho_{g_2}(f_2) \},$$

and the relative lower order is given by

$$\lambda_g(f) = \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \}.$$

In the special case when $g(\mathfrak{w}) = \exp(\mathfrak{w})$, these definitions reduce to the classical notions of order and lower order, as discussed in [13].

Furthermore, if the relative order and relative lower order of a bicomplex valued entire function f with respect to g are equal, then f is said to exhibit regular relative growth with respect to g .

Example 1.2. Let $f(z_1 + i_2 z_2) = \exp(z_1 + i_2 z_2)$ and $g(z_1 + i_2 z_2) = \exp(5(z_1 + i_2 z_2))$ be two bicomplex valued entire functions. Then

$$\begin{aligned} f(z_1 + i_2 z_2) &= \exp(z_1 + i_2 z_2) \\ &= \exp(z_1 - i_1 z_2)e_1 + \exp(z_1 + i_1 z_2)e_2 \\ &= f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2 \end{aligned}$$

where f_1, f_2 are complex valued functions.

Similarly

$$\begin{aligned} g(z_1 + i_2 z_2) &= \exp(5(z_1 + i_2 z_2)) \\ &= \exp(5z_1 - 5i_1 z_2)e_1 + \exp(5z_1 + 5i_1 z_2)e_2 \\ &= g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2 \end{aligned}$$

where g_1, g_2 are complex valued functions.

Then for $\|z_1 + i_2 z_2\| = r$ we can write

$$|z_1 - i_1 z_2| = r \quad \text{and} \quad |z_1 + i_1 z_2| = r.$$

$$M_{f_1}(r) = \max_{|z_1 - i_1 z_2| = r} |f_1(r)| = e^r,$$

$$M_{g_1}(r) = \max_{|z_1 - i_1 z_2| = r} |g_1(r)| = e^{5r}$$

$$\implies r = M_{g_1}^{-1}(e^{5r})$$

$$\implies 5r = \log M_{g_1}(r)$$

$$\implies r = \frac{1}{5} \log M_{g_1}(r)$$

Now,

$$M_{g_1}^{-1}(e^{5r}) = r.$$

Therefore,

$$M_{g_1}^{-1}(R) = \frac{1}{5} \log R,$$

where $R = e^{5r} \implies r = \frac{1}{5} \log R$.

So, $M_{g_1}^{-1}(M_{f_1}(r)) = \frac{1}{5} \log M_{f_1}(r)$.

Then,

$$\begin{aligned} \rho_{g_1}(f_1) &= \limsup_{r \rightarrow \infty} \frac{\log M_{g_1}^{-1}(M_{f_1}(r))}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log \left(\frac{1}{5} \log M_{f_1}(r) \right)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M_{f_1}(r) + O(1)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} e^r + O(1)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log r}{\log r} = 1. \end{aligned}$$

Similarly we can find that $\rho_{g_2}(f_2) = 1$.

So, $\rho_g(f) = \max\{\rho_{g_1}(f_1), \rho_{g_2}(f_2)\} = 1$.

Example 1.3. If f is a polynomial in $\mathbb{C}_2$ and $g(z) = \exp(z)$ where $z \in \mathbb{C}_2$, then $\rho_g(f) = 0$.

Let

$$\begin{aligned} f(z_1 + i_2 z_2) &= \sum_{k=0}^n (a_k + i_2 b_k)(z_1 + i_2 z_2)^k \\ &= \left[\sum_{k=0}^n (a_k - i_1 b_k)(z_1 - i_1 z_2)^k \right] e_1 + \left[\sum_{k=0}^n (a_k + i_1 b_k)(z_1 + i_1 z_2)^k \right] e_2 \\ &= f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2 \end{aligned}$$

Now,

$$\begin{aligned} g(z_1 + i_2 z_2) &= \exp(z_1 + i_1 z_2) \\ &= \exp(z_1 - i_1 z_2)e_1 + \exp(z_1 + i_1 z_2)e_2 \\ &= g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2 \end{aligned}$$

