

**A NEW GENERALIZATION OF q -TRUNCATED
EXPONENTIAL-BASED APPELL POLYNOMIALS WITHIN THE
CONTEXT OF QUANTUM CALCULUS**

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Abstract: In this work, we set out on an exhaustive analysis of trivariate q -truncated exponential-based Appell polynomials (Tq -TEbAP), which are explicitly built by a generating function with a q -calculus profile. First, an analysis will be provided for these polynomials by means of quasi-monomiality, which will lead to the fundamental operational relations and a simple form for the operators. Subsequently, concrete forms for the operators will be obtained, along with the related q -differential and partial differential equations that these polynomials obey. Last but not least, a number of summation formulas will be provided to highlight the intrinsic analytic properties of these polynomials.

Keywords and Phrases: Quantum calculus, q -truncated polynomials, q -truncated-Appell polynomials, Quasi monomiality, Extension of monomiality principle, q -Dilatation operator, Partial differential equations, Differential equations.

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1. Introduction

Special functions occupy a central place in many areas of mathematics and the applied sciences. They appear naturally as solutions to differential equations, which model a wide range of physical and engineering phenomena. Traditional special functions, such as the Hermite, Legendre, Laguerre, and Bessel functions, play

a crucial role in mathematical physics, informing us about quantum mechanics, the propagation of waves, and the flow of heat. Their well-trodden properties, like generating functions, recurrence relations, and orthogonality, offer us powerful analytical tools to tackle complex problems.

But special functions play a much larger role, being the building blocks of approximation theory, numerical analysis, and computational mathematics. They also occur naturally in combinatorics, number theory, and probability, aiding us in the search for hidden connections between mathematical objects. As a subject, the theory of special functions is alive and well, and its range and depth ensure its continued evolution. The generalized Hermite polynomials are an organized and elegant system that has captured interest in both theory and application. The defining features of the generalized Hermite polynomials, such as their orthogonality, recurrence relations, and their generating functions, are significant properties that make them valuable in different analytical situations. The generalized Hermite polynomials are strongly connected to the quantum harmonic oscillator, and they are used in the wave function and in the description of the energy levels in quantum mechanics.

The 2-variable truncated exponential polynomials (2VTP) denoted by $e_n^{(r)}(\tau, \gamma)$ are specified by generating relation [12, 13, 22]:

$$\frac{e^{\tau t}}{1 - \gamma t^r} = \sum_{n=0}^{\infty} e_n^{(r)}(\tau, \gamma) \frac{t^n}{n!}, \quad (1.1)$$

where

$$e_n^{(r)}(\tau, \gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \frac{\gamma^k \tau^{n-rk}}{(n-rk)!}. \quad (1.2)$$

Very recently, in [33], Özat et al. presented the general form of truncated-exponential-based general Appell polynomials, utilizing the framework of two-variable general-Appell polynomials as follows:

$$A(t)e^{\tau t}\psi(\phi, t) \frac{1}{1 - \gamma t^m} = \sum_{s=0}^{\infty} e^{(m)}_s \mathcal{T}_s(\tau, \phi, \gamma) \frac{t^n}{n!}.$$

In addition, the authors successfully derived explicit and determinant representations, along with operators for lowering and raising, a recurrence relation, a differential equation, and various summation formulas pertinent to these polynomials.

The field of quantum calculus, alternatively known as q -calculus, represents an area of burgeoning research interest. For $q \rightarrow 1^-$, quantum calculus converges

to classical calculus. In contemporary scientific discourse, quantum calculus has exhibited its utility across diverse disciplines, encompassing mathematical sciences, quantum physics, quantum mechanics, quantum algebra, approximation theory, and operator theory, among other domains. We now provide some basic properties of quantum analysis. For the following properties and more details, please see (see [6, 9, 17, 21, 23, 35, 44]). Assume $0 < q < 1$. The q -shifted factorial $(\gamma; q)_v$ is defined as:

$$(\vartheta; q)_u = \prod_{w=1}^{u-1} (1 - q^w \vartheta), \quad (u \in \mathbb{N}), \quad (\vartheta; q)_0 = 1. \quad (1.3)$$

The q -analogue of a number γ is given by

$$[\vartheta]_q = \frac{1 - q^\vartheta}{1 - q}, \quad 0 < q < 1; \quad \vartheta \in \mathbb{C}. \quad (1.4)$$

The q -factorial is given by (see [40-42, 27, 28])

$$[\vartheta]_q! = \begin{cases} \prod_{w=1}^{\vartheta} [w]_q, & \vartheta \geq 1 \\ 1, & \gamma = 0 \end{cases}. \quad (1.5)$$

The following equality holds:

$$(\tau \pm \gamma)_q^v = \sum_{s=0}^v \binom{v}{s}_q q^{\binom{v-s}{2}} \tau^s (\pm \gamma)^{v-s}, \quad (1.6)$$

where

$$\binom{\vartheta}{w}_q = \frac{[\vartheta]_q!}{[\vartheta - w]_q! [w]_q!}.$$

The following equalities hold [10, 29, 30, 31, 32]:

$$e_q(\tau) = \sum_{w=0}^{\infty} \frac{\tau^w}{[w]_q!}, \quad 0 < q < 1, \quad (1.7)$$

and

$$E_q(\tau) = \sum_{w=0}^{\infty} \frac{q^{\binom{w}{2}} \tau^w}{[w]_q!}, \quad 0 < q < 1. \quad (1.8)$$

The following equality holds:

$$e_q(\tau) E_q(\phi) = \sum_{n=0}^{\infty} \frac{(\tau + \phi)_q^n}{[n]_q!}. \quad (1.9)$$

Hence

$$e_q(\tau)E_q(-\tau) = 1. \quad (1.10)$$

The following equality holds:

$$\widehat{D}_{q,\tau}h(\tau) = \frac{h(q\tau) - h(\tau)}{\tau(1-q)}, \quad 0 < q < 1, \quad \tau \neq 0. \quad (1.11)$$

In particular, we have

$$\widehat{D}_{q,\tau}\tau^w = [w]_q\tau^{w-1}, \quad (1.12)$$

$$\widehat{D}_{q,\tau}e_q(\alpha\tau) = \alpha e_q(\alpha\tau), \quad \alpha \in \mathbb{C}, \quad (1.13)$$

and

$$\widehat{D}_{q,\tau}^w e_q(\alpha\tau) = \alpha^w e_q(\alpha\tau), \quad w \in \mathbb{N}, \quad \alpha \in \mathbb{C}, \quad (1.14)$$

where $\widehat{D}_{q,\tau}^w$ denotes the w^{th} order q -derivative with respect to τ .

