

ON THE CATEGORIES OF NEUTROSOPHIC SETS AND NEUTROSOPHIC TOPOLOGICAL SPACES

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Abstract: This paper develops a categorical framework for neutrosophic sets and neutrosophic topological spaces. We first redefine the category $\mathbf{NSet}$ of neutrosophic sets and investigate its fundamental categorical properties, proving in particular that it is concretely cartesian closed. We also establish relationships among several natural subcategories of $\mathbf{NSet}$. We then introduce the category $\mathbf{NTop}$ of neutrosophic topological spaces and prove that it is a topological category. Furthermore, we show that the category of intuitionistic fuzzy topological spaces embeds into $\mathbf{NTop}$, and we identify important subcategories, including $\mathbf{CNNTop}$ and the category $\mathbf{SNTop}$ of stratified neutrosophic topological spaces, together with their coreflective properties. We identify the Sierpiński object in $\mathbf{NTop}$ and prove that it serves as the open-set classifier for neutrosophic topological spaces. Finally, we establish that the neutrosophic Sierpiński space characterizes $\mathbf{NTop}$ as a category of sets with structures and show that the category $\mathbf{NTop}_0$ of T_0 neutrosophic topological spaces is the epireflective hull of the Sierpiński object in $\mathbf{NTop}$.

Keywords and Phrases: Neutrosophic set, Neutrosophic topology, Category, Functor, Sierpiński object.

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1. Introduction

Neutrosophic set theory, introduced by Smarandache [23], extends Atanassov's

intuitionistic fuzzy set theory [5] by assigning to each element three independent membership functions, namely the truth-membership, indeterminacy-membership and falsity-membership functions. The independence of these three components enables neutrosophic sets to model incomplete, inconsistent and indeterminate information more effectively than intuitionistic fuzzy sets. Owing to this additional flexibility, neutrosophic set theory has found numerous applications in areas such as decision making [19], medical image processing [14] and many other disciplines involving uncertain information.

For practical applications, Wang *et al.* [26] introduced the notion of single-valued neutrosophic sets, which has since become one of the most widely studied forms of neutrosophic sets. Subsequently, several algebraic and topological aspects of neutrosophic sets have been investigated. Kim *et al.* [13] studied single-valued neutrosophic relations, while Smarandache [24] introduced various notions of neutrosophic topologies. Further developments include neutrosophic topological K -algebras [2, 3], intuitionistic fuzzy topological K -algebras [4], and more recent studies reported in [9, 15, 21]. Lupiáñez [18] showed that the natural extension of an intuitionistic fuzzy topology to a family of neutrosophic sets does not necessarily produce a neutrosophic topology, thereby revealing that the transition from intuitionistic fuzzy topology to neutrosophic topology is far from straightforward.

Category theory has proved to be an effective framework for studying fuzzy structures from a unified viewpoint. Various categorical aspects of fuzzy topology and intuitionistic fuzzy topology have been investigated extensively. Lee *et al.* [16] established that the category of Chang's fuzzy topological spaces [7] is a bireflective subcategory of the category of intuitionistic fuzzy topological spaces. Hur *et al.* [10] studied the category of intuitionistic H -fuzzy sets from the viewpoint of topological universes.

A categorical study of neutrosophic sets was initiated by Hur *et al.* [11], who introduced the category $\mathbf{NSet}(H)$ of neutrosophic H -sets. However, in their formulation the three membership functions satisfy the constraints $T_A \leq N(F_A)$ and $I_A \geq N(F_A)$ where N denotes the involution on the Heyting algebra H . Consequently, the three membership functions are not independent. Since the independence of truth, indeterminacy and falsity memberships is one of the defining characteristics of neutrosophic theory, these restrictions limit the categorical treatment of general neutrosophic sets. To the best of our knowledge, a systematic categorical framework for neutrosophic sets and neutrosophic topological spaces preserving this fundamental independence has not yet been developed.

Another important aspect of categorical topology is the role played by the Sierpiński object. In classical topology, the topology of a space is completely de-

terminated by its continuous maps into the Sierpiński space, which therefore serves as an open-set classifier [20]. Similar characterizations have been established for fuzzy topology. Srivastava [25] identified a Sierpiński object in the category $\mathbf{FTop}$ and characterized $\mathbf{FTop}$ as a category of sets with structures. Lowen and Srivastava [17] further proved that $\mathbf{FTS}_0$, the category of T_0 fuzzy topological spaces, is the epireflective hull of the Sierpiński object. Although several aspects of neutrosophic topology have been investigated, the identification of a Sierpiński object for neutrosophic topological spaces, its role as an open-set classifier, and the resulting categorical characterization of $\mathbf{NTop}$ have, to the best of our knowledge, not been addressed in the existing literature.

The objective of the present paper is to develop a categorical foundation for neutrosophic sets and neutrosophic topological spaces that fully respects the independence of the three membership functions. To achieve this, we first redefine the category $\mathbf{NSet}$ of neutrosophic sets without imposing additional constraints on the membership functions. We investigate its basic categorical properties, proving that it is a topological category and is concretely cartesian closed, and establish relationships among several of its natural subcategories.

Building upon this framework, we introduce the category $\mathbf{NTop}$ of neutrosophic topological spaces and prove that it is a topological category. We explicitly construct products and coproducts in $\mathbf{NTop}$, investigate the subcategories of intuitionistic-induced neutrosophic topologies and strong intuitionistic-induced neutrosophic topologies, and show that the category of intuitionistic fuzzy topological spaces embeds naturally into $\mathbf{NTop}$. We further introduce the category $\mathbf{SNTop}$ of stratified neutrosophic topological spaces and prove that it is a coreflective subcategory of $\mathbf{NTop}$.

Finally, we identify the Sierpiński object in $\mathbf{NTop}$ and prove that it serves as the open-set classifier for neutrosophic topological spaces. We establish a characterization of $\mathbf{NTop}$ as a category of sets with structures and show that the category $\mathbf{NTop}_0$ of T_0 neutrosophic topological spaces is precisely the epireflective hull of the Sierpiński object in $\mathbf{NTop}$.

2. Preliminaries

In this section, we recall the basic concepts and results from neutrosophic set theory and category theory that will be used throughout the paper.

Definition 2.1. [6] *The objects of the category of fuzzy subsets $\mathbf{Set}([0, 1])$ are all pairs (X, A) where X is a set and $A : X \rightarrow [0, 1]$ is a function from X to the unit interval. The maps of $\mathbf{Set}([0, 1])$ are defined by $\mathbf{Hom}_{\mathbf{Set}([0, 1])}((X, A), (Y, B)) = \{f \in \mathbf{Hom}_{\mathbf{SET}}(X, Y) \mid A(x) \leq B(f(x)), \forall x \in X\}$.*

Definition 2.2. [16] *IFTop* be the category with all intuitionistic fuzzy topological spaces as objects and continuous maps as morphisms.