Then for $\|z_1 + i_2 z_2\| = r$ we can write

$$\begin{aligned} |z_1 - i_1 z_2| &= r \quad \text{and} \quad |z_1 + i_1 z_2| = r. \\ M_{f_1}(r) &= \max_{|z_1 - i_1 z_2| = r} |f_1(z)| \sim |a_n - i_1 b_n| r^n \quad \text{as } r \rightarrow \infty \end{aligned}$$

$$\text{and} \quad M_{g_1}(r) = \max_{|z_1 - i_1 z_2| = r} |g_1(z)| = e^r.$$

Therefore,

$$\begin{aligned} \rho_{g_1}(f_1) &= \limsup_{r \rightarrow \infty} \frac{\log M_{g_1}^{-1} M_{f_1}(r)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log M_{g_1}^{-1} (|a_n - i_1 b_n| r^n)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log \log (|a_n - i_1 b_n| r^n)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log(n \log r) + O(1)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log n + \log \log r}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log \log r}{\log r} = 0 \end{aligned}$$

Similarly, $\rho_{g_2}(f_2) = 0$.

Hence,

$$\rho_g(f) = \max\{\rho_{g_1}(f_1), \rho_{g_2}(f_2)\} = 0.$$

In this section, we present a lemma which will be needed in the sequel.

Lemma 1.4. ([1, 2]) Suppose that f is a non-constant entire function of complex variable, $\alpha > 1$ and $0 < \beta < \alpha$. Then,

$$M_f(\alpha r) > \beta M_f(r)$$

for all sufficiently large r .

Lemma 1.5. ([1, 2]) Let f be a non-constant entire complex valued function. Then, for any positive integer p and for all $\sigma > 1$

$$[M_f(r)]^p < M_f(r^\sigma).$$

2. Main Theorems and Proofs

Theorem 2.1. Let f, g and h be bicomplex valued entire functions having relative orders $\rho_h(f)$ and $\rho_h(g)$ respectively. Then

$$(i) \quad \rho_h(f \pm g) \leq 4 \max\{\rho_h(f), \rho_h(g)\}$$

$$(ii) \quad \rho_h(f \cdot g) \leq 4 \max\{\rho_h(f), \rho_h(g)\}$$

Proof. If $\rho_h(f \pm g) = 0$, then the result is obvious.

We suppose that $\rho_h(f \pm g) > 0$.

Let $\rho_h(f)$ and $\rho_h(g)$ be finite and $\max\{\rho_{h_i}(f_i), \rho_{h_i}(g_i)\} = A_i, \quad i = 1, 2$.

Now, f, g, h are bicomplex valued entire functions, then we can write

$$f(z) = f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2$$

$$g(z) = g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2$$

$$h(z) = h_1(z_1 - i_1 z_2)e_1 + h_2(z_1 + i_1 z_2)e_2$$

where f_i, g_i, h_i are complex valued entire functions for $i = 1, 2$.

Now for any $\epsilon > 0$ from the definition of $\rho_{h_i}(f_i)$ we have for all large $r_1, r_2, \dots, r_n$

$$M_{f_i}(r_n) < M_{h_i} \left(r_n^{\rho_{h_i}(f_i) + \epsilon} \right), \quad i = 1, 2.$$

$$i.e., \quad M_{f_i}(r_n) < M_{h_i} \left(r_n^{A_i + \epsilon} \right), \quad i = 1, 2. \quad (2.1)$$

Also for arbitrary $\epsilon > 0$ from the definition of $\rho_{h_i}(g_i)$, we have for all large $r_1, r_2, \dots, r_n$

$$\begin{aligned} M_{g_i}(r_n) &< M_{h_i} \left(r_n^{\rho_{h_i}(g_i) + \epsilon} \right), \quad i = 1, 2. \\ \text{i.e.,} \quad M_{g_i}(r_n) &< M_{h_i} \left(r_n^{A_i + \epsilon} \right), \quad i = 1, 2, \end{aligned} \quad (2.2)$$

since

$$M_{f_i \pm g_i}(r) \leq M_{f_i}(r) + M_{g_i}(r), \quad \forall i = 1, 2.$$

We obtain from (2.1) and (2.2), for all large $r_1, r_2, \dots, r_n$, that

$$\begin{aligned} M_{f_i \pm g_i}(r_n) &\leq M_{f_i}(r_n) + M_{g_i}(r_n) \\ &< 2M_{h_i} \left(r_n^{A_i + \epsilon} \right) \\ &< M_{h_i} \left(3r_n^{A_i + \epsilon} \right) \quad [\text{By Lemma 1.4}] \\ &< M_{h_i} \left(r_n^{A_i + 3\epsilon} \right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get from above that

$$\rho_{h_i}(f_i \pm g_i) \leq A_i = \max\{\rho_{h_i}(f_i), \rho_{h_i}(g_i)\} \quad \forall i = 1, 2.$$

i.e.,

$$\rho_{h_1}(f_1 \pm g_1) \leq \max\{\rho_{h_1}(f_1), \rho_{h_1}(g_1)\}, \quad (2.3)$$

$$\rho_{h_2}(f_2 \pm g_2) \leq \max\{\rho_{h_2}(f_2), \rho_{h_2}(g_2)\}. \quad (2.4)$$