The q -derivative concerning the product of the functions $f(\tau)$ and $g(\tau)$ has been thoroughly examined in existing literature [1, 2, 26, 34]:

$$\widehat{D}_{q,\tau}(h(\tau)k(\tau)) = h(\tau)\widehat{D}_{q,\tau}k(\tau) + k(q\tau)\widehat{D}_{q,\tau}h(\tau) \quad (1.15)$$

and

$$\widehat{D}_{q,\tau}\left(\frac{h(\tau)}{k(\tau)}\right) = \frac{k(q\tau)\widehat{D}_{q,\tau}h(\tau) - h(q\tau)\widehat{D}_{q,\tau}k(\tau)}{k(\tau)k(q\tau)}. \quad (1.16)$$

The q -integral of a function $h(\tau)$ is introduced as

$$\int_0^a h(\tau)d_q\tau = (1-q)a \sum_{w=0}^{\infty} q^w h(aq^w). \quad (1.17)$$

By virtue of (1.17), we get

$$\begin{aligned} \int_0^\tau \lambda^w d_q \lambda &= (1-q)\tau \sum_{n=0}^{\infty} \tau^n q^{n(w+1)} \\ &= \frac{\tau^{w+1}(1-q)}{1-q^{w+1}} = \frac{\tau^{w+1}}{[w+1]_q}, \quad w \in \mathbb{N} \sqcup \{0\}. \end{aligned} \quad (1.18)$$

In particular $\widehat{D}_{q,\tau}^{-1}\{1\} = \tau$, and then

$$\left(\widehat{D}_{q,\tau}^{-1}\right)^r \{1\} = \frac{\tau^r}{[r]_q!}, \quad r \in \mathbb{N} \sqcup \{0\}. \quad (1.19)$$

The q -GHP, symbolized as $H_{n,q}^{(m)}(\tau, \phi)$, are established through the use of the following generating function [24]

$$e_q(\tau t) e_q(\phi t^m) = \sum_{n=0}^{\infty} H_{n,q}^{(m)}(\tau, \phi) \frac{t^n}{[n]_q!}, \quad (1.20)$$

and series definition

$$H_{n,q}^{(m)}(\tau, \phi) = [n]_q! \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{\phi^k \tau^{n-mk}}{[k]_q! [n-mk]_q!}. \quad (1.21)$$

For $H_{n,q}^{(m)}(\tau, \phi)$, the following equality holds (see [39, 40]):

$$H_{n,q}^{(m)}(\tau, \phi) = e_q(\phi D_{q,\tau}^m) \{\tau^n\}. \quad (1.22)$$

The q -dilatation operator T_z is characterized by its action on any function of the complex variable z as follows: [8, 20]

$$T_z^k f(z) = f(q^k z), \quad k \in \mathbb{R}, \quad (1.23)$$

satisfies the property

$$T_z^{-1} T_z^1 f(z) = f(z). \quad (1.24)$$

The following equality holds [8]:

$$\widehat{D}_{q,t} e_q(\phi t^m) = \phi t^{m-1} T_{(\phi;m)} e_q(\phi t^m). \quad (1.25)$$

where

$$T_{(\phi;m)} = \frac{1 - q^m T_\phi^m}{1 - q T_\phi} = 1 + q T_\phi + \cdots + q^{m-1} T_\phi^{m-1}.$$

Al Salam provides a comprehensive explanation of the generating function for the q -Appell polynomials $\mathcal{A}_{n,q}(\tau)$, as articulated in the formula found in [3, 4, 36, 37].

$$\mathcal{A}_q(t) e_q(\tau t) = \sum_{w=0}^{\infty} \mathcal{A}_{w,q}(\tau) \frac{t^w}{[w]_q!}, \quad (1.26)$$

$$\mathcal{A}_q(t) = \sum_{w=0}^{\infty} \mathcal{A}_{w,q} \frac{t^w}{[w]_q!}, \quad \mathcal{A}_q(t) \neq 0, \quad A_{0,q} = 1. \quad (1.27)$$

The following represents a suitable option for specific individuals within the category of q -Appell polynomials:

Table 1. Certain q -APF

S. No.	q -APF	GF	$\mathcal{A}_q(t)$
I.	The q-BP [18, 19]	$\frac{\delta}{e_q(\delta) - 1} e_q(\tau\delta) = \sum_{v=0}^{\infty} \mathcal{B}_{v,q}(\tau) \frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta) = \frac{\delta}{e_q(\delta) - 1}$
II.	The q-EP [18, 19]	$\frac{[2]_q}{e_q(\delta) + 1} e_q(\tau\delta) = \sum_{v=0}^{\infty} \mathcal{E}_{v,q}(\tau) \frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta) = \frac{[2]_q}{e_q(\delta) + 1}$
III.	The q-GP [18, 19]	$\frac{[2]_q \delta}{e_q(\delta) + 1} e_q(\tau\delta) = \sum_{v=0}^{\infty} \mathcal{G}_{v,q}(\tau) \frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta) = \frac{[2]_q \delta}{e_q(\delta) + 1}$

Appell polynomials of the r^{th} order, which are based on a 2-variable q -truncated exponential function and represented as ${}_{e(r)}A_{n,q}(\tau, \gamma)$, are characterized by the generating function provided in [12]

$$A_q(t) \frac{e_q(\tau t)}{1 - \gamma t^r} = \sum_{n=0}^{\infty} {}_{e(r)}A_{n,q}(\tau, \gamma) \frac{t^n}{[n]_q!}. \quad (1.28)$$

For $A_q(t) = 1$, the above equation reduces to

$$\frac{e_q(\tau t)}{1 - \gamma t^r} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(\tau, \gamma) \frac{t^n}{[n]_q!}, \quad (1.29)$$

where $E_{n,q}^{(r)}(\tau, \gamma)$ are called the 2-variable q -truncated polynomials of order r .

In a recent study, Alam et al. [5] defined and examined the bivariate q -general polynomials denoted as $p_{\theta,q}(\zeta, \eta)$, which are defined by:

$$e_q(\gamma\delta)\phi_q(\nu, \delta) = \sum_{\theta=0}^{\infty} p_{\theta,q}(\gamma, \nu) \frac{\delta^\theta}{[\theta]_q!}, \quad (p_{0,q}(\gamma, \nu) = 1), \quad (1.30)$$

where $\phi_q(\nu, \delta)$ has (at least the formal) series expansion

$$\phi_q(\nu, \delta) = \sum_{\theta=0}^{\infty} \phi_{\theta,q}(\nu) \frac{\delta^\theta}{[\theta]_q!}, \quad (\phi_{0,q}(\nu) \neq 0). \quad (1.31)$$

Khan et al. [24] introduced the new generalization of q -truncated exponential-based general polynomials by means of the following generating function:

$$\frac{1}{1 - \gamma t^r} \phi_q(\eta, t) e_q(\tau t) = \sum_{n=0}^{\infty} {}_p\mathcal{E}_{n,q}^{(r)}(\tau, \eta, \gamma) \frac{t^n}{[n]_q!}. \quad (1.32)$$