Definition 2.3. [23] (Neutrosophic set) Let X be a non-empty set. Then A is called a neutrosophic set (in short, NS) in X , if A has the form $A = (T_A, I_A, F_A)$, where $T_A : X \rightarrow]0^-, 1^+[$, $I_A : X \rightarrow]0^-, 1^+[$ and $F_A : X \rightarrow]0^-, 1^+[$ are respectively the truth, indeterminacy and falsity membership functions. It's a point to be noted that the functions T_A , I_A and F_A are independent of each other. A neutrosophic set A in X is said to be a single valued neutrosophic set (SVN Set) if $T_A(x), I_A(x), F_A(x) \in [0, 1]$. The set of all neutrosophic sets on X will be denoted by $NS(X)$.

In the remainder of this paper, the term neutrosophic set will always mean a single-valued neutrosophic set.

Definition 2.4. [13] (Inclusion) A neutrosophic set A is contained in another neutrosophic set B , or say $A \subseteq B$ if and only if $T_A \leq T_B, I_A \geq I_B, F_A \geq F_B$. More precisely, $A \subseteq B$ if and only if $\forall x \in X$

$$T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$$

Two neutrosophic sets are said to be equal if and only if $A \subseteq B$ and $A \supseteq B$.

Definition 2.5. [13] Let A and B be two neutrosophic sets in X .

1. The union of A and B is a neutrosophic set C , denoted $C = A \cup B$, where $\forall x \in X$

$$T_C(x) = \vee\{T_A(x), T_B(x)\}, I_C(x) = \wedge\{I_A(x), I_B(x)\}, F_C(x) = \wedge\{F_A(x), F_B(x)\}$$

2. The intersection of A and B is a neutrosophic set D , denoted $D = A \cap B$, where $\forall x \in X$

$$T_D(x) = \wedge\{T_A(x), T_B(x)\}, I_D(x) = \vee\{I_A(x), I_B(x)\}, F_D(x) = \vee\{F_A(x), F_B(x)\}$$

Definition 2.6. [12] Let $A \in NS(X)$, $B \in NS(Y)$ and f be a mapping from X to Y .

1. Define $f^\rightarrow$, the neutrosophic mapping from $NS(X)$ to $NS(Y)$ induced by f as

$$f^\rightarrow(A) = \{ \langle y, T_{f^\rightarrow(A)}(y), I_{f^\rightarrow(A)}(y), F_{f^\rightarrow(A)}(y) \rangle \mid y \in Y \}$$

where,

$$T_{f^\rightarrow(A)}(y) = \bigvee_{x \in f^{-1}(y)} \{T_A(x)\}, I_{f^\rightarrow(A)}(y) = \bigwedge_{x \in f^{-1}(y)} \{I_A(x)\} \text{ and}$$

$$F_{f^\rightarrow(A)}(y) = \bigwedge_{x \in f^{-1}(y)} \{F_A(x)\}$$

2. Define $f^{\leftarrow}$, the inverse neutrosophic mapping from $NS(Y)$ to $NS(X)$ induced by f as

$$\begin{aligned} f^{\leftarrow}(B) &= \{ \langle x, T_{f^{\leftarrow}(B)}(x), I_{f^{\leftarrow}(B)}(x), F_{f^{\leftarrow}(B)}(x) \rangle \mid x \in X \} \\ &= \{ \langle x, T_B(f(x)), I_B(f(x)), F_B(f(x)) \rangle \mid x \in X \} \end{aligned}$$

Definition 2.7. [22] A family τ of neutrosophic sets of a non-empty set X is said to be a neutrosophic topology, if the following conditions hold:

1. $(0, 1, 1), (1, 0, 0) \in \tau$
2. $A_1 \cap A_2 \in \tau$ for $A_1, A_2 \in \tau$
3. $\cup A_i \in \tau$ for $\{A_i\} \subseteq \tau$.

The base and subbase for a neutrosophic topology are defined analogous to the base and subbase in crisp topology [12].

Definition 2.8. [12] Let (X, τ_1) and (Y, τ_2) be two neutrosophic topological spaces and $f : X \rightarrow Y$ be a mapping. Then $f^{\rightarrow} : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be a continuous function between (X, τ_1) and (Y, τ_2) if $f^{\leftarrow}(A) \in \tau_1, \forall A \in \tau_2$.

Definition 2.9. [1] An object A in a concrete category $\mathcal{C}$ over X with the forgetful functor F is called discrete whenever, for each object B , every X -morphism $F(A) \rightarrow F(B)$ is a $\mathcal{C}$ -morphism.

Definition 2.10. [1] A functor is an embedding iff it is faithful and injective on objects.

Definition 2.11. [1] Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ and let B be a $\mathcal{B}$ -object. An $\mathcal{A}$ -reflection for B is a $\mathcal{B}$ -morphism $r : B \rightarrow A$ from a $\mathcal{B}$ -object B to A with the following universal property: for any $\mathcal{B}$ -morphism $f : B \rightarrow A'$ from B to some $\mathcal{A}$ -object A' , there exists a unique $\mathcal{A}$ -morphism $f' : A \rightarrow A'$ such that $f = f' \circ r$. $\mathcal{A}$ is called a reflective subcategory of $\mathcal{B}$ provided that each $\mathcal{B}$ -object has an $\mathcal{A}$ -reflection.

Definition 2.12. [1] Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ and let B be a $\mathcal{B}$ -object. An $\mathcal{A}$ -coreflection for B is a $\mathcal{B}$ -morphism $c : A \rightarrow B$ from a $\mathcal{A}$ -object A to B with the following universal property: for any $\mathcal{B}$ -morphism $f : A' \rightarrow B$ from some $\mathcal{A}$ -object A' to B there exists a unique $\mathcal{A}$ -morphism $f' : A' \rightarrow A$ such that $f = c \circ f'$. $\mathcal{A}$ is called a coreflective subcategory of $\mathcal{B}$ provided that each $\mathcal{B}$ -object has an $\mathcal{A}$ -coreflection.

Definition 2.13. [1] Let $(\mathcal{A}, U)$ be a construct. A source $(f_i : A \rightarrow A_i)_{i \in I}$ in

$\mathcal{A}$ is called *initial* provided that a function $f : U(B) \rightarrow U(A)$ is an $\mathcal{A}$ -morphism whenever each composite $f_i \circ f : U(B) \rightarrow U(A_i)$ is an $\mathcal{A}$ -morphism.