(i) Now,

$$\begin{aligned} \rho_h(f \pm g) &= \max\{\rho_{h_1}(f_1 \pm g_1), \rho_{h_2}(f_2 \pm g_2)\} \\ &\leq \rho_{h_1}(f_1 \pm g_1) + \rho_{h_2}(f_2 \pm g_2) \\ &\leq \max\{\rho_{h_1}(f_1), \rho_{h_1}(g_1)\} + \max\{\rho_{h_2}(f_2), \rho_{h_2}(g_2)\} \quad [\text{Using (2.3) and (2.4)}] \\ &\leq (\rho_{h_1}(f_1) + \rho_{h_1}(g_1)) + (\rho_{h_2}(f_2) + \rho_{h_2}(g_2)) \\ &= (\rho_{h_1}(f_1) + \rho_{h_2}(f_2)) + (\rho_{h_1}(g_1) + \rho_{h_2}(g_2)) \\ &\leq 2 \max\{\rho_{h_1}(f_1), \rho_{h_2}(f_2)\} + 2 \max\{\rho_{h_1}(g_1), \rho_{h_2}(g_2)\} \\ &= 2\{\rho_h(f) + \rho_h(g)\} \\ &\leq 4 \max\{\rho_h(f), \rho_h(g)\}. \end{aligned}$$

(ii) For the second part we know that

$$\begin{aligned} \forall i = 1, 2, \quad M_{h_i \cdot g_j}(r_n) &\leq M_{f_i}(r_n) \cdot M_{g_j}(r_n) \\ &< [M_{h_i}(r_n^{A_i + \epsilon})]^2 \quad [\text{From (2.1) and (2.2)}] \end{aligned}$$

$$< M_{h_i}(r_n^{\sigma(A_i+\epsilon)}) \quad \text{for } \sigma > 1 \text{ [Using Lemma 1.5].}$$

$$\rho_{h_1}(f_1 \cdot g_1) \leq A_1 = \max\{\rho_{h_1}(f_1), \rho_{h_1}(g_1)\}$$

and

$$\rho_{h_2}(f_2 \cdot g_2) \leq A_2 = \max\{\rho_{h_2}(f_2), \rho_{h_2}(g_2)\}.$$

Now

$$\begin{aligned} \rho_h(f \cdot g) &= \max\{\rho_{h_1}(f_1 \cdot g_1), \rho_{h_2}(f_2 \cdot g_2)\} \\ &\leq \rho_{h_1}(f_1 \cdot g_1) + \rho_{h_2}(f_2 \cdot g_2) \end{aligned}$$

From the previous part of this proof we can conclude that

$$\rho_h(f \cdot g) \leq 4 \max\{\rho_h(f), \rho_h(g)\}.$$

Theorem 2.2. *Let f , g and h be any three bicomplex valued entire functions. Then*

$$(i) \quad \lambda_h(f \pm g) \leq 4 \max\{\lambda_h(f), \lambda_h(g)\}$$

$$(ii) \quad \lambda_h(f \cdot g) \leq 4 \max\{\lambda_h(f), \lambda_h(g)\}$$

Proof. If $\lambda_h(f \pm g) = 0$, then the result is obvious.

We suppose that $\lambda_h(f \pm g) > 0$.

Let $\lambda_h(f)$ and $\lambda_h(g)$ be finite and $\max\{\lambda_{h_i}(f_i), \lambda_{h_i}(g_i)\} = A_i$, $i = 1, 2$.

Now, f, g, h are bicomplex valued entire functions, then we can write

$$f(z) = f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2$$

$$g(z) = g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2$$

$$h(z) = h_1(z_1 - i_1 z_2)e_1 + h_2(z_1 + i_1 z_2)e_2$$

where f_i, g_i, h_i are complex valued entire functions for $i = 1, 2$.

