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On evaluation of certain Ramanujan's theta functions

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Abstract: In this paper, we have established modular identities with the help of Ramanujan's modular equations and using these identities we have evaluated certain theta functions defined by Ramanujan.

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Introduction, Notations and Definitions

In the Second Notebook of Ramanujan [3], he has recorded several results in the theories of elliptic functions to alternative bases. In Entries 51 - 72 of chapter 25 of the Second Notebook, Ramanujan states twenty three elegant P - Q identities. The main purpose of this paper is to establish certain P - Q identities by making use of modular equations found in the Second Notebook and then to evaluate Ramanujan's theta functions for some particular values of q by using these P - Q identities.

For real or complex q ($|q| < 1$), let
$[\alpha; q]_\infty = \prod_{k=0}^{\infty}(1 - \alpha q^k)$,
where α is any complex number.
Also,
$[a_1, a_2, ..., a_r; q]_\infty = [a_1; q]_\infty [a_2; q]_\infty ... [a_r; q]_\infty$.
Ramanujan defined the general theta function as,

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1}$$

which by an appeal of Jacobi's triple product identity [2; App. II (II.28)],yields

$$f(a, b) = [ab, -a, -b; ab]_\infty. \tag{2}$$

The most important special cases of (1) are,

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q;-q]_\infty}{[q;-q]_\infty} = [q^2;q^2]_\infty [-q;q^2]_\infty^2, \tag{3}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty}, \tag{4}$$

$$f(-q) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n-1)/2} = [q;q]_\infty, \tag{5}$$

and

$$\chi(-q) = [q;q^2]_\infty. \tag{6}$$

Let
$z_r = z(r,x) =_2 F_1[1/r,(r-1)/r;1;x]$
and

$$q_r = q_r(x) = exp\left[-\pi cosec\frac{\pi}{r}\frac{{}_2F_1[1/r,(r-1)/r;1;1-x]}{{}_2F_1[1/r,(r-1)/r;1;x]}\right], \tag{7}$$

where r = 2,3,4,6 and $|x| < 1$.
${}_2F_1[a,b;c;x] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$,
with $(a)_k = a(a+1)(a+2)...(a+k-1); (a)_0 = 1$.
Let n denote a fixed natural number and assume that

$$n\frac{{}_2F_1[1/r,(r-1)/r;1;1-\alpha]}{{}_2F_1[1/r,(r-1)/r;1;\alpha]} = \frac{{}_2F_1[1/r,(r-1)/r;1;1-\beta]}{{}_2F_1[1/r,(r-1)/r;1;\beta]} \tag{8}$$

where r = 2,3,4 and 6. Then a modular equation of degree 'n' in the theory of elliptic function of signature 'r' is a relation between α and β induced by (8). We often say that β has degree n over α and $m(r) = z(r,\alpha)/z(r,\beta)$ is called the multiplier.

we shall use the following modular equations due to Ramanujan in our analysis
(i) If β and the multiplier m have degree 3, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}, \tag{9}$$

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}, \tag{10}$$

[1;Entry (17.3.21) p. 391]

(ii) If β and the multiplier m have degree 5, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \tag{11}$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}, \tag{12}$$

[1;Entry (17.3.22) p. 391]

(iii) If β and the multiplier m have degree 9, then

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}, \tag{13}$$

$$\frac{3}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}, \tag{14}$$

[1;Entry (17.3.24) p. 391]

We shall also make use of the following results due to Ramanujan,

(a)

$$\phi(q) = \sqrt{z}, \tag{15}$$

[3;chapter 17, entry 10(i)]

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{x(1-x)/q}}, \tag{16}$$

$$\chi(-q) = \frac{\sqrt[6]{2}\,\sqrt[12]{1-x}}{\sqrt[24]{x/q}}, \tag{17}$$

$$\chi(-q^2) = \frac{\sqrt[3]{2}\,\sqrt[24]{1-x}}{\sqrt[12]{x/q}}, \tag{18}$$

[3;chapter 17, entry 12(v),(vi),(vii)]