Definition 2.14. [1] Let $(\mathcal{A}, U)$ be a construct. A U -structured source is a source in SET of the form $(f_i : X \rightarrow U(A_i))_{i \in I}$. An initial lift for a U -structured source is an initial source $(\bar{f}_i : A \rightarrow A_i)_{i \in I}$ in $\mathcal{A}$ such that $U(A) = X$ and $U(\bar{f}_i) = f_i$ for all $i \in I$.

Definition 2.15. [1] A construct $(\mathcal{A}, U)$ is called *topological* provided that every U -structured source has a unique initial lift.

The dual notion of a topological category is called a cotopological category.

Proposition 2.16. [1] A construct $\mathcal{A}$ is topological iff it's cotopological.

Proposition 2.17. [1] A topological category is complete and cocomplete.

Proposition 2.18. [1] For a topological category $\mathcal{A}$ over a cartesian closed category X , the following are equivalent:

1. $\mathcal{A}$ is concretely cartesian closed
2. $\mathcal{A}$ is cartesian closed and every $\mathcal{A}$ -morphism with a discrete codomain has a discrete domain.

Definition 2.19. $(X, \tau) \in NTop$ is said to be T_0 if for every distinct $x, y \in X$, there exists a neutrosophic set $U \in \tau$ such that $(T_U(x), I_U(x), F_U(x)) \neq (T_U(y), I_U(y), F_U(y))$. $NTop_0$ will denote the category of all T_0 neutrosophic topological spaces.

Definition 2.20. [1] Let $(\mathbb{A}, F)$ be a construct. A source $(f_i : A \rightarrow A_i)_{i \in I}$ in $\mathbb{A}$ is called *initial* provided that a function $f : F(B) \rightarrow F(A)$ is an $\mathbb{A}$ -morphism whenever each composite $f_i \circ f : F(B) \rightarrow F(A_i)$ is an $\mathcal{A}$ -morphism. An initial lift of a source $(f_i : X \rightarrow (X_i, A_i))_{i \in I}$ is a structure A on X such that $(f_i : (X, A) \rightarrow (X_i, A_i))_{i \in I}$ is an initial source.

Definition 2.21. [1] An object S of the category $\mathbb{A}$ is said to be a *Sierpiński object* if for any $X \in \mathbb{A}$, the family of all morphisms from X to S is initial.

Definition 2.22. [1] Let $\mathbb{A}$ be a full subcategory of $\mathbb{C}$. $\mathbb{A}$ is said to be *epireflective* in $\mathbb{C}$ if $\mathbb{A}$ is reflective in $\mathbb{C}$ and the $\mathbb{A}$ -reflection of each $\mathbb{C}$ -object is an epimorphism. $\mathbb{A}$ is said to be the *epireflective hull* in $\mathbb{C}$ of a class $\mathcal{B}$ of $\mathbb{C}$ -objects if $\mathbb{A}$ is the smallest epireflective subcategory of $\mathbb{C}$ containing $\mathcal{B}$.

3. The category of neutrosophic sets

In this section, we introduce $NSet$, the category of single-valued neutrosophic sets, and investigate its fundamental categorical properties. We also establish the categorical relationships among SET , $SET(I)$, and a newly introduced subcategory of $NSet$.

Definition 3.1. Let (X, A) and (Y, B) be neutrosophic sets and $f : X \rightarrow Y$ be a mapping. Then $f : (X, A) \rightarrow (Y, B)$ is said to be a morphism between (X, A) and (Y, B) if $T_A(x) \leq T_B(f(x))$, $I_A(x) \geq I_B(f(x))$ and $F_A(x) \geq F_B(f(x))$, $\forall x \in X$.

In particular, for any neutrosophic set (X, A) , $Id : (X, A) \rightarrow (X, A)$ is a morphism, called the identity morphism on (X, A) . The proof of the following proposition is straightforward.

Proposition 3.2. Let $(X, A), (Y, B)$ and (Z, C) be neutrosophic sets and let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be mappings. If $f : (X, A) \rightarrow (Y, B)$ and $g : (Y, B) \rightarrow (Z, C)$ are morphisms, then $g \circ f : (X, A) \rightarrow (Z, C)$ is a morphism.

By Definition 3.1 and Proposition 3.2, the collection of all (single-valued) neutrosophic sets together with the corresponding morphisms forms the category $NSet$. The following results hold in this redefined category. Since their proofs are analogous to those in [11], they are omitted.

Theorem 3.3. $NSet$ is topological over SET .

Theorem 3.4. $NSet$ is cartesian closed.

Proposition 3.5. Let $A = (X, (T_A, I_A, F_A))$ and $B = (Y, (T_B, I_B, F_B))$ be two objects of $NSet$. A morphism $f : (X, (T_A, I_A, F_A)) \rightarrow (Y, (T_B, I_B, F_B))$ is an isomorphism in $NSet$ iff f is a bijection and $T_A(x) = T_B(f(x))$, $I_A(x) = I_B(f(x))$ and $F_A(x) = F_B(f(x))$, $\forall x \in X$.

Proposition 3.6. For any set X , if D is a neutrosophic set on X such that $T_D(x) = 0$, $I_D(x) = 1$ and $F_D(x) = 1$, $\forall x \in X$, then D is a discrete object in $NSet$.

Theorem 3.7. $NSet$ is concretely cartesian closed over SET .

Proof. Let $f : (X, A) \rightarrow (Y, D)$ be an $NSet$ morphism with (Y, D) a discrete object. Then we have $T_D = 0, I_D = 1$ and $F_D = 1$. Since f is a morphism, $T_A(x) \leq T_D(f(x)) = 0$, $I_A(x) \geq I_D(f(x)) = 1$ and $F_A(x) \geq F_D(f(x)) = 1$. Therefore, $A = (0, 1, 1)$, a discrete object. Hence it follows from Theorem 3.4 that $NSet$ is concretely cartesian closed.

Definition 3.8. Consider the class of neutrosophic sets with complete data, namely, the neutrosophic sets $(X, A) \in NSet$ satisfying $T_A(x) + F_A(x) \geq 1, \forall x \in X$. The

collection of all such neutrosophic sets, together with the morphisms inherited from $NSet$, forms a full subcategory of $NSet$, denoted by $CTNSet$.

Definition 3.9. Let $\mathcal{A}$ be the class of all neutrosophic sets in $CTNSet$ with $I_A(x) = 0, T_A(x), F_A(x) \in \{0, 1\}$ and either $T_A(x) = 1$ or $F_A(x) = 1 \forall x \in X$. Then $\mathcal{A}$ will form a full subcategory of $CTNSet$ and can be identified with the category SET of crisp sets.