Now for any $\epsilon > 0$ from the definition of $\lambda_{h_i}(f_i)$ we have a sequence $\{r'_n\} \rightarrow \infty$ such that

$$M_{f_i}(r'_n) < M_{h_i} \left(r_n^{\lambda_{h_i}(f_i)+\epsilon} \right), \quad i = 1, 2.$$

$$i.e., \quad M_{f_i}(r'_n) < M_{h_i} \left(r_n^{\lambda_{h_i}(f_i)+\epsilon} \right), \quad i = 1, 2. \quad (2.5)$$

Also, for arbitrary $\epsilon > 0$ from the definition of $\lambda_{h_i}(g_i)$, we have a sequence $\{r''_n\} \rightarrow \infty$ such that

$$M_{g_i}(r''_n) < M_{h_i} \left(r_n^{\lambda_{h_i}(g_i)+\epsilon} \right), \quad i = 1, 2.$$

$$i.e., \quad M_{g_i}(r_n'') < M_{h_i}\left(r_n''^{A_i+\epsilon}\right), \quad i = 1, 2. \quad (2.6)$$

Let $r_n = \max\{r_n', r_n''\}$. Then $\{r_n\} \rightarrow \infty$ and both (2.5) & (2.6) hold for r_n .

$$\text{Since } M_{f_i \pm g_i}(r) \leq M_{f_i}(r) + M_{g_i}(r), \quad \forall i = 1, 2.$$

We obtain from (2.5) and (2.6), for a sequence of values of $r_n \rightarrow \infty$, that

$$\begin{aligned} M_{f_i \pm g_i}(r_n) &\leq M_{f_i}(r_n) + M_{g_i}(r_n) \\ &< 2M_{h_i}\left(r_n^{A_i+\epsilon}\right) \\ &< M_{h_i}\left(3r_n^{A_i+\epsilon}\right) \quad [By \text{ Lemma 1.4}] \\ &< M_{h_i}\left(r_n^{A_i+3\epsilon}\right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get from above that

$$\lambda_{h_i}(f_i \pm g_i) \leq A_i = \max\{\lambda_{h_i}(f_i), \lambda_{h_i}(g_i)\} \quad \forall i = 1, 2.$$

$$\lambda_{h_1}(f_1 \pm g_1) \leq \max\{\lambda_{h_1}(f_1), \lambda_{h_1}(g_1)\} \quad (2.7)$$

$$\lambda_{h_2}(f_2 \pm g_2) \leq \max\{\lambda_{h_2}(f_2), \lambda_{h_2}(g_2)\} \quad (2.8)$$

(i) Now,

$$\begin{aligned} \lambda_h(f \pm g) &= \max\{\lambda_{h_1}(f_1 \pm g_1), \lambda_{h_2}(f_2 \pm g_2)\} \\ &\leq \lambda_{h_1}(f_1 \pm g_1) + \lambda_{h_2}(f_2 \pm g_2) \\ &\leq \max\{\lambda_{h_1}(f_1), \lambda_{h_1}(g_1)\} + \max\{\lambda_{h_2}(f_2), \lambda_{h_2}(g_2)\} \quad [Using (2.7) \text{ and } (2.8)] \\ &\leq (\lambda_{h_1}(f_1) + \lambda_{h_1}(g_1)) + (\lambda_{h_2}(f_2) + \lambda_{h_2}(g_2)) \\ &= (\lambda_{h_1}(f_1) + \lambda_{h_2}(f_2)) + (\lambda_{h_1}(g_1) + \lambda_{h_2}(g_2)) \\ &\leq 2 \max\{\lambda_{h_1}(f_1), \lambda_{h_2}(f_2)\} + \max\{\lambda_{h_1}(g_1), \lambda_{h_2}(g_2)\} \\ &= 2\{\lambda_h(f) + \lambda_h(g)\} \\ &\leq 4 \max\{\lambda_h(f), \lambda_h(g)\}. \end{aligned}$$