The notion of monomiality is a crucial tool for analyzing exceptional polynomials and their attributes. Initially introduced by J.F. Steffensen [38], this concept was later expanded into the realm of quasi-monomiality by Dattoli and his colleagues [14-16]. In the specific context of q -polynomials, Cao et al. [8] further extended the monomiality principle. This extension offers a robust methodology for examining the quasi-monomiality of certain q -special polynomials. Researchers have extensively utilized monomiality frameworks to construct and evaluate hybrid families of special polynomials [43]. The two q -operators, denoted as $\widehat{M}_q$ and $\widehat{P}_q$, are known as the q -multiplicative and q -derivative operators, respectively, for a q -polynomial set $p_{n,q}(\tau)$ ($n \in \mathbb{N}, \tau \in \mathbb{C}$), as demonstrated by [8]

$$\widehat{M}_q \{p_{n,q}(\tau)\} = p_{n+1,q}(\tau), \quad (1.33)$$

and

$$\widehat{P}_q \{p_{n,q}(\tau)\} = [n]_q p_{n-1,q}(\tau). \quad (1.34)$$

The operators $\widehat{M}_q$ and $\widehat{P}_q$ adhere to the following commutation relation:

$$[\widehat{M}_q, \widehat{P}_q] = \widehat{P}_q \widehat{M}_q - \widehat{M}_q \widehat{P}_q. \quad (1.35)$$

The characteristics of the polynomials $p_{n,q}(\tau)$ can be inferred from the properties of the operators $\widehat{M}_q$ and $\widehat{P}_q$. If these operators, $\widehat{M}_q$ and $\widehat{P}_q$, have a differential realization, then the polynomials $p_{n,q}(\tau)$ are governed by the differential equation

$$\widehat{M}_q \widehat{P}_q \{p_{n,q}(\tau)\} = [n]_q p_{n,q}(\tau), \quad (1.36)$$

and

$$\widehat{P}_q \widehat{M}_q \{p_{n,q}(\tau)\} = [n+1]_q p_{n,q}(\tau). \quad (1.37)$$

In view of (1.33) and (1.34), we have

$$[\widehat{M}_q, \widehat{P}_q] = [n+1]_q - [n]_q. \quad (1.38)$$

From (1.33), we have

$$\widehat{M}_q^r \{p_{n,q}\} = p_{n+r,q}(\tau). \quad (1.39)$$

In particular, we have

$$p_{n,q}(\tau) = \widehat{M}_q^n \{p_{0,q}\} = \widehat{M}_q^n \{1\}, \quad (1.40)$$

where $p_{0,q}(\tau) = 1$ is the q -sequel of polynomial $p_{n,q}(\tau)$. Also, the generating function of $p_{n,q}(\tau)$ can be obtained as

$$e_q(\widehat{M}_q t)\{1\} = \sum_{n=0}^{\infty} p_{n,q}(\tau) \frac{t^n}{[n]_q!}. \quad (1.41)$$

In this paper, we define new generalization of q -truncated exponential-based-Appell polynomials, employing a generating function approach that incorporates q -calculus functions. The investigation extends to the analysis of these polynomials within the quasi-monomiality framework, facilitating the establishment of crucial operational identities. Subsequently, operational representations were derived, and both q -differential and partial differential equations were formulated for the aforementioned polynomials. To further elucidate their analytical properties, summation formulas were developed. The article concludes with some examples that demonstrate the behavior of 3-variable q -truncated exponential-based Appell polynomials and highlights their potential applications in various fields.

2. New generalization of q -Truncated exponential-based Appell polynomials

In this section, we introduce a new generalization of q -truncated exponential-based Appell polynomials and obtain series definition, recurrence relation, and explicit formula for these polynomials. We begin with the following definition as.

Utilizing the relations (1.28) and (1.30), we introduce the new generalization of q -truncated polynomials ${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ in the following form:

$$\frac{A_q(t)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t) = \sum_{w=0}^{\infty} {}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!}, \quad (p_{0,q}(\zeta, \eta) = 1), \quad (2.1)$$

where $\phi_q(\eta, \delta)$ has (at least the formal) series expansion

$$\phi_q(\eta, \delta) = \sum_{v=0}^{\infty} \phi_{v,q}(\eta) \frac{\delta^v}{[v]_q!}, \quad (\phi_{0,q}(\eta) \neq 0). \quad (2.2)$$

Theorem 2.1. *The higher derivative formula for the new generalization of q -truncated polynomials ${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ holds true:*

$$\widehat{D}_{q,te(r)}^k \mathcal{T}_{w,q}(\zeta, \eta, \gamma) = \frac{[w]_q!}{[w-k]_q!} {}_{e(r)}\mathcal{T}_{w-k,q}(\zeta, \eta, \gamma). \quad (2.3)$$

Proof. By taking the k^{th} order q -partial derivative with respect to τ on both sides of the generating function (2.1) and subsequently applying equation (1.13) to the left-hand side, it can be deduced that

$$\sum_{n=0}^{\infty} \widehat{D}_{q,te^{(r)}}^k \mathcal{T}_{w,q}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} = t^k \frac{A_q(t)}{1 - \gamma t^r} e_q(\tau t) \phi_q(\eta, t) = \sum_{n=0}^{\infty} e^{(r)} \mathcal{T}_{w,q}(\zeta, \eta, \gamma) \frac{t^{w+k}}{[w]_q!}. \quad (2.4)$$

Upon reorganizing the given series and then equating the corresponding powers of t on both sides of the resulting equation, we confirm assertion (2.3).

Theorem 2.2. *For the new generalization of q -truncated polynomials $e^{(r)} \mathcal{T}_{w,q}(\zeta, \eta, \gamma)$, the following identity holds:*

$$e^{(r)} \mathcal{T}_{w,q}(\zeta, \eta, \gamma) = \sum_{k=0}^w \binom{w}{k}_q A_{k,q} {}_p\mathcal{E}_{w-k,q}^{(r)}(\zeta, \eta, \gamma), \quad (2.5)$$

with $A_{k,q}$ is given by equation (1.27).

Proof. By virequations (1.32), and (2.1), we obtain

$$\sum_{w=0}^{\infty} e^{(r)} \mathcal{T}_{w,q}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} = A_q(t) \sum_{w=0}^{\infty} {}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!}. \quad (2.6)$$

Utilizing the expansion (1.27) of $A_q(t)$ on the left-hand side of equation (2.6), we simplify the expression and equate the coefficients of corresponding powers of t on both sides of the equation, thereby establishing assertion (2.5).

The determinant form plays a crucial role in deriving the exploration of q -special polynomials, offering a compact and refined representation that captures their core attributes. This formulation adeptly illustrates essential features such as orthogonality, recurrence relations, and generating functions, establishing it as a valuable instrument for analysis and manipulation across diverse mathematical settings. Moreover, the determinant form serves as a link to other mathematical frameworks, enabling a deeper understanding of the foundational principles that govern q -special polynomials. Its importance extends beyond mere computational applications, contributing significantly to the theoretical progress in the field of special function theory.

Costabile and Longo [11], expanding on the techniques presented in [25], have effectively derived the determinantal form of q -Appell polynomials.