Definition 3.10. Let $\mathcal{A}$ be the class of all neutrosophic sets in $CTNSet$ satisfying $I_A(x) = 0, T_A(x), F_A(x) \in \{0, 1\}$ and $T_A(x) = 1$ or $F_A(x) = 1, \forall x \in X$. Then $\mathcal{A}$, together with the morphisms inherited from $CTNSet$, forms a full subcategory of $CTNSet$. Moreover, $\mathcal{A}$ can be identified with the category SET of crisp sets.

Theorem 3.11. $\mathcal{A}$ is a reflective subcategory of $CTNSet$.

Proof. Let $(X, A) \in CTNSet$, where $A = (T_A, I_A, F_A)$.

Define (X, A_R) by $A_R = ([T_A], 0, 1 - [T_A])$ where $[]$ is the ceiling function. Then by definition $(X, A_R) \in \mathcal{A}$. It is easy to check that $id : (X, A) \leftarrow (X, A_R)$ is a $CTNSet$ morphism.

Let $(Y, A') \in \mathcal{A}$ and $f : (X, A) \rightarrow (Y, A')$ be a $CTNSet$ morphism. Then, $T_A(x) \leq T_{A'}(f(x)), I_A(x) \geq I_{A'}(f(x)) = 0$ and $F_A(x) \geq F_{A'}(f(x))$.

Now we show that $f : (X, A_R) \rightarrow (Y, A')$ is an $\mathcal{A}$ -morphism. For $x \in X$, if $T_{A_R}(x) = 0$ then $F_{A_R}(x) = 1$ and obviously $T_{A_R}(x) \leq T_{A'}(f(x))$ and $F_{A_R}(x) \geq F_{A'}(f(x))$.

Now suppose $T_{A_R}(x) > 0$, i.e., $T_{A_R}(x) = 1$ and $F_{A_R}(x) = 0$. Then, $T_A(x) > 0$ and $T_A(x) \leq T_{A'}(f(x)) \implies T_{A'}(f(x)) = 1 \implies F_{A'}(f(x)) = 0$. Therefore $T_{A_R}(x) \leq T_{A'}(f(x))$ and $F_{A_R}(x) \geq F_{A'}(f(x))$. Thus f is an $\mathcal{A}$ -morphism. Since $f = f \circ id$, $\mathcal{A}$ is a reflective subcategory of $CTNSet$.

The proofs of the following two propositions are straightforward and are therefore omitted.

Proposition 3.12. $F : CTNSet \rightarrow SET(I)$ defined by $F((X, (T_A, I_A, F_A))) = (X, T_A)$ is a covariant functor.

Proposition 3.13. $G : SET(I) \rightarrow CTNSet$ defined by $G((X, T_B)) = (X, T_B, 0, 1 - T_B)$ is a covariant functor.

Theorem 3.14. The functor $F : CTNSet \rightarrow SET(I)$ is left adjoint to the functor $G : SET(I) \rightarrow CTNSet$.

Proof. For any $(X, A) \in CTNSet$, $1_X : (X, A) \rightarrow GF((X, A))$ is a morphism in $CTNSet$. For, we have $A = (T_A, I_A, F_A)$ and $GF(A) = (T_A, 0, 1 - T_A)$. Since $(X, A) \in CTNSet$, $T_A + F_A \geq 1$, thus $F_A \geq 1 - T_A$.

For $(Y, B) \in SET(I)$, suppose that $f : (X, A) \rightarrow G(Y, B) = (Y, (T_B, 0, 1 - T_B))$

is a CTNSet morphism. Then, we have $T_A(x) \leq T_B(f(x))$ and $F_A(x) \geq 1 - T_B(f(x))$.

Since $T_A(x) \leq T_B(f(x))$, $f : F((X, A)) = (X, T_A) \rightarrow (Y, B)$ is a morphism in SET(I). Hence F is left adjoint to G .

4. The Category of Neutrosophic Topological Spaces

In this section, we introduce the category **NTop** of neutrosophic topological spaces and prove that it is a topological category. We then investigate its fundamental categorical properties, show that **IFTop** embeds naturally into **NTop**, and introduce the subcategory of consistent neutrosophic topological spaces.

Definition 4.1. *The collection of all neutrosophic topological spaces, together with continuous functions as morphisms, forms a category denoted by NTop. The full subcategory of NTop consisting of consistent neutrosophic topological spaces is denoted by CNNTop.*

Theorem 4.2. *NTop is topological over SET.*

Proof. Let (X_j, τ_j) be a family of objects in NTop indexed by a class J . For $X \in SET$, let $(f_j : X \rightarrow X_j)_j$ be a U-structured source in SET. Define τ on X as the neutrosophic topology generated by the subbase $\{f_j^\leftarrow(B) : B \in \tau_j, j \in J\}$. Then, it is obvious that $f_j^\rightarrow : (X, \tau) \rightarrow (X_j, \tau_j)$ is an NTop morphism $\forall j \in J$. Assume that (Z, τ') is an NTop object and $g : Z \rightarrow X$ is a mapping such that $f_j^\rightarrow \circ g^\rightarrow : (Z, \tau') \rightarrow (X_j, \tau_j)$ is an NTop morphism $\forall j \in J$. Then it follows easily that $g^\rightarrow : (Z, \tau') \rightarrow (X, \tau)$ is an NTop morphism. Thus $f_i^\rightarrow : (X, \tau) \rightarrow (X_j, \tau_j)$ is the unique initial lift for the U-structured source $(f_j : X \rightarrow X_j)_j$.

The following corollaries are immediate consequences of Proposition 2.16.

Corollary 4.3. *NTop is cotopological over SET.*

Corollary 4.4. *NTop is complete and cocomplete.*

Since NTop is complete and cocomplete, products and coproducts of arbitrary families of objects exist in NTop. The following two propositions give explicit constructions of these objects.

Proposition 4.5. *Let $(X_j, \tau_j)_{j \in J}$ be a family of objects in NTop. Let $X = \prod_{j \in J} X_j$ and $p_j : X \rightarrow X_j$ be ordinary projection maps, $\forall j \in J$. Define τ_X on X as the neutrosophic topology generated by the subbase $\{p_j^\leftarrow(U) : U \in \tau_j, j \in J\}$. Then, the source $(p_j^\rightarrow : X \rightarrow X_j)$ is the product of the family $(X_j, \tau_j)_{j \in J}$.*

Proof. Let $(f_j^\rightarrow : (Y, \tau) \rightarrow (X_j, \tau_j))_{j \in J}$ be a source. Define $F : Y \rightarrow X$ as $F(y) = \prod_{j \in J} f_j(y)$, observe that $f_j = p_j \circ F, \forall j \in J$. Now we show that $F^\rightarrow$ is a morphism in NTop. Let $U \in \tau_X$ be a subbasic element, i.e., $U = p_j^\leftarrow(V)$, for some $j \in J$ and $V \in \tau_j$. Then, $F^\leftarrow(U) = F^\leftarrow(p_j^\leftarrow(V)) = (p_j \circ F)^\leftarrow(V) = f_j^\leftarrow(V)$. Since

$f_j^\rightarrow$ is continuous, $f_j^\leftarrow(V) \in \tau$, hence $F^\rightarrow$ is a morphism in $N\text{Top}$.