(ii) For the second part we know that

$$\begin{aligned} \forall i = 1, 2, \quad M_{h_i \cdot g_j}(r_n) &\leq M_{f_i}(r_n) \cdot M_{g_j}(r_n) \\ &< [M_{h_i}(r_n^{A_i+\epsilon})]^2 \quad [From (2.5) \text{ and } (2.6)] \\ &< M_{h_i}(r_n^{\sigma(A_i+\epsilon)}) \quad \text{for } \sigma > 1 \quad [Using \text{ Lemma 1.5}]. \end{aligned}$$

$$\lambda_{h_1}(f_1 \cdot g_1) \leq A_1 = \max\{\lambda_{h_1}(f_1), \lambda_{h_1}(g_1)\}$$

and

$$\lambda_{h_2}(f_2 \cdot g_2) \leq A_2 = \max\{\lambda_{h_2}(f_2), \lambda_{h_2}(g_2)\}.$$

Now

$$\begin{aligned} \lambda_h(f \cdot g) &= \max\{\lambda_{h_1}(f_1 \cdot g_1), \lambda_{h_2}(f_2 \cdot g_2)\} \\ &\leq \lambda_{h_1}(f_1 \cdot g_1) + \lambda_{h_2}(f_2 \cdot g_2) \end{aligned}$$

From the previous part of this proof we can conclude that

$$\lambda_h(f \cdot g) \leq 4 \max\{\lambda_h(f), \lambda_h(g)\}.$$

Theorem 2.3. *Let f, g, h be any three bicomplex entire functions having relative order. Then*

$$\lambda_{g \pm h}(f) \geq \min\{\lambda_g(f), \lambda_h(f)\}.$$

Proof. Case 1: If $\lambda_{g \pm h}(f) = \infty$, then the result is obvious.

Case 2: Let one of $\lambda_g(f)$ and $\lambda_h(f)$ be infinite.

Without loss of generality, let $\lambda_g(f) = \infty$ and $\lambda_h(f) < \infty$, then

$$\min\{\lambda_g(f), \lambda_h(f)\} = \lambda_h(f).$$

Since, $g = (g \pm h) \mp h$, by Theorem 2.2 we have

$$\lambda_g(f) \leq 4 \max\{\lambda_{g \pm h}(f), \lambda_h(f)\}.$$

Therefore, $\lambda_{g \pm h}(f) = \infty$, since $\lambda_g(f) = \infty$ and $\lambda_h(f) < \infty$.

Thus, the theorem is true.

Case 3: Let $\lambda_g(f)$ and $\lambda_h(f)$ both be infinite, then from **Case 2** we can say that $\lambda_{g \pm h}(f) = \infty$. Hence, the theorem is true.

Case 4: Let all $\lambda_{g \pm h}(f)$, $\lambda_g(f)$ and $\lambda_h(f)$ be finite. Then two subcases arise.

Subcase I: Let $\lambda_g(f) > \lambda_h(f)$.

Now, f, g, h are bicomplex valued entire functions. Then we can write

$$f(z) = f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2,$$

$$g(z) = g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2,$$

$$h(z) = h_1(z_1 - i_1 z_2)e_1 + h_2(z_1 + i_1 z_2)e_2,$$

where f_i, g_i, h_i are complex-valued entire functions for $i = 1, 2$.

Further, let

$$A_i = \min \{ \lambda_{g_i}(f_i), \lambda_{h_i}(f_i) \}, \quad i = 1, 2.$$

Now for any $\epsilon > 0$, from the definition of $\lambda_{g_i}(f_i)$ and $\lambda_{h_i}(f_i)$ we have sequences $\{r'_n\} \rightarrow \infty$ and $\{r''_n\} \rightarrow \infty$ such that

$$M_{g_i} \left(r_n^{\lambda_{g_i}(f_i) - \epsilon} \right) < M_{f_i}(r'_n) \quad \text{and} \quad M_{h_i} \left(r_n^{\lambda_{h_i}(f_i) - \epsilon} \right) < M_{f_i}(r''_n), \quad \text{for } i = 1, 2. \quad (2.9)$$

Let $r_n = \max\{r'_n, r''_n\}$. Then $\{r_n\} \rightarrow \infty$ and (2.9) holds for r_n . From the above, for $\{r_n\} \rightarrow \infty$, we get

$$M_{g_i} \left(r_n^{A_i - \epsilon} \right) < M_{f_i}(r_n), \quad \text{and} \quad M_{h_i} \left(r_n^{A_i - \epsilon} \right) < M_{f_i}(r_n), \quad \text{for } i = 1, 2. \quad (2.10)$$