(b) If $AB = \pi$, then

$$\sqrt{A}\phi(e^{-A^2}) = \sqrt{B}\phi(e^{-B^2}). \tag{19}$$

[3;chapter 16, entry 27(i)]

Modular Identities

In this section establish modular identities which will be used in next section. **(i)** In modular

equations (9) and (10), β is of degree 3 over α, so from (15),(16),(17) and (18) we have:

$$\phi(q) = \sqrt{z_1}, \phi(q^3) = \sqrt{z_3}, \tag{20}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\alpha(1-\alpha)/q}}, \chi(q^3) = \frac{\sqrt[6]{2}}{\sqrt[24]{\beta(1-\beta)/q^3}}, \tag{21}$$

$$\chi(-q) = \frac{\sqrt[6]{2}\,\sqrt[12]{1-\alpha}}{\sqrt[24]{\alpha/q}}, \chi(-q^3) = \frac{\sqrt[6]{2}\,\sqrt[12]{1-\beta}}{\sqrt[24]{\beta/q^3}}, \tag{22}$$

$$\chi(-q^2) = \frac{\sqrt[3]{2}\,\sqrt[24]{1-\alpha}}{\sqrt[12]{\alpha/q}}, \text{ and } \chi(-q^6) = \frac{\sqrt[3]{2}\,\sqrt[24]{1-\beta}}{\sqrt[12]{\beta/q^3}}. \tag{23}$$

Now, let,

$$P = \frac{\phi(q)}{\phi(q^3)} = \sqrt{m}, \tag{24}$$

$$Q = \frac{\chi(q^3)}{q^{1/12}\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}, \tag{25}$$

$$R = \frac{q^{1/12}\chi(-q)}{\chi(-q^3)} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/12}\left(\frac{\beta}{\alpha}\right)^{1/24}, \tag{26}$$

$$S = \frac{q^{1/6}\chi(-q^2)}{\chi(-q^6)} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/24}\left(\frac{\beta}{\alpha}\right)^{1/12}, \tag{27}$$

Thus we have,

$$u = \frac{\chi(q^3)\chi(-q)}{\chi(q)\chi(-q^3)} = QR = \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}, \tag{28}$$

$$v = \frac{q^{1/4}\chi(-q^2)\chi(q)}{\chi(-q^6)\chi(q^3)} = \frac{S}{Q} = \left(\frac{\beta}{\alpha}\right)^{1/8}, \tag{29}$$

Now, eliminating α, β and m from (9) and (10) using (24) and (25) we get,

$$P^8 + P^4Q^{12} = P^4 + 9Q^{12} \tag{30}$$

Eliminating α, β and m from (9) and (10) using (24) and (28) we get,

$$(P^4u^4 - 1)(P^4u^4 - 9) = P^4(u^4 - 1)^2. \tag{31}$$

Again, eliminating α, β and m from (9) and (10) using (24) and (29) we get,

$$(P^4 - v^4)(P^4 - 9v^4) = P^4(v^4 - 1)^2. \tag{32}$$

(ii) In (11) and (12), β is degree 5 over α, so from (15),(16),(17) and (18) we have,

$$\phi(q) = \sqrt{z_1}, \phi(q^5) = \sqrt{z_5}, \tag{33}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\alpha(1-\alpha)/q}}, \chi(q^5) = \frac{\sqrt[6]{2}}{\sqrt[24]{\beta(1-\beta)/q^5}}, \tag{34}$$

$$\chi(-q) = \frac{\sqrt[6]{2}\sqrt[12]{1-\alpha}}{\sqrt[24]{\alpha/q}}, \chi(-q^5) = \frac{\sqrt[6]{2}\sqrt[12]{1-\beta}}{\sqrt[24]{\beta/q^5}}, \tag{35}$$

$$\chi(-q^2) = \frac{\sqrt[3]{2}\sqrt[24]{1-\alpha}}{\sqrt[12]{\alpha/q}}, \text{ and } \chi(-q^{10}) = \frac{\sqrt[3]{2}\sqrt[24]{1-\beta}}{\sqrt[12]{\beta/q^5}}. \tag{36}$$