Proposition 4.6. Let $(X_j, \tau_j)_{j \in J}$ be a family of objects in $N\text{Top}$. Let $X = \coprod_{j \in J} X_j = \cup X_j \times \{j\}$ and $i_j : X_j \rightarrow X$ be the natural injective functions. Define τ_X on X by $U \in \tau_X$ iff $i_j^\leftarrow(U) \in \tau_j, \forall j \in J$. Then, the source $(i_j^\rightarrow : X_j \rightarrow X)$ is the coproduct of the family $(X_j, \tau_j)_{j \in J}$.

Proof. Let $(f_j^\rightarrow : (X_j, \tau_j) \rightarrow (Y, \tau))_{j \in J}$ be a source in $N\text{Top}$. Define $F : X \rightarrow Y$ as $F(x \times \{j\}) = f_j(x), \forall j \in J$, then $F \circ i_j = f_j, \forall j \in J$. Now we show $F^\rightarrow$ is a morphism in $N\text{Top}$. Suppose $U \in \tau$, by the continuity of $f_j^\rightarrow, f_j^\leftarrow(U) \in \tau_j$. We have $(F \circ i_j)^\leftarrow(U) \in \tau_j \implies i_j^\leftarrow \circ F^\leftarrow(U) \in \tau_j \implies F^\leftarrow(U) \in \tau_j$.

Definition 4.7. For an intuitionistic fuzzy topology τ , the neutrosophic topology generated by $\{J(A) : A \in \tau\} \cup \{(0, 1, 1)\}$ as a subbase is called the intuitionistic-induced neutrosophic topology associated with τ , and is denoted by $N(\tau)$. The collection of all intuitionistic-induced neutrosophic topological spaces, together with the morphisms inherited from $N\text{Top}$, forms a subcategory of $N\text{Top}$, denoted by $IIN\text{-Top}$.

Example 4.8. Let X be a set and τ be a intuitionistic fuzzy topology on X defined by $\tau = \{(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2})\}$. Then, $\{J(A) : A \in \tau\} = \{(1, 0, 0), (0, 0, 1), (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), (\frac{1}{2}, 0, \frac{1}{2})\}$ is not a neutrosophic topology on X . $N(\tau) = \{(1, 0, 0), (0, 0, 1), (0, 1, 1), (0, \frac{1}{6}, 1), (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{6}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{3})\}$ is a neutrosophic topology on X .

Proposition 4.9. Let $f^\rightarrow : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a morphism in $IF\text{Top}$. Then, $f^\rightarrow : (X, N(\tau_1)) \rightarrow (Y, N(\tau_2))$ is a morphism in $N\text{Top}$.

Proof. Let A be a subbasic element of $N(\tau_2)$, i.e.,

$$A = (\mu_A, I_A, \nu_A) = J(A') = (\mu_{A'}, 1 - (\mu_{A'} + \nu_{A'}), \nu_{A'}), \text{ for some } A' \in \tau_2.$$

Since $f^\rightarrow$ is continuous in $IF\text{STop}$, $f^\leftarrow(A') = (\mu_{A'}(f(x)), \nu_{A'}(f(x))) \in \tau_1$, thus we have

$$\begin{aligned} & J((\mu_{A'}(f(x)), \nu_{A'}(f(x)))) \in N(\tau_1) \\ & \text{i.e., } (\mu_{A'}(f(x)), 1 - (\mu_{A'}(f(x)) + \nu_{A'}(f(x))), \nu_{A'}(f(x))) \in N(\tau_1) \\ & \implies (\mu_A(f(x)), 1 - (\mu_A(f(x)) + \nu_A(f(x))), \nu_A(f(x))) = f^\leftarrow(A) \in N(\tau_1) \end{aligned}$$

Hence $f^\rightarrow$ is a morphism in $N\text{Top}$.

The following theorem is an immediate consequence of Proposition 4.9.

Theorem 4.10. Define $N : IF\text{Top} \rightarrow N\text{Top}$, where $N(\tau)$ is the intuitionistic-induced neutrosophic topology associated with τ and $N(f^\rightarrow) = f^\rightarrow$. Then N defines a functor.

Theorem 4.11. *The functor N is full and faithful.*

Proof. The faithfulness of N is obvious. Now we prove that N is a full functor.

Let $f^\rightarrow : (X, N(\tau_1)) \rightarrow (Y, N(\tau_2))$ be a morphism in $N\text{Top}$. Let $B = (\mu_B, \nu_B) \in \tau_2$, then $J(B) = (\mu_B, 1 - (\mu_B + \nu_B), \nu_B) \in N(\tau_2)$. Since $f^\rightarrow$ is continuous in $N\text{Top}$, $f^\leftarrow((\mu_B, 1 - (\mu_B + \nu_B), \nu_B)) = (\mu_B(f(x)), 1 - (\mu_B + \nu_B)(f(x)), \nu_B(f(x))) \in N(\tau_1)$

$$\implies (\mu_B(f(x)), 1 - (\mu_B + \nu_B)(f(x)), \nu_B(f(x))) = \bigcup_{j \in J} \bigcap_{i \in \{1, \dots, n_j\}} J(A_{j_i}), \text{ where } A_{j_i} \in \tau_1$$

$$\implies \mu_B(f(x)) = \bigvee_{j \in J} \bigwedge_{i \in \{1, \dots, n_j\}} \mu_{A_{j_i}} \text{ and } \nu_B(f(x)) = \bigwedge_{j \in J} \bigvee_{i \in \{1, \dots, n_j\}} \nu_{A_{j_i}} \implies (\mu_B(f(x)), \nu_B(f(x))) \in \tau_1. \text{ Hence } f^\rightarrow : (X, \tau_1) \rightarrow (Y, \tau_2) \text{ is an IFTop morphism.}$$

The following theorem shows that the functor N defines an embedding of IFTop into NTop.

Theorem 4.12. *The functor N is an embedding.*

Proof. By Theorem 4.11 we have N is full and faithful. It remains to show that N is injective on objects. Let $\tau_1, \tau_2 \in \text{IFTop}$ such that $\tau_1 \neq \tau_2$.

WLOG let $A = (\mu_A, \nu_A) \in \tau_1$ be such that $A \notin \tau_2$. Then, $J(A) = (\mu_A, 1 - (\mu_A + \nu_A), \nu_A) \in N(\tau_1)$.