Since

$$M_{g_i \pm h_i}(r) \leq M_{g_i}(r) + M_{h_i}(r), \quad \text{for } i = 1, 2.$$

From (2.9), (2.10) and Lemma 1.4, for $\{r_n\} \rightarrow \infty$, we get

$$\begin{aligned} M_{g_i \pm h_i} \left(r_n^{A_i - \epsilon} \right) &< M_{g_i} \left(r_n^{A_i - \epsilon} \right) + M_{h_i} \left(r_n^{A_i - \epsilon} \right) \\ \implies M_{g_i \pm h_i} \left(r_n^{A_i - \epsilon} \right) &< 2M_{f_i}(r_n) \\ \implies M_{g_i \pm h_i} \left(\frac{1}{3} r_n^{A_i - \epsilon} \right) &< M_{f_i}(r_n) \\ \implies M_{g_i \pm h_i} \left(r_n^{A_i - 3\epsilon} \right) &< M_{f_i}(r_n), \quad \text{where } i = 1, 2. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get from above

$$\lambda_{g_i \pm h_i}(f_i) \geq A_i = \min \{ \lambda_{g_i}(f_i), \lambda_{h_i}(f_i) \}, \quad \forall i = 1, 2.$$

Therefore,

$$\lambda_{g_1 \pm h_1}(f_1) \geq \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \} \quad \text{and} \quad \lambda_{g_2 \pm h_2}(f_2) \geq \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \}. \quad (2.11)$$

Also, $\lambda_{g \pm h}(f) = \max\{\lambda_{g_1 \pm h_1}(f_1), \lambda_{g_2 \pm h_2}(f_2)\}$. Then from (2.11) we get

$$\lambda_{g \pm h}(f) \geq \max \{ \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \}, \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \} \}. \quad (2.12)$$

Now, since

$$\begin{aligned} \lambda_g(f) &> \lambda_h(f) \\ \text{i.e., } \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \} &> \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \}, \end{aligned} \quad (2.13)$$

two cases arise.

Subcase I.a): If $\lambda_{g_1}(f_1) > \lambda_{g_2}(f_2)$, then from (2.13) we have

$$\lambda_{g_1}(f_1) > \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \} > \lambda_{h_1}(f_1).$$

Similarly $\lambda_{g_1}(f_1) > \lambda_{h_2}(f_2)$.

Subcase I.b): If $\lambda_{g_1}(f_1) < \lambda_{g_2}(f_2)$, then from (2.13) we have

$$\lambda_{g_2}(f_2) > \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \} > \lambda_{h_2}(f_2).$$

So, from equation (2.12) and **Subcase I.a)**, we get

$$\lambda_{g \pm h}(f) \geq \max \{ \lambda_{g_1}(f_1), \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \} \} \geq \lambda_{g_1}(f_1). \quad (2.14)$$

Similarly from equation (2.12) and **Subcase I.b)**, we get

$$\lambda_{g \pm h}(f) \geq \lambda_{g_2}(f_2). \quad (2.15)$$

From (2.14) and (2.15), we have

$$\begin{aligned} \lambda_{g \pm h}(f) &\geq \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \} \\ \text{i.e., } \lambda_{g \pm h}(f) &\geq \lambda_g(f) \geq \min \{ \lambda_g(f), \lambda_h(f) \}. \end{aligned}$$

Subcase II: Let $\lambda_g(f) = \lambda_h(f)$, then from **Subcase I**, the result follows.

This proves the theorem.

Theorem 2.4. *Let f, g, h be any three bicomplex valued entire functions having relative order. Then*

$$\lambda_{g \cdot h}(f) \geq \min \{ \lambda_g(f), \lambda_h(f) \}.$$

Proof. Case 1: If $\lambda_{g \cdot h}(f) = \infty$, then the result is obvious.

Case 2: Let one of $\lambda_g(f)$ and $\lambda_h(f)$ be infinite.

Without loss of generality, let $\lambda_g(f) = \infty$ and $\lambda_h(f) < \infty$, then

$$\min \{ \lambda_g(f), \lambda_h(f) \} = \lambda_h(f).$$

Again,

$$\lambda_g(f) = \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \}.$$

Let $\lambda_{g_1}(f_1) = \infty$.