Let,

$$P = \frac{\phi(q)}{\phi(q^5)} = \sqrt{m}, \tag{37}$$

$$Q = \frac{\chi(q^5)}{q^{1/6}\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}, \tag{38}$$

$$R = \frac{q^{1/6}\chi(-q)}{\chi(-q^5)} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/12}\left(\frac{\beta}{\alpha}\right)^{1/24}, \tag{39}$$

$$S = \frac{q^{1/3}\chi(-q^2)}{\chi(-q^{10})} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/24}\left(\frac{\beta}{\alpha}\right)^{1/12}. \tag{40}$$

Thus we have,

$$u = \frac{\chi(q^5)\chi(-q)}{\chi(q)\chi(-q^5)} = QR = \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}, \tag{41}$$

$$v = \frac{q^{1/2}\chi(-q^2)\chi(q)}{\chi(-q^{10})\chi(q^5)} = \frac{S}{Q} = \left(\frac{\beta}{\alpha}\right)^{1/8}. \tag{42}$$

Now, eliminating α, β and m from (11) and (12) using (37) and (38) we get,

$$(P^2Q^6 + 1) = Q^6 + \frac{5}{P^2}. \tag{43}$$

Eliminating α, β and m from (9) and (10) using (37) and (41) we get,

$$(P^2u^2 - 1)(P^2u^2 - 5) = P^2(u^2 - 1)^2. \tag{44}$$

Again, eliminating α, β and m from (11) and (12) by making use of (37) and (42) we get,

$$(P^2 - v^2)(P^2 - 5v^2) = P^2(v^2 - 1)^2. \tag{45}$$

(iii) In (13) and (14), β is degree 9 over α, so from (15),(16),(17) and (18) we have,

$$\phi(q) = \sqrt{z_1}, \phi(q^9) = \sqrt{z_9}, \tag{46}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\alpha(1-\alpha)/q}}, \chi(q^9) = \frac{\sqrt[6]{2}}{\sqrt[24]{\beta(1-\beta)/q^9}}, \tag{47}$$

$$\chi(-q) = \frac{\sqrt[6]{2}\sqrt[12]{1-\alpha}}{\sqrt[24]{\alpha/q}}, \chi(-q^9) = \frac{\sqrt[6]{2}\sqrt[12]{1-\beta}}{\sqrt[24]{\beta/q^9}}, \tag{48}$$

$$\chi(-q^2) = \frac{\sqrt[3]{2}\sqrt[24]{1-\alpha}}{\sqrt[12]{\alpha/q}}, \text{ and } \chi(-q^{18}) = \frac{\sqrt[3]{2}\sqrt[24]{1-\beta}}{\sqrt[12]{\beta/q^9}}. \tag{49}$$

Let,

$$P = \frac{\phi(q)}{\phi(q^9)} = \sqrt{m}, \tag{50}$$

$$Q = \frac{\chi(q^9)}{q^{1/3}\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}, \tag{51}$$

$$R = \frac{q^{1/3}\chi(-q)}{\chi(-q^9)} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/12}\left(\frac{\beta}{\alpha}\right)^{1/24}, \tag{52}$$

$$S = \frac{q^{2/3}\chi(-q^2)}{\chi(-q^{18})} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/24}\left(\frac{\beta}{\alpha}\right)^{1/12}. \tag{53}$$

Thus we have,

$$u = \frac{\chi(q^9)\chi(-q)}{\chi(q)\chi(-q^9)} = QR = \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}, \tag{54}$$

$$v = \frac{q\chi(-q^2)\chi(q)}{\chi(-q^{18})\chi(q^9)} = \frac{S}{Q} = \left(\frac{\beta}{\alpha}\right)^{1/8}. \tag{55}$$