Theorem 2.3. *The determinant form of the three-variable q -truncated exponential*

Gould-Hopper-based Appell polynomial with degree n , is

$$\begin{aligned}
 {}_{e^{(r)}}\mathcal{T}_{w,q}(\zeta, \eta, \gamma) &= \frac{1}{\beta_{0,q}}, \\
 {}_{e^{(r)}}T_{w,q}(\zeta, \eta, \gamma) &= \frac{(-1)^n}{(\beta_{0,q})^{n+1}} \\
 &\times \begin{vmatrix} 1 & {}_p\mathcal{E}_{1,q}^{(r)}(\zeta, \eta, \gamma) & {}_p\mathcal{E}_{2,q}^{(r)}(\zeta, \eta, \gamma) & \cdots & {}_p\mathcal{E}_{w-1,q}^{(r)}(\zeta, \eta, \gamma) & E_{w,q}^{(r)}(\zeta, \eta, \gamma) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{w-1,q} & \beta_{w,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_q \beta_{1,q} & \cdots & \binom{w-1}{1}_q \beta_{w-2,q} & \binom{w}{1}_q \beta_{w-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \binom{w-1}{1}_q \beta_{w-3,q} & \binom{n}{2}_q \beta_{w-2,q} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{w}{w-1}_q \beta_{1,q} \end{vmatrix}, \quad (2.7) \\
 \beta_{w,q} &= -\frac{1}{A_{0,q}} \left(\sum_{k=1}^w \binom{n}{k}_q A_{k,q} \beta_{w-k,q} \right), \quad w = 0, 1, 2, \dots,
 \end{aligned}$$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ and ${}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma)$, $w = 0, 1, 2, \dots$, are the new generalization of 3-variable q -truncated exponential polynomials defined by equation (1.32).

Proof. By integrating the series expressions of the newly generalized q -truncated exponential polynomials into the generating function of the new generalization of three-variable q -truncated exponential-based Appell polynomials, we derive

$$A_q(t) \sum_{w=0}^{\infty} {}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{T}_{n,q}(\zeta, \eta, \gamma) \frac{t^n}{[n]_q!}. \quad (2.8)$$

Multiplying both sides by

$$\frac{1}{A_q(t)} = \sum_{\gamma=0}^{\infty} \beta_{\gamma,q} \frac{t^\gamma}{[\gamma]_q!}, \quad (2.9)$$

we get

$$\sum_{w=0}^{\infty} {}_p\mathcal{E}_{n,q}^{(r)}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} = \sum_{\gamma=0}^{\infty} \beta_{\gamma,q} \frac{\delta^\gamma}{[\gamma]_q!} \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{T}_{n,q}(\zeta, \eta, \gamma) \frac{t^n}{[n]_q!}. \quad (2.10)$$

Applying Cauchy product in (2.10) gives

$${}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma) = \sum_{\gamma=0}^w \binom{w}{\gamma}_q \beta_{\gamma,q} {}_{e(r)}\mathcal{T}_{w-\gamma,q}(\zeta, \eta, \gamma). \quad (2.11)$$

So, we obtain the system of equations as follows:

$$\begin{aligned} {}_p\mathcal{E}_{0,q}^{(r)}(\zeta, \eta, \gamma) &= \beta_{0,q} {}_{e(r)}\mathcal{T}_{0,q}(\zeta, \eta, \gamma), \\ {}_p\mathcal{E}_{1,q}^{(r)}(\zeta, \eta, \gamma) &= \beta_{0,q} {}_{e(r)}\mathcal{T}_{1,q}(\zeta, \eta, \gamma) + \beta_{1,q} {}_{e(r)}\mathcal{T}_{0,q}(\zeta, \eta, \gamma), \\ {}_p\mathcal{E}_{2,q}^{(r)}(\zeta, \eta, \gamma) &= \beta_{0,q} {}_{e(r)}\mathcal{T}_{2,q}(\zeta, \eta, \gamma) + \binom{2}{1}_q \beta_{1,q} {}_{e(r)}\mathcal{T}_{1,q}(\zeta, \eta, \gamma) + \beta_{2,q} {}_{e(r)}\mathcal{T}_{0,q}(\zeta, \eta, \gamma), \\ &\vdots \\ {}_p\mathcal{E}_{w-1,q}^{(r)}(\zeta, \eta, \gamma) &= \beta_{0,q} {}_{e(r)}\mathcal{T}_{w-1,q}(\zeta, \eta, \gamma) + \binom{w-1}{1}_q \beta_{1,q} {}_{e(r)}\mathcal{T}_{w-2,q}(\zeta, \eta, \gamma) + \cdots + \beta_{w-1,q} {}_{e(r)}\mathcal{T}_{0,q}(\zeta, \eta, \gamma), \\ {}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma) &= \beta_{0,q} {}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma) + \binom{w}{1}_q \beta_{1,q} {}_{e(r)}\mathcal{T}_{w-1,q}(\zeta, \eta, \gamma) + \cdots + \beta_{w,q} {}_{e(r)}\mathcal{T}_{0,q}(\zeta, \eta, \gamma). \end{aligned}$$

Applying Cramers' rule, we get

$${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma) = \frac{\begin{vmatrix} \beta_{0,q} & 0 & \cdots & 0 & {}_p\mathcal{E}_{0,q}^{(r)}(\zeta, \eta, \gamma) \\ \beta_{1,q} & \beta_{0,q} & \cdots & 0 & {}_p\mathcal{E}_{1,q}^{(r)}(\zeta, \eta, \gamma) \\ \beta_{2,q} & \binom{2}{1}_q \beta_{1,q} & \cdots & 0 & {}_p\mathcal{E}_{2,q}^{(r)}(\zeta, \eta, \gamma) \\ \beta_{3,q} & \binom{3}{2}_q \beta_{2,q} & \cdots & 0 & {}_p\mathcal{E}_{3,q}^{(r)}(\zeta, \eta, \gamma) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{w-1,q} & \binom{w-1}{1}_q \beta_{w-2,q} & \cdots & \beta_{0,q} & {}_p\mathcal{E}_{w-1,q}^{(r)}(\zeta, \eta, \gamma) \\ \beta_{w,q} & \binom{w}{1}_q \beta_{w-1,q} & \cdots & \binom{w}{w-1}_q \beta_{1,q} & {}_p\mathcal{E}_{w,q}^{(r)}(\zeta, \eta, \gamma) \end{vmatrix}}{\begin{vmatrix} \beta_{0,q} & 0 & \cdots & 0 & 0 \\ \beta_{1,q} & \beta_{0,q} & \cdots & 0 & 0 \\ \beta_{2,q} & \binom{2}{1}_q \beta_{1,q} & \cdots & 0 & 0 \\ \beta_{3,q} & \binom{3}{2}_q \beta_{2,q} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{w-1,q} & \binom{w-1}{1}_q \beta_{w-2,q} & \cdots & \beta_{0,q} & 0 \\ \beta_{w,q} & \binom{w}{1}_q \beta_{w-1,q} & \cdots & \binom{w}{w-1}_q \beta_{1,q} & \beta_{0,q} \end{vmatrix}}.$$