Now

$$\lambda_h(f) = \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \} < \infty.$$

Therefore,

$$\lambda_{h_1}(f_1) < \infty \quad \text{and} \quad \lambda_{h_2}(f_2) < \infty.$$

In particular,

$$\lambda_{g_1}(f_1) > \lambda_{h_1}(f_1).$$

Now for complex-valued functions f_1, g_1, h_1 , we know that

$$\lambda_{g_1 \cdot h_1}(f_1) \geq \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \}.$$

Therefore, $\lambda_{g_1 \cdot h_1}(f_1) \geq \lambda_{h_1}(f_1)$.

Again, in $\mathbb{C}_1$,

$$\lambda_{g_1}(f_1) \geq \lambda_{h_1}(f_1) \implies \lambda_{g_1 \cdot h_1}(f_1) \geq \lambda_{g_1}(f_1).$$

Since $\lambda_{g_1}(f_1) = \infty$, we have $\lambda_{g_1 \cdot h_1}(f_1) = \infty$.

From the definition of multicomplex order,

$$\lambda_{g \cdot h}(f) \geq \lambda_{g_1 \cdot h_1}(f_1) \implies \lambda_{g \cdot h}(f) = \infty.$$

Thus, the theorem is true.

Case 3: Let $\lambda_g(f)$ and $\lambda_h(f)$ both be infinite, then from **Case 2** we can say that $\lambda_{g \cdot h}(f) = \infty$. Hence, the theorem is true.

Case 4: Let all $\lambda_{g \cdot h}(f)$, $\lambda_g(f)$ and $\lambda_h(f)$ be finite. Then two subcases arise.

Subcase I: Let $\lambda_g(f) > \lambda_h(f)$. Now, f, g, h are bicomplex entire functions. Then we can write

$$f(z) = f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2,$$

$$g(z) = g_1(z_1 - i_1 z_2)e_1 + g_2(z_1 + i_1 z_2)e_2,$$

$$h(z) = h_1(z_1 - i_1 z_2)e_1 + h_2(z_1 + i_1 z_2)e_2,$$

where f_i, g_i, h_i are complex-valued entire functions for $i = 1, 2$.

Further, let $A_i = \min \{ \lambda_{h_i}(f_i), \lambda_{g_i}(f_i) \}$ for $i = 1, 2$.

Now for any $\epsilon > 0$, from the definition of $\lambda_{g_i}(f_i)$ and $\lambda_{h_i}(f_i)$, we have sequences $\{r'_n\} \rightarrow \infty$ and $\{r''_n\} \rightarrow \infty$ such that

$$M_{g_i} \left(r'^{\lambda_{g_i}(f_i) - \epsilon}_n \right) < M_{f_i}(r'_n) \quad \text{and} \quad M_{h_i} \left(r''^{\lambda_{h_i}(f_i) - \epsilon}_n \right) < M_{f_i}(r''_n) \quad \text{where } i = 1, 2. \quad (2.16)$$

Let $r_n = \max \{ r'_n, r''_n \}$. Then $\{r_n\} \rightarrow \infty$ and (2.16) holds for r_n . From the above, for $\{r_n\} \rightarrow \infty$, we get

$$M_{g_i} \left(r_n^{A_i - \epsilon} \right) < M_{f_i}(r_n) \quad \text{and} \quad M_{h_i} \left(r_n^{A_i - \epsilon} \right) < M_{f_i}(r_n), \quad \text{where } i = 1, 2.$$

Since

$$\begin{aligned} M_{g_i \cdot h_i} (r_n^{A_i - \epsilon}) &\leq M_{g_i} (r_n^{A_i - \epsilon}) \cdot M_{h_i} (r_n^{A_i - \epsilon}) < [M_{f_i}(r_n)]^2 \\ \text{i.e., } [M_{g_i \cdot h_i} (r_n^{A_i - \epsilon})]^{\frac{1}{2}} &< M_{f_i}(r_n). \end{aligned} \quad (2.17)$$

From (2.17) and for any $\delta > 1$, and for all sufficiently large values of $r_1, r_2, \dots, r_n$, we get

$$M_{g_i \cdot h_i} (r_n^{\delta(A_i - \epsilon)}) < M_{f_i}(r_n). \quad (2.18)$$