Now, eliminating α, β and m from (13) and (14) using (50) and (51) we get,

$$(PQ^3 + 1) = Q^3 + \frac{3}{P}. \tag{56}$$

Eliminating α, β and m from (13) and (14) using (50) and (54) we get,

$$(Pu - 1)(Pu - 3) = P(u - 1)^2. \tag{57}$$

Again, eliminating α, β and m from (13) and (14) by making use of (50) and (55) we get,

$$(P - v)(P - 3v) = P(v - 1)^2. \tag{58}$$

Evaluation of theta functions

(i) Taking $A = \frac{\sqrt{\pi}}{\sqrt[4]{3}}$ and $B = \sqrt{\pi}\sqrt[4]{3}$ in (19) we get,

$$\frac{\phi(e^{-\pi/\sqrt{3}})}{\phi(e^{-\pi\sqrt{3}})} = \sqrt[4]{3}. \tag{59}$$

Now for $q = e^{-\pi/\sqrt{3}}$in (24) we obtain,

$$P = \frac{\phi(e^{-\pi/\sqrt{3}})}{\phi(e^{-\pi\sqrt{3}})} = \sqrt[4]{3}. \tag{60}$$

Putting this value of P in (30), we obtain,

$$Q = \frac{\chi(e^{-\pi\sqrt{3}})}{e^{-\pi/12\sqrt{3}}\chi(e^{-\pi/\sqrt{3}})} = 1. \tag{61}$$

Putting this value of P in (31) we get,

$$u^4 = 2 \pm \sqrt{3}. \tag{62}$$

Taking negative sign in (62) we have

$$u = \frac{\chi(e^{-\pi\sqrt{3}})\chi(-e^{-\pi/\sqrt{3}})}{\chi(e^{-\pi/\sqrt{3}})\chi(-e^{-\pi\sqrt{3}})} = \sqrt[4]{2-\sqrt{3}}. \tag{63}$$

Again, putting the value of P in (32) we obtain,

$$v^4 = 2 \pm \sqrt{3}. \tag{64}$$

Taking positive sign in (64) we have,

$$v = \frac{e^{-\pi/4\sqrt{3}}\chi(-e^{-2\pi/\sqrt{3}})\chi(e^{-\pi/\sqrt{3}})}{\chi(-e^{-2\pi\sqrt{3}})\chi(e^{-\pi\sqrt{3}})} = \sqrt[4]{2+\sqrt{3}}. \tag{65}$$

From (61) and (63) we have,

$$\frac{e^{-\pi/12\sqrt{3}}\chi(-e^{-\pi/\sqrt{3}})}{\chi(-e^{-\pi\sqrt{3}})} = \sqrt[4]{2-\sqrt{3}}. \tag{66}$$

From (61) and (65) we have

$$\frac{e^{-\pi/6\sqrt{3}}\chi(-e^{-2\pi/\sqrt{3}})}{\chi(-e^{-2\pi\sqrt{3}})} = \sqrt[4]{2+\sqrt{3}}. \tag{67}$$

(ii) Taking $A = \frac{\sqrt{\pi}}{\sqrt[4]{5}}$ and $B = \sqrt{\pi}\sqrt[4]{5}$ in (19) we get,
For $q = e^{-\pi/\sqrt{5}}$, (37) yields,

$$\frac{\phi(e^{-\pi/\sqrt{5}})}{\phi(e^{-\pi\sqrt{5}})} = \sqrt[4]{5}. \tag{68}$$

$$P = \frac{\phi(e^{-\pi/\sqrt{5}})}{\phi(e^{-\pi\sqrt{5}})} = \sqrt[4]{5}. \tag{69}$$

Putting this value of P in (43), we obtain,

$$Q = \frac{\chi(e^{-\pi\sqrt{5}})}{e^{-\pi/6\sqrt{5}}\chi(e^{-\pi/\sqrt{5}})} = 1. \tag{70}$$