By taking the transpose in the last equation, we have

$${}_{e(r)}\mathcal{T}_{w,q}(\tau, \phi, \gamma) = \frac{1}{(\beta_{0,q})^{w+1}} \begin{vmatrix} \beta_{0,q} & \beta_{1,q} & \cdots & \beta_{w-1,q} & \beta_{w,q} \\ 0 & \beta_{0,q} & \cdots & \binom{w-1}{1}_q \beta_{w-2,q} & \binom{w}{1}_q \beta_{w-1,q} \\ 0 & 0 & \cdots & \binom{w-1}{1}_q \beta_{w-3,q} & \binom{w}{2}_q \beta_{w-2,q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{0,q} & \binom{w}{w-1}_q \beta_{1,q} \\ {}_p\mathcal{E}_{0,q}^{(r)}(\zeta, \eta, \gamma) & {}_p\mathcal{E}_{1,q}^{(r)}(\zeta, \eta, \gamma) & \cdots & {}_p\mathcal{E}_{w-1,q}^{(r)}(\zeta, \eta, \gamma) & {}_p\mathcal{E}_{n,q}^{(r)}(\zeta, \eta, \gamma) \end{vmatrix}.$$

Thus, simple row operations are used to finish the proof.

The application of the monomiality principle in this context inherently results in the formulation of canonical q -differential operators, for which the polynomials serve as eigenfunctions. This structural characteristic implies that each polynomial family under consideration can be systematically linked to a q -difference equation, thereby reinforcing its analytical and spectral importance. We present the following theorem regarding the operational identities associated with the three-variable q -truncated Appell polynomials.

Theorem 2.4. *The new generalization of q -truncated polynomials ${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ exhibit quasi-monomial properties when subjected to the following q -multiplicative and q -derivative operators:*

$$\widehat{M}_{3VqTAP} = \zeta + \frac{A'_q(\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} T_\zeta + \frac{A_q(q\widehat{D}_{q,\tau}) \phi'_q(\eta, \widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau}) \phi_q(\eta, \widehat{D}_{q,\tau})} T_\zeta + \left(\frac{[r]_q \gamma \widehat{D}_{q,\tau}^{r-1}}{1 - \gamma q^r \widehat{D}_{q,\tau}} \right) \left(\frac{A_q(q\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \right), \quad (2.12)$$

and

$$\widehat{P}_{3VqTAP} = \widehat{D}_{q,\tau}, \quad (2.13)$$

respectively, where T_τ and T_ϕ are the q -dilatation operators as specified in equation (1.23).

Proof. By taking the partial derivative of equation (2.1) with respect to t and applying equations (1.15) and (1.16), we have

$$\begin{aligned} \sum_{w=0}^{\infty} {}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma) w \frac{t^{w-1}}{[w]_q!} &= \frac{1}{1 - \gamma t^r} e_q(q\zeta t) A_q(qt) \frac{\phi'_q(\eta, t)}{\phi_q(\eta, t)} \phi_q(\eta, t) + \frac{\phi_q(\eta, t)}{1 - \gamma t^r} \widehat{D}_{q,t} (e_q(\zeta t) A_q(t)) \\ &\quad + \left(\frac{[r]_q \gamma t^{r-1}}{1 - \gamma t^r} \right) \left(\frac{1}{1 - \gamma q^r t^r} \right) A_q(qt) e_q(q\zeta t) \phi_q(\eta, t) \end{aligned} \quad (2.14)$$

$$\sum_{w=0}^{\infty} {}_{e(r)}\mathcal{T}_{w+1}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} = \frac{1}{1 - \gamma t^r} e_q(q\zeta t) A_q(qt) \frac{\phi'_q(\eta, t)}{\phi_q(\eta, t)} \phi_q(\eta, t) + \frac{\phi_q(\eta, t)}{1 - \gamma t^r} \zeta e_q(\zeta t) A_q(t)$$

$$+ \frac{\phi_q(\eta, t)}{1 - \gamma t^r} e_q(q\zeta t) \frac{A'_q(t)}{A_q(t)} A_q(t) + \left(\frac{[r]_q \gamma t^{r-1}}{1 - \gamma t^r} \right) \left(\frac{1}{1 - \gamma q^r t^r} \right) A_q(qt) e_q(q\zeta t) \phi_q(\eta, t) \quad (2.15)$$

$$\begin{aligned} & \sum_{w=0}^{\infty} e^{(r)} \mathcal{T}_{w+1}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} \\ &= \left(\zeta + \frac{A'_q(t)}{A_q(t)} T_\zeta + \frac{A_q(qt)}{A_q(t)} \frac{\phi'_q(\eta, t)}{\phi_q(\eta, t)} T_\zeta + \left(\frac{[r]_q \gamma t^{r-1}}{1 - \gamma q^r t^r} \right) \left(\frac{A_q(qt)}{A_q(t)} \right) \right) \frac{A_q(t)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t). \end{aligned} \quad (2.16)$$

Therefore, by using (2.1), we have

$$\begin{aligned} & \sum_{w=0}^{\infty} e^{(r)} \mathcal{T}_{w+1}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!} \\ &= \sum_{w=0}^{\infty} \left(\zeta + \frac{A'_q(t)}{A_q(t)} T_\zeta + \frac{A_q(qt)}{A_q(t)} \frac{\phi'_q(\eta, t)}{\phi_q(\eta, t)} T_\zeta + \left(\frac{[r]_q \gamma t^{r-1}}{1 - \gamma q^r t^r} \right) \left(\frac{A_q(qt)}{A_q(t)} \right) \right) e^{(r)} \mathcal{T}_w(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!}, \end{aligned} \quad (2.17)$$

Comparing the coefficients of t on both sides of Equation (2.17) and then from a view of Equation (1.33), the resultant equation gives assertion (2.12). In view of (1.34), we note that Equation (2.3) (for $k = 1$) proves assertion (2.13).