Since $g_i \cdot h_i$ satisfies the equation (2.18), and since $\epsilon > 0$ is arbitrary, letting $\lambda \rightarrow 1^+$, we obtain

$$\lambda_{g_i \cdot h_i}(f_i) \geq \min \{ \lambda_{g_i}(f_i), \lambda_{h_i}(f_i) \}, \quad \text{for } i = 1, 2$$

i.e.,

$$\begin{aligned} \lambda_{g_1 \cdot h_1}(f_1) &\geq \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \}, \\ \lambda_{g_2 \cdot h_2}(f_2) &\geq \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \}. \end{aligned}$$

Now,

$$\lambda_{g \cdot h}(f) = \max \{ \lambda_{g_1 \cdot h_1}(f_1), \lambda_{g_2 \cdot h_2}(f_2) \} \geq \lambda_{g_1 \cdot h_1}(f_1) \geq \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \} \quad (2.19)$$

Similarly,

$$\lambda_{g \cdot h}(f) \geq \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \} \quad (2.20)$$

From (2.19) and (2.20), we have

$$\lambda_{g \cdot h}(f) \geq \max \{ \min \{ \lambda_{g_1}(f_1), \lambda_{h_1}(f_1) \}, \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \} \} \quad (2.21)$$

Since

$$\begin{aligned} \lambda_g(f) &> \lambda_h(f) \\ \text{i.e., } \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \} &> \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \}, \end{aligned} \quad (2.22)$$

two cases arise.

Subcase I.a): If $\lambda_{g_1}(f_1) > \lambda_{g_2}(f_2)$, then from (2.22) we have

$$\lambda_{g_1}(f_1) > \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \} > \lambda_{h_1}(f_1).$$

Similarly $\lambda_{g_1}(f_1) > \lambda_{h_2}(f_2)$.

Subcase I.b): If $\lambda_{g_1}(f_1) < \lambda_{g_2}(f_2)$, then from (2.22) we have

$$\lambda_{g_2}(f_2) > \max \{ \lambda_{h_1}(f_1), \lambda_{h_2}(f_2) \} > \lambda_{h_2}(f_2).$$

Similarly $\lambda_{g_2}(f_2) > \lambda_{h_1}(f_1)$.

So, from equation (2.21) and **Subcase I.a**), we get

$$\lambda_{g \cdot h}(f) \geq \max \{ \lambda_{g_1}(f_1), \min \{ \lambda_{g_2}(f_2), \lambda_{h_2}(f_2) \} \} \geq \lambda_{g_1}(f_1). \quad (2.23)$$

Similarly from equation (2.21) and **Subcase I.b**), we get

$$\lambda_{g \cdot h}(f) \geq \lambda_{g_2}(f_2). \quad (2.24)$$

From (2.23) and (2.24), we have

$$\begin{aligned} \lambda_{g \cdot h}(f) &\geq \max \{ \lambda_{g_1}(f_1), \lambda_{g_2}(f_2) \} \\ \text{i.e., } \lambda_{g \cdot h}(f) &\geq \lambda_g(f) \geq \min \{ \lambda_g(f), \lambda_h(f) \}. \end{aligned}$$

Subcase II: Let $\lambda_g(f) = \lambda_h(f)$, then from **Subcase I**, the result follows.

This completes the proof.

3. Future Prospect

The distribution of values of a bicomplex-valued entire function can be determined by its order. But for the bicomplex-valued meromorphic function, we cannot measure the growth rate by using the maximum modulus, since a meromorphic function can hit the infinity on a disc of finite radius. To overcome the situation one may think of Nevanlinna's value distribution theory to study the growth of a bicomplex meromorphic function and it is a wide area for future research.

4. Conclusion

The introduction of bicomplex (quaternions) numbers is a very useful mathematical tool that enhances the analysis, rendering it much easier and much more productive, and our paper includes relative order and relative lower order, which plays an important role in finding the growth of any holomorphic function with respect to another holomorphic function.

Even though there are various proofs and approaches for proving the sum and product theorem in connection with the relative order and relative lower order of the entire function in $\mathbb{C}$ and several complex variables $\mathbb{C}_n$, in this paper our target is to prove this theorem in the bicomplex plane and in future to extend it in multicomplex plane.

Acknowledgement

Authors are grateful to the anonymous referee(s) for careful checking in details and also for helpful comments towards the improvement of the paper. The third author sincerely acknowledges the financial support under FRG(Faculty Research Grant) during April, 2026- March, 2027 rendered by University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India

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