Putting this value of P in (44) we get,

$$u^2 = \frac{2 \pm \sqrt{2\sqrt{5}-2}}{\sqrt{5}-1}. \tag{71}$$

Taking negative sign we have,

$$u = \frac{\chi(e^{-\pi\sqrt{5}})\chi(-e^{-\pi/\sqrt{5}})}{\chi(e^{-\pi/\sqrt{5}})\chi(-e^{-\pi\sqrt{5}})} = \sqrt{\frac{2-\sqrt{(2\sqrt{5}-2)}}{(\sqrt{5}-1)}}. \tag{72}$$

Putting the value of P in (45) we get,

$$v^2 = \frac{2 \pm \sqrt{2\sqrt{5}-2}}{\sqrt{5}-1}. \tag{73}$$

Taking positive sign in (73) we have,

$$v = \frac{e^{-\pi/2\sqrt{5}}\chi(-e^{-2\pi/\sqrt{5}})\chi(e^{-\pi/\sqrt{5}})}{\chi(-e^{-2\pi\sqrt{5}})\chi(e^{-\pi\sqrt{5}})} = \sqrt{\frac{2+\sqrt{(2\sqrt{5}-2)}}{(\sqrt{5}-1)}}. \tag{74}$$

Also, from (70) and (72) we have,

$$u = \frac{e^{-\pi/3\sqrt{6}}\chi(-e^{-\pi/\sqrt{5}})}{\chi(-e^{-\pi\sqrt{5}})} = \sqrt{\frac{2-\sqrt{(2\sqrt{5}-2)}}{(\sqrt{5}-1)}}. \tag{75}$$

Also, from (70) and (73) we have,

$$\frac{e^{-\pi/3\sqrt{5}}\chi(-e^{-2\pi/\sqrt{5}})}{\chi(-e^{-2\pi\sqrt{5}})} = \sqrt{\frac{2+\sqrt{(2\sqrt{5}-2)}}{(\sqrt{5}-1)}}. \tag{76}$$

(iii) Taking $A = \frac{\sqrt{\pi}}{\sqrt{3}}$ and $B = \sqrt{3\pi}$ in (19) we get,

$$\frac{\phi(e^{-\pi/3})}{\phi(e^{-3\pi})} = \sqrt{3}. \tag{77}$$

For $q = e^{-\pi/3}$ in (50) yields,

$$P = \frac{\phi(e^{-\pi/3})}{\phi(e^{-3\pi})} = \sqrt{3}. \tag{78}$$

Putting this value of P in (56) we obtain,

$$Q = \frac{\chi(e^{-3\pi})}{e^{-\pi/9}\chi(e^{-\pi/3})} = 1. \tag{79}$$

Putting this value of P in (57) we find,

$$u = \frac{1 \pm \sqrt{2\sqrt{3}-3}}{\sqrt{3}-1}. \tag{80}$$

Taking negative sign we have,

$$u = \frac{\chi(e^{-3\pi})\chi(-e^{-\pi/3})}{\chi(e^{-\pi/3})\chi(-e^{-3\pi})} = \frac{1-\sqrt{(2\sqrt{3}-3)}}{(\sqrt{3}-1)}. \tag{81}$$

Again putting the value of P in (58) and taking positive sign we find,

$$v = \frac{e^{-\pi/3}\chi(-e^{-2\pi/3})\chi(e^{-\pi/3})}{\chi(-e^{-6\pi})\chi(e^{-3\pi})} = \frac{1+\sqrt{(2\sqrt{3}-3)}}{(\sqrt{3}-1)}. \tag{82}$$

From (79) and (81) we find

$$\frac{e^{-\pi/9}\chi(-e^{-\pi/3})}{\chi(-e^{-3\pi})} = \frac{1-\sqrt{(2\sqrt{3}-3)}}{(\sqrt{3}-1)}. \tag{83}$$

Also, from (79) and (82) we have

$$\frac{e^{-2\pi/9}\chi(-e^{-2\pi/3})}{\chi(-e^{-6\pi})} = \frac{1+\sqrt{(2\sqrt{3}-3)}}{(\sqrt{3}-1)}. \tag{84}$$

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