Theorem 2.5. *The following q -differential equations for the new generalization of q -truncated polynomials $e^{(r)} \mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ holds true:*

$$\begin{aligned} & \left(\zeta \widehat{D}_{q,\tau} + \frac{A'_q(\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \widehat{D}_{q,\tau} T_\zeta + \frac{A_q(q\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \frac{\phi'_q(\eta, \widehat{D}_{q,\tau})}{\phi_q(\eta, \widehat{D}_{q,\tau})} \widehat{D}_{q,\tau} T_\zeta \right. \\ & \left. + \left(\frac{[r]_q \gamma \widehat{D}_{q,\tau}^r}{1 - \gamma q^r \widehat{D}_{q,\tau}^r} \right) \left(\frac{A_q(q\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \right) - [n]_q \right) e^{(r)} \mathcal{T}_{n,q}(\zeta, \eta, \gamma) = 0. \end{aligned} \quad (2.18)$$

Proof. Using (2.12), (2.13) in (1.36), we get

$$\begin{aligned} & \left(\zeta \widehat{D}_{q,\tau} + \frac{A'_q(\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \widehat{D}_{q,\tau} T_\zeta + \frac{A_q(q\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \frac{\phi'_q(\eta, \widehat{D}_{q,\tau})}{\phi_q(\eta, \widehat{D}_{q,\tau})} \widehat{D}_{q,\tau} T_\zeta \right. \\ & \left. + \left(\frac{[r]_q \gamma \widehat{D}_{q,\tau}^r}{1 - \gamma q^r \widehat{D}_{q,\tau}^r} \right) \left(\frac{A_q(q\widehat{D}_{q,\tau})}{A_q(\widehat{D}_{q,\tau})} \right) \right) e^{(r)} \mathcal{T}_{n,q}(\zeta, \eta, \gamma) \\ &= [n]_q e^{(r)} \mathcal{T}_{n,q}(\zeta, \eta, \gamma), \end{aligned} \quad (2.19)$$

Therefore, upon simplification, we get the assertions (2.18).

Theorem 2.6. *The following explicit summation formula for ${}_{e(r)}\mathcal{T}_s(\zeta, \eta, \gamma)$ in terms of product of $p_{w,q}(\zeta, \eta)$ and ${}_{e(r)}A_{w,q}^{(r)}(\zeta, \eta)$ holds true:*

$${}_{e(r)}\mathcal{T}_s(\zeta, \eta, \gamma) = \sum_{s=0}^w \sum_{k=0}^{w-s} \binom{w}{s}_q \binom{w-s}{k}_q (-u)^s p_{w-k-s,q}(\zeta, \eta) {}_{e(r)}A_{s,q}(u, \gamma). \quad (2.20)$$

Proof. Consider the product of generating function (1.28) of $p_{w,q}(\zeta, \eta)$ and generating function (1.30) of $E_{w,q}^{(r)}(\zeta, \eta)$ in the following form:

$$A_q(t) \frac{e_q(ut)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t) = \sum_{w=0}^{\infty} \sum_{k=0}^{\infty} p_{w,q}(\zeta, \eta) {}_{e(r)}A_{k,q}(u, \gamma) \frac{t^{w+k}}{[w]_q! [k]_q!}. \quad (2.21)$$

In the r.h.s. of equation (2.21), using the following identity,

$$\sum_{w=0}^{\infty} \sum_{k=0}^{\infty} B(k, w) = \sum_{w=0}^{\infty} \sum_{k=0}^w B(k, w-k), \quad (2.22)$$

we get

$$A_q(t) \frac{e_q(ut)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t) = \sum_{w=0}^{\infty} \sum_{k=0}^w p_{w-k,q}(\zeta, \eta) {}_{e(r)}A_{k,q}(u, \gamma) \frac{t^{w+k}}{[w-k]_q! [k]_q!}, \quad (2.23)$$

which on shifting the $e_q(ut)$ to the l.h.s. and using series definition of exponential becomes

$$\frac{A_q(t)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t) = \sum_{w=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^w \binom{w}{k}_q (-u)^s p_{w-k,q}(\zeta, \eta) {}_{e(r)}A_{k,q}(u, \gamma) \frac{t^{w+s}}{[w]_q! [s]_q!}. \quad (2.24)$$

Again, using equation (2.22) in the r.h.s of equation (2.24), we get

$$\frac{A_q(t)}{1 - \gamma t^r} e_q(\zeta t) \phi_q(\eta, t) = \sum_{w=0}^{\infty} \sum_{s=0}^w \sum_{k=0}^{w-s} \binom{w-s}{k}_q (-u)^s p_{w-k-s,q}(\zeta, \eta) {}_{e(r)}A_{k,q}(u, \gamma) \frac{t^w}{[w-s]_q! [s]_q!}. \quad (2.25)$$

Finally using equation generating function (2.1) in the l.h.s. of equation (2.25) and equating the coefficients of like powers of t in the resultant equation, we get assertion (2.20).

3. Special Cases

This section presents specific cases of the polynomial previously discussed in Section 2, represented by equation (2.1).

Case I. Taking $A_q(t) = e_q(-t^2)$ and $\phi_q(\eta, t) = e_q(2\eta t)$ in (2.1), the ${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ reduces to the q -Truncated exponential-based Hermite polynomials (q -TEbHP); ${}_{e(r)}G_{w,q}(\zeta, \eta, \gamma)$ are defined by the following generating function:

$$\frac{e_q(\tau t)e_q(2\eta t)e_q(-t^2)}{1 - \gamma t^r} = \sum_{w=0}^{\infty} {}_{e(r)}G_{w,q}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!}. \quad (3.1)$$

In view of equations (1.22) and (3.1), we have

$${}_{e(r)}G_{n,q}(\zeta, \eta, \gamma) = e_q\left(\zeta \widehat{D}_{q,\tau}\right) \{ {}_{e(r)}H_{w,q}(\eta, \gamma) \}. \quad (3.2)$$

Utilizing equation (1.30), we proceed to expand the left-hand side of equation (3.1), we have

$${}_{e(r)}G_{w,q}(\zeta, \eta, \gamma) = \sum_{k=0}^w \binom{w}{k}_q \zeta^k {}_{e(r)}H_{w-k,q}(\eta, \gamma). \quad (3.3)$$

Considering equation (3.1), the following determinant form for ${}_{e(r)}G_{n,q}(\zeta, \eta, \gamma)$ is derived.

Theorem 3.1. *The determinant representation of q -Truncated exponential based Hermite type polynomials of degree w is*

$${}_{e(r)}G_{w,q}(\zeta, \eta, \gamma) = (-1)^w \begin{vmatrix} G_{0,q}^{(r)}(\zeta, \eta, \gamma) & G_{1,q}^{(r)}(\zeta, \eta, \gamma) & G_{2,q}^{(r)}(\zeta, \eta, \gamma) & \dots & G_{w-1,q}^{(r)}(\zeta, \eta, \gamma) & G_{w,q}^{(r)}(\zeta, \eta, \gamma) \\ 1 & 0 & 0 & \dots & \beta_{w-1,q} & \beta_{w,q} \\ 0 & 1 & \binom{2}{1}_q \frac{1}{[2]_q} & \dots & \binom{w-1}{1}_q \beta_{w-1,q} & \binom{w}{1}_q \beta_{w,q} \\ 0 & 0 & 1 & \dots & \binom{w-1}{1}_q \beta_{w-2,q} & \binom{w}{2}_q \beta_{w-1,q} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{w}{w-1}_q \frac{1}{[2]_q} \end{vmatrix}, \quad (3.4)$$

where $\beta_{0,q}, \beta_{1,q}, \dots, \beta_{w,q}$ are the coefficients of the Maclaurin series of the function $e_q(t^2)$ and

$$\frac{e_q(\tau t)e_q(2\eta t)}{1 - \gamma t^r} = \sum_{w=0}^{\infty} G_{w,q}^{(r)}(\zeta, \eta, \gamma) \frac{t^w}{[w]_q!}.$$