Suppose that $J(A) \in N(\tau_2)$, then,

$$J(A) = (\mu_A, 1 - (\mu_A + \nu_A), \nu_A) = \bigcup_{k \in K} \bigcap_{i \in \{1, \dots, n_k\}} A_{k_i},$$

where A_{k_i} 's belong to the subbase of $N(\tau_2)$.

$$\text{Thus, } \mu_A = \bigvee_{k \in K} \bigwedge_{i \in \{1, \dots, n_k\}} \mu_{A_{k_i}} \text{ and } \nu_A = \bigwedge_{k \in K} \bigvee_{i \in \{1, \dots, n_k\}} \nu_{A_{k_i}},$$

where $\mu_{A_{k_i}}$ and $\nu_{A_{k_i}}$ are respectively the truth and falsity membership functions of A_{k_i} . Thus we have $(\mu_{A_{k_i}}, \nu_{A_{k_i}}) \in \tau_2$, where $N((\mu_{A_{k_i}}, \nu_{A_{k_i}})) = A_{k_i}$.

Since τ_2 is a topology $\bigcup_{k \in K} \bigcap_{i \in \{1, \dots, n_k\}} (\mu_{A_{k_i}}, \nu_{A_{k_i}}) \in \tau_2$, therefore $(\mu_A, \nu_A) \in \tau_2$, a contradiction.

Hence N is injective on objects. It follows by Definition 2.10 and Theorem 4.11 that N is an embedding.

Theorem 4.13. *$F : \text{CNNTop} \rightarrow \text{IFTop}$ defined by $F(\tau) = \{(\mu_A, \nu_A) | (\mu_A, I_A, \nu_A) \in \tau\}$ and $F(f^\rightarrow) = f^\rightarrow$ is a covariant functor.*

Proof. Proving $F(\tau)$ is in fact an intuitionistic fuzzy topological space is straightforward and thus omitted. Let $(X, \tau_1), (Y, \tau_2) \in \text{CNNTop}$ and $f^\rightarrow : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a morphism in CNNTop . We will show that $f^\rightarrow : (X, F(\tau_1)) \rightarrow (Y, F(\tau_2))$ is a morphism in IFTop.

Let $A = (\mu_A, \nu_A) \in F(\tau_2)$, then $\exists A' \in \tau_2$ such that $A' = (\mu_{A'}, I_{A'}, \nu_{A'})$ where, $\mu_A = \mu_{A'}$ and $\nu_A = \nu_{A'}$. By the continuity of $f^\rightarrow$ in $CNNTop$, we have $f^\leftarrow(A') \in \tau_1$.

$$\begin{aligned} f^\leftarrow(A') &= (\mu_{A'}(f(x)), I_{A'}(f(x)), \nu_{A'}(f(x))) \in \tau_1 \\ \implies F(f^\leftarrow(A')) &= (\mu_{A'}(f(x)), \nu_{A'}(f(x))) = (\mu_A(f(x)), \nu_A(f(x))) \in F(\tau_1) \\ \implies f^\leftarrow(A) &\in F(\tau_1) \end{aligned}$$

Hence $f^\rightarrow$ is continuous in $IFTop$.

Remark 4.14. Let $(X, \tau_1), (X, \tau_2) \in CNNTop$ with $\tau_1 = \{(0, 1, 1), (1, 0, 0), (\frac{1}{2}, 0, \frac{1}{4})\}$ and $\tau_2 = \{(0, 1, 1), (1, 0, 0), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\}$. In this case $\tau_1 \neq \tau_2$ but $F(\tau_1) = F(\tau_2)$. Hence the functor F is not injective on objects.

Remark 4.15. Let $X = \{a, b\}$, $\tau_1 = \{(1, 0, 0), (0, 1, 1), ((a, \frac{1}{2}, 0, 0), (b, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}))\}$ and $\tau_2 = \{(1, 0, 0), (0, 1, 1), ((a, \frac{1}{2}, \frac{1}{2}, 0), (b, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}))\}$ be two consistent neutrosophic topologies on X . Then, $F(\tau_1) = F(\tau_2) = \{(1, 0), (0, 1), ((a, \frac{1}{2}, 0), (b, \frac{1}{2}, \frac{1}{2}))\}$ and $id : X \rightarrow X$ is a morphism in $IFTop$ which do not have a pre-image under F in $CNNTop$. Hence the functor F is not full.

Definition 4.16. For $\tau \in CNNTop$, let $SN(\tau)$ denote the neutrosophic topology generated by $\{J(A) : A \in F(\tau)\} \cup \tau$. The topology $SN(\tau)$ is called the strong intuitionistic-induced neutrosophic topology associated with τ . The collection of all strong intuitionistic-induced neutrosophic topological spaces, together with the morphisms inherited from $NTop$, forms a subcategory of $NTop$, denoted by $SINTop$.

It follows that $IINTop$ is a subcategory of $SINTop$. The following theorem shows that $SINTop$ is a coreflective subcategory of $CNNTop$.

Theorem 4.17. $SINTop$ is a coreflective subcategory of $CNNTop$.

Proof. For $(X, \tau) \in CNNTop$, take $(X, \tau_s) \in SINTop$ as the strong intuitionistic induced neutrosophic topological space of τ on X . That is, τ_s is the neutrosophic topology generated by the subbase $\{J(A) : A \in F(\tau)\} \cup \tau$.

Since $\tau \subseteq \tau_s$, $id^\rightarrow : (X, \tau_s) \rightarrow (X, \tau)$ is a $CNNTop$ morphism.

Let $(Y, \tau') \in SINTop$ and $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau)$ be a $CNNTop$ morphism. We will show that $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau_s)$ is a $SINTop$ morphism.

Let A be an element of the subbase of τ_s , i.e., $A \in \{J(A) : A \in F(\tau)\} \cup \tau$. If $A \in \tau$ then $f^\leftarrow(A) \in \tau'$, since $f^\rightarrow$ is continuous in $CNNTop$.

Else, $A \in \{J(A) : A \in F(\tau)\}$. Then, $A = J((\mu_{A'}, \nu_{A'}))$, where $(\mu_{A'}, \nu_{A'}) \in F(\tau)$ for some $(\mu_{A'}, I_{A'}, \nu_{A'}) \in \tau$, which implies $A = (\mu_{A'}, 1 - (\mu_{A'} + \nu_{A'}), \nu_{A'})$ where $\mu_A = \mu_{A'}$ and $\nu_A = \nu_{A'}$. Therefore $(\mu_{A'}, I_{A'}, \nu_{A'}) \in \tau$.