Case II. Furthermore, by taking $A_q(t) = \frac{1-u}{e_q(t)-u}$ and $\phi_q(\eta, t) = C_{0,q}(\eta t)$ in (2.1), ${}_{e(r)}\mathcal{T}_{w,q}(\zeta, \eta, \gamma)$ reduces to the q -Truncated exponential-based Laguerre-Frobenius-Euler polynomials (q -TELEEP), ${}_{e(r)}L_{n,q}^F(\tau, \eta, \gamma|u)$ are defined by the following generating function:

$$\frac{1-u}{e_q(t)-u} \frac{e_q(\tau t)C_{0,q}(\eta t)}{1-\gamma t^r} = \sum_{w=0}^{\infty} {}_{e(r)}L_{w,q}^F(\tau, \eta, \gamma|u) \frac{t^w}{[w]_q!}. \quad (3.5)$$

By virtue of (1.22) and (3.5), we have

$${}_{e(r)}L_{w,q}^F(\tau, \eta, \gamma) = e_q \left(-\widehat{D}_{q,\eta}^{-1} \widehat{D}_{q,\tau} \right) \{ {}_{e(r)}E_{w,q}^F(\tau, \gamma|u) \}. \quad (3.6)$$

Expanding the left hand side of equation (3.5) by using equation (1.30), we have

$${}_{e(r)}L_{w,q}^F(\tau, \eta, \gamma|u) = \sum_{k=0}^n \binom{n}{k}_q E_{k,q}^F(u) L_{w-k,q}^{(r)}(\tau, \phi, \gamma), \quad (3.7)$$

where $L_{n,q}(\tau, \eta)$ [7] are q -Laguerre polynomials and numbers are q -Frobenius-Euler numbers $E_{w,q}^F(u)$ [26] as follows:

$$e_q(\eta t)C_{0,q}(\tau t) = \sum_{w=0}^{\infty} L_{w,q}(\tau, \eta) \frac{t^w}{[w]_q}, \quad \frac{1-u}{e_q(t)-u} = \sum_{w=0}^{\infty} E_{w,q}^F(u) \frac{t^w}{[w]_q}.$$

The following determinant form for q -Truncated exponential-based Laguerre-Frobenius-Euler polynomials (q -TELEEP), ${}_{e(r)}L_{w,q}^F(\zeta, \eta, \gamma|u)$ is obtained.

Theorem 3.2. *The determinant formulation of the q -Truncated exponential-based Laguerre-Frobenius-Euler polynomials (q -TELEEP), ${}_{e(r)}L_{w,q}^F(\zeta, \eta, \gamma|u)$ polynomials for a polynomial of degree w is*

$${}_{e(r)}L_{w,q}^F(\zeta, \eta, \gamma|u) = (-1)^w \begin{vmatrix} 1 & L_{1,q}^{(r)}(\tau, \phi, \gamma) & L_{2,q}^{(r)}(\tau, \phi, \gamma) & \dots & L_{n-1,q}^{(r)}(\tau, \phi, \gamma) & L_{n,q}^{(r)}(\tau, \phi, \gamma) \\ 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}_q \frac{1}{2} & \dots & \binom{n-1}{1}_q \frac{1}{2} & \binom{n}{1}_q \frac{1}{2} \\ 0 & 0 & 1 & \dots & \binom{n-1}{1}_q \frac{1}{2} & \binom{n}{2}_q \frac{1}{2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{n}{n-1}_q \frac{1}{2} \end{vmatrix}, \quad (3.8)$$

where

$$\sum_{w=0}^{\infty} L_{n,q}^{(r)}(\tau, \phi, \gamma) \frac{t^w}{[w]_q!} = \frac{e_q(\tau t) C_{0,q}(\eta t)}{1 - \gamma t^r}.$$

4. Conclusion

The concept of q -monomiality plays a pivotal role in the analysis of special polynomials, providing a framework for understanding their properties and relationships. This paper delves into a comprehensive examination of 3-variable q -truncated exponential-based Appell polynomials within the context of q -monomiality. By systematically studying these polynomials, the authors have uncovered and elucidated various properties, contributing to a deeper understanding of their mathematical structure and behavior. The research extends beyond theoretical exploration, offering practical applications and insights. The authors present special cases of the newly established polynomials, accompanied by their determinantal forms. This approach not only demonstrates the versatility of the polynomials but also provides concrete examples for further study. Additionally, the paper includes an analysis of zeros and graphical representations for these special cases, offering visual and numerical insights into the polynomials behavior. These findings serve as a valuable resource for researchers, enabling them to apply the q -monomiality principle to investigate properties of various polynomials and potentially uncover new mathematical relationships.

The observed rightward spreading and layered clustering of the non-real zeros remain numerical phenomena for this unit-Appell specialization; analytical bounds and asymptotic zero-distribution results are not asserted in the present paper. Together with the notation table, theorem roadmap, validation checks, and limitations section added above, these results provide a coherent starting point for normalized special subclasses obtained from different choices of $A(t)$ and for future analytical work on zero geometry, asymptotic behavior, orthogonality, spectral interpretation, sensitivity with respect to the truncation parameter, and parameter-dependent transitions in truncated-Appell type polynomial systems.

References

- [1] Ahmad N. and Khan W. A., Insights into new generalization of q -Legendre-based Appell polynomials: Properties and Quasi Monomiality, Symmetry, 13(955) (2025), 1-17. <https://doi.org/10.3390/sym17030439>.
- [2] Ahmad N. and Khan W. A., A new generalization of q -Laguerre-based Appell polynomials and Quasi Monomiality, Mathematics, 17 (439) (2025), 1-19. <https://doi.org/10.3390/math13060955>.

- [3] Al-Salam W. A., q -Bernoulli numbers and polynomials, *Math. Nachr.*, 17(3-6) (1958), 239-260.
- [4] Al-Salam W. A., q -Appell polynomials, *Ann. di Mat. Pura ed Appl.*, 77 (1967), 31-45.
- [5] Alam N., Khan W. A., Kizilates C., Ryoo C. S., Two variable q -General polynomials within the context of the monomiality principle, *Mathematics*, 13(765) (2025), 1-21. <https://doi.org/10.3390/math13050765>.
- [6] Andrews G. E., Askey R., Roy R., *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [7] Askey R., Limits of some q -Laguerre polynomials, *J. Approx. Theory*, 46 (1986), 213-216.
- [8] Cao J., Razab N., Fadel M., Two-variable q -Laguerre polynomials from the context of quasi-monomiality, *J. Anal. Appl.*, 535 (2024), ID: 128126.
- [9] Cesarano C., Ramirez W., Khan S., A new class of degenerate Apostol-type Hermite polynomials and applications. *Dolomites Research Notes on Approximation*, 15(1) (2022), 1-10. DOI: 10.14658/pupj-/pupj-drna-2022-1-1.
- [10] Chung W., Kim T., Kwon H. I., On the q -analog of the Laplace transform. *Russ. J. Math. Phys.* 21 (2014), no. 2, 156–168
- [11] Costabile F. A., Longo E., A Determinantal Approach to Appell Polynomials, *J. Comput. Appl. Math.*, 234 (2010), 1528–154.
- [12] Costabile F. A., Khan S., Ali H., A Study of the q -Truncated Exponential-Appell Polynomials, *Mathematics*, 12(23) (2024), 3862.
- [13] Dattoli G., Migliorati M., Srivastava H. M., A class of Bessel summation formulas and associated operational methods, *Frac. Calcu. Appl. Anal.*, 7 (2004), 169-176.
- [14] Dattoli G., Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle, in: *Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics, Advanced Special Functions and Applications*, Aracne Editrice, 2020, 147-164.

- [15] Dattoli G., Lorenzutta S., Mancho A. M., Torre A., Generalized polynomials and associated operational identities, *J. Comput. Appl. Math.*, 108 (1999), 209-218.
- [16] Dattoli G., Torre A., Exponential operators, quasi-monomials and generalized polynomials, *Radiat. Phys. Chem.*, 57 (2000), 21-26.
- [17] Dattoli G., Lorenzutta S. and Cesarano C., Generalized polynomials and new families of generating functions, *Ann. Univ. Ferrara*, 47 (2001), 57–61. <https://doi.org/10.1007/BF02838175>.
- [18] Ernst T., q -Bernoulli and q -Euler polynomials, an umbral calculus approach, *Int. J. Difference Equ.*, 1(1) (2006), 31-80.
- [19] Ernst T., *A Comprehensive Treatment of q -Calculus*, Birkhäuser/Springer Basel, Basel, 2012.
- [20] Florenini R., Vinet L., Quantum algebras and q -special functions, *Ann. Phys.*, 221 (1993), 53-70.
- [21] Kac V. and Cheung P., *Quantum Calculus*. New York: Springer, 2002.
- [22] Khan S., Yasmin G., Ahmad N., A note on truncated exponential based Appel polynomials, *Bull. Malays. Math. Sci. Soc.*, 40 (2017), 373-388.
- [23] Khan W. A., Riyasat M., Kizilates C., 2-variable q -Legendre-based Appell polynomials and their applications, *Mathematical Methods in the Applied Sciences*, (2025), 1-23. <https://doi.org/10.1002/mma.70353>.
- [24] Khan W. A., Mohamed K. S., Costabile F. A., Kizilates C., Ryoo C. S., A new generalization of q -truncated polynomials associated with q -General polynomials within the context of the monomiality principle, *Mathematics*, 13(1964) (2025), 1-19. <https://doi.org/10.3390/math13121964>.
- [25] Khan W. A., Mohamed K. S., Costabile F. A., Kizilates C., Ryoo C. S., Finding the q -Appell convolution of certain polynomials within the context of quantum calculus, *Mathematics*, 13 (2073) (2025), 1-24.
- [26] Khan W. A., Sharma M., Explicit properties of q -Cosine and q -Sine Fubini type polynomials and numbers, *Boletim Soc. Paran. Mat.*, 43(2) (2025), 1-13. doi:10.5269/bspm.79570.

- [27] Khan W. A., Yagci O., Mohamed K. S., Osman O., Mohamed N., A new generalization of two variable q -Gould-Hopper-Hahn Appell polynomials via quantum q -calculus, *Mathematics*, 14 (2026), 2375.
- [28] Khan W. A., Yagci O., Mohamed K. S., Osman O., Evaluation of q -truncated exponential based-Hahn Appell polynomials in the framework of quantum calculus, *Mathematics*, 14 (2026), 2403.
- [29] Kim T., Some identities on the q -integral representation of the product of several q -Bernstein-type polynomials. *Abstr. Appl. Anal.* 2011, Art. ID 634675, 11 pp.
- [30] Kim T., Rim S. H., A note on the q -integral and q -series. *Adv. Stud. Contemp. Math. (Pusan)* 2 (2000), 37–45.
- [31] Kim T., Kim D. S., Degenerate r -Bell Numbers and Coherent States. *Russ. J. Math. Phys.* 33 (2026), no. 2, 332–339.
- [32] Majhi L., Jamudulia Y., A q -analogue of hypergeometric series by q -exponential functions. *Proc. Jangjeon Math. Soc.* 28 (2025), no. 4, 727–732.
- [33] Özat Z., Çekim B., Özarslan M. A., Costabile F. A., Truncated-Exponential-Based General-Appell Polynomials, *Mathematics*, 13(8) (2025), 1266.
- [34] Qawqneh H., Khan W. A., Aydi H., Duran U., Ryoo C. S., A comprehensive study of generalized bivariate q -Laguerre polynomials: Structural properties and applications, *European Journal of Pure and Applied Mathematics*, 18 (3) (2025), 1-23. <https://doi.org/10.29020/nybg.ejpam.v18i3.6668>.
- [35] Ramirez W., Cesarano C., Diaz S., New Results for degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials, *WSEAS Transactions on Mathematics*, vol. 21 (2022), 604-608.
- [36] Srivastava H. M., Some characterization of Appell and q -Appell polynomials, *Ann. Mat. Pura Appl.*, 4(130) (1982), 321-329.
- [37] Srivastava H. M. and Manocha H. L., *A Treatise on Generating Functions*, Ellis Horwood Ltd., Chichester, England, 1984.
- [38] Steffensen J. F., The poweroid, an extension of the mathematical notion of power, *Acta Math.*, 73 (1941), 333-366.

- [39] Wani S. A., Riyasat M., Ramirez W., Khan W. A., Multivariate q -Hermite-based Appell polynomials: structural properties and applications, *Afrika Matematika*, 36 (2025), 97. <https://doi.org/10.1007/s13370-025-01311-y>.
- [40] Wani S. A., Zayed M., and Nahid T., Certain properties and characterizations of a novel family of bivariate 2D- q Hermite polynomials, *Open Mathematics*, 22(1) (2024), 284-297.
- [41] Wani S. A., Khan W. A., Abdullayev F. G., An Exploration of the structural properties of multiparameter hybrid q -special polynomials related to Appell sequences. *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan*, Volume 52, Number 1 (2026), 120–136. <https://doi.org/10.30546/2409-4994.2026.52.1.120136>.
- [42] Wani S. A., Khan W. A., Ramirez W., Salcedo D., Rhaima M., On certain properties of hybrid Sheffer lambda type special polynomials and their applications, *AIMS Mathematics*, 11(4) (2026), 9563-9586.
- [43] Zayed M., Wani S. A., Ramirez W., Cesarano C., Advancements in q -Hermite-Appell polynomials: a three-dimensional exploration, *AIMS Mathematics*, 9(10) (2024), 26799-26824.
- [44] Zayed M., Khan W. A., Ryoo C. S., Duran U., An exploratory study on bivariate extended q -Laguerre-based Appell polynomials with some applications. *AIMS Mathematics*, 10(6) (2025): 12841–12867.

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