Since $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau)$ is a morphism in $CNNTop$,

$$f^\leftarrow((\mu_{A'}, I_{A'}, \nu_{A'})) = (\mu_{A'}(f(x)), I_{A'}(f(x)), \nu_{A'}(f(x))) \in \tau'$$

Since $\tau' \in \text{SINTop}$, $(\mu_{A'}(f(x)), (1 - (\mu_{A'} + \nu_{A'})(f(x)), \nu_{A'}(f(x))) \in \tau'$. Thus $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau_s)$ is a SINTop morphism and $f^\rightarrow = id^\rightarrow \circ f^\rightarrow$. Hence SINTop is a coreflective subcategory of CNNTop .

Corollary 4.18. *Let $(X_j, \tau_j)_{j \in J}$ be a family of consistent neutrosophic topological spaces and $X = \prod_{j \in J} X_j$. Then $SI(\prod_{j \in J} \tau_j) = \prod_{j \in J} SI(\tau_j)$.*

We now consider a drawback concerning constant functions in fuzzy and neutrosophic topological spaces, namely, that constant functions need not be continuous. It was shown in [12] that constant functions between neutrosophic topological spaces are not necessarily continuous. This issue can be resolved by including all constant neutrosophic sets in the topology. In other words, we include in the topology all neutrosophic sets whose truth, indeterminacy, and falsity membership functions are constant.

Let $(\bar{a}, \bar{b}, \bar{c})$ denote the neutrosophic set on X defined by $T_A(x) = a, I_A(x) = b, F_A(x) = c, \forall x \in X$. Let (X, τ) be a neutrosophic topological space. The neutrosophic topology on X generated by $\tau \cup \{(\bar{a}, \bar{b}, \bar{c}) : a, b, c \in [0, 1]\}$ is called the stratification of τ , and is denoted by $S(\tau)$. It can be verified that every constant function between stratified neutrosophic topological spaces is continuous.

Let SNTop denote the category of all stratified neutrosophic topological spaces. Then SNTop is a full subcategory of NTop .

The following proposition shows that stratification is inherited by domains of continuous maps into stratified neutrosophic topological spaces.

Proposition 4.19. *Let (X, τ_1) and (Y, τ_2) be neutrosophic topological spaces. If (Y, τ_2) is stratified and $f^\rightarrow : (X, \tau_1) \rightarrow (Y, \tau_2)$ is a continuous map, then (X, τ_1) is stratified.*

Proof. Let $(\bar{a}, \bar{b}, \bar{c})$ be a constant neutrosophic set. Since $f^\leftarrow((\bar{a}, \bar{b}, \bar{c})) = (\bar{a}, \bar{b}, \bar{c})$, the proof of the proposition follows.

Theorem 4.20. *SNTop is a coreflective subcategory of NTop .*

Proof. Let $(X, \tau) \in \text{NTop}$ and $S(\tau)$ be the stratification of τ . Since $\tau \subseteq S(\tau)$, $id^\rightarrow : (X, S(\tau)) \rightarrow (X, \tau)$ is a NTop morphism.

Let $(Y, \tau') \in \text{SNTop}$ and $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau)$ be a NTop morphism. We will show that $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau)$ is also a SNTop morphism.

Let A be an element of the subbase of $S(\tau)$, i.e., $A \in \tau$ or A is a constant set. If $A \in \tau$ then $f^\leftarrow(A) \in \tau'$, since $f^\rightarrow$ is continuous in NTop .

Else, A is a constant neutrosophic set. Since τ' is stratified and $f^\leftarrow(U)$ is a constant set whenever U is a constant neutrosophic set, we get $f^\leftarrow(U) \in \tau'$. Thus $f^\rightarrow : (Y, \tau') \rightarrow (X, \tau)$ is a morphism in SNTop and $f^\rightarrow = id^\rightarrow \circ f^\rightarrow$. Hence SNTop is a coreflective subcategory of NTop .

Corollary 4.21. *Let $(X_j, \tau_j)_{j \in J}$ be a family of neutrosophic topological spaces and $X = \prod_{j \in J} X_j$. Then $S(\prod_{j \in J} \tau_j) = \prod_{j \in J} S(\tau_j)$.*

5. Sierpiński Object in NTop

In this section, we identify the Sierpiński object in NTop and investigate its properties.

Let $N_s = I \times I \times I$ and $\tau_s = \{(1, 0, 0), (0, 1, 1), (id_1, id_2, id_3)\}$, where $id_i : I^3 \rightarrow I$ is defined by $id_i(a_1, a_2, a_3) = a_i; i = 1, 2, 3$. Clearly, (N_s, τ_s) is a neutrosophic topological space. We denote the neutrosophic set (id_1, id_2, id_3) by ID . The following proposition shows that (N_s, τ_s) serves as an open-set classifier in NTop.

Proposition 5.1. *Let $(X, \tau) \in NTop$. Then $(T_U, I_U, F_U) \in \tau$ if and only if the map $U : (X, \tau) \rightarrow (N_s, \tau_s)$, defined by $U(x) = (T_U(x), I_U(x), F_U(x))$, is continuous.*

Proof. Assume (T_U, I_U, F_U) is open in (X, τ) .

$U^\leftarrow(ID)(x) = (T_{ID}(U(x)), I_{ID}(U(x)), F_{ID}(U(x))) = (T_U(x), I_U(x), F_U(x)), \forall x \in X$. Since $U^\leftarrow(ID) = (T_U, I_U, F_U) \in \tau$, $U : (X, \tau) \rightarrow (N_s, \tau_s)$ is continuous. Conversely, continuity of $U : (X, \tau) \rightarrow (N_s, \tau_s)$ implies $U^\leftarrow(ID) = U \in \tau$.

Theorem 5.2. *(N_s, τ_s) is a Sierpiński object in NTop.*

Proof. Assume $(X, \tau_X) \in NTop$. Let $\mathcal{H}$ denote the family of all continuous functions from (X, τ_X) to (N_s, τ_s) . We need to show that the family $\mathcal{H}$ is initial. Let $(Z, \tau_z) \in NTop$ and $h : Z \rightarrow X$ be an ordinary function such that $\forall f \in \mathcal{H}$, $f \circ h : (Z, \tau_z) \rightarrow (N_s, \tau_s)$ is continuous. It requires to show that $h : (Z, \tau_z) \rightarrow (X, \tau_X)$ is continuous. Suppose $(T_A, I_A, F_A) \in \tau_X$, then by Proposition 5.1, $A : (X, \tau_X) \rightarrow (N_s, \tau_s)$ is continuous, thus, $A \in \mathcal{H}$. Since $A \in \mathcal{H}$, $A \circ h$ is continuous by assumption and by Proposition 5.1, we get $(T_{A \circ h}, I_{A \circ h}, F_{A \circ h}) \in \tau_z$, where $T_{A \circ h} = P_1(A \circ h)$, $I_{A \circ h} = P_2(A \circ h)$ and $F_{A \circ h} = P_3(A \circ h)$. Since $h^\leftarrow(A) = (T_{A \circ h}, I_{A \circ h}, F_{A \circ h})$, we get $h : (Z, \tau_z) \rightarrow (X, \tau_X)$ is continuous.

The following theorem characterizes NTop as a category of sets with structures.

Theorem 5.3. *A category $\mathbb{A}$ of sets with structures is isomorphic to NTop if and only if there exists an object (N_s, τ_s) in $\mathbb{A}$ with underlying set $N_s = I \times I \times I$, satisfying the following conditions:*

1. *(N_s, τ_s) is a Sierpiński object in $\mathbb{A}$.*
2. *An initial lift exists for every family of the form $\{f_i : X \rightarrow (N_s, \tau_s)\}_{i \in I}$.*
3. *Define $(N_s, \tau_s)^J = (N_s^J, \tau)$, where τ is the initial lift induced by all projections $N_s^J \rightarrow (N_s, \tau_s)$. Then the map $join : (N_s, \tau_s)^J \rightarrow (N_s, \tau_s)$, defined by $join(V) = \bigcup_{j \in J} V(j); V : J \rightarrow N_s$, is an $\mathbb{A}$ -morphism for every set J .*

4. The map $meet : (N_s, \tau_s)^J \rightarrow (N_s, \tau_s)$, defined by $meet(V) = \bigcap_{j \in J} V(j); V : J \rightarrow N_s$, is an $\mathbb{A}$ -morphism for every finite set J .
5. For any $(X, \tau) \in \mathbb{A}$ and every initial family $\mathcal{F}$ of $\mathbb{A}$ -morphisms from (X, τ) to (N_s, τ_s) , the initial lift of $\{f : X \rightarrow (N_s, \tau_s) \mid f \in \mathcal{F}\}$ contains every $\mathbb{A}$ -morphism $h : (X, \tau) \rightarrow (N_s, \tau_s)$.

Proof. Let $\mathbb{A}$ be isomorphic to $NTop$. It can be easily verified that $NTop$ satisfies (1) and (2). To show $join : (N_s, \tau_s)^J \rightarrow (N_s, \tau_s)$ is continuous, it suffices to prove that the neutrosophic set $(P_1 \circ join, P_2 \circ join, P_3 \circ join)$ is open in N_s^J and then the continuity follows by Proposition 5.1. Since the neutrosophic topology τ is generated by the projection functions and $P_j^{\leftarrow}(ID) = P_j$, we get P_j is open in N_s^J . Also, observe that $join = \bigcup_{j \in J} P_j$, and therefore, $join$ is open in N_s^J . Similarly, the continuity of $meet$ follows since $inf = \bigcap_{j \in J} P_j$.

Let $(X, \tau) \in NTop$ and $\mathcal{F}$ be an initial family of continuous functions from (X, τ) to (N_s, τ_s) . Let τ_l be the initial lift of $\{f : X \rightarrow (N_s, \tau_s) \mid f \in \mathcal{F}\}$. Observe that $\forall f \in \mathcal{F}, f \circ id = f$, where id is the identity function from (X, τ_l) to (X, τ) . Since $\mathcal{F}$ is initial, we get id is continuous and hence $\tau \subseteq \tau_l$.

Conversely, assume that $\mathbb{A}$ satisfies properties (1) through (5). To prove $\mathbb{A}$ and $NTop$ are isomorphic, we define two functors between $\mathbb{A}$ and $NTop$ which are inverse to each other. For $(X, u) \in \mathbb{A}$, define $H((X, u)) = (X, \tau_u)$, where τ_u is the initial lift of the family $\mathcal{F} = \{f : X \rightarrow (N_s, \tau_s) \mid f : (X, u) \rightarrow (N_s, \tau_s) \text{ is an } \mathbb{A}\text{-morphism}\}$ and $H(f) = f$, where f is an $\mathbb{A}$ -morphism. Since $f^{\leftarrow}(ID) = (T_{ID}(f(x)), I_{ID}(f(x)), F_{ID}(f(x))) = (P_1 \circ f(x), P_2 \circ f(x), P_3 \circ f(x)) = f(x)$, it clearly follows that $\tau_u = \mathcal{F}$. We will show that $\mathcal{F} \in NTop$. Consider a family $\{g_j : j \in J\}$ in $\mathcal{F}$, define the function $g : (X, u) \rightarrow (N_s, \tau_s)^J$ by $g(x)(j) = g_j(x)$. For $j \in J$, projection map $P_j : N_s^J \rightarrow N_s$, $P_j \circ g(x) = g_j(x)$, therefore $P_j \circ g = g_j$. Using property (2) we can deduce that g is an $\mathbb{A}$ -morphism. Now we use property (3) to show $\bigvee_{j \in J} g_j \in \mathcal{F}$. For $x \in X$, $meet \circ g(x) = meet(g(x)) = \bigvee_{j \in J} g(x)(j) = \bigvee_{j \in J} g_j(x)$. Therefore, $meet \circ g = \bigvee_{j \in J} g_j$. Since $meet$ and g are $\mathbb{A}$ -morphisms, we get $\bigvee_{j \in J} g_j \in \mathcal{F}$. Similarly, property (4) implies that $\bigwedge_{j \in J} g_j \in \mathcal{F}$, where J is a finite set. Hence $\mathcal{F} \in NTop$. Now let $f : (X, u) \rightarrow (Y, v)$ be an $\mathbb{A}$ -morphism, we have to prove that $f : (X, \tau_u) \rightarrow (Y, \tau_v)$ is continuous. Let $(T_A, I_A, F_A) \in \tau_v$, then $A : (Y, v) \rightarrow (N_s, \tau_s)$ is an $\mathbb{A}$ -morphism and hence $A \circ f : (X, u) \rightarrow (N_s, \tau_s)$ is an $\mathbb{A}$ -morphism. Thus $(T_{A \circ f}, I_{A \circ f}, F_{A \circ f}) \in \tau_u$ implying f is continuous.

Now let's define a functor $G : NTop \rightarrow \mathbb{A}$. For $(X, \tau) \in NTop$, define $G(f) = f$ and $G((X, \tau)) = (X, a_\tau)$, where a_τ is the initial lift of the family $\{g : X \rightarrow$

$(N_s, \tau_s)|g \in \tau\}$, whose existence is guaranteed by (2). By definition, $\{g : (X, a_\tau) \rightarrow (N_s, \tau_s)|g \in \tau\}$ is an initial family.

Let $f : (X, \tau) \rightarrow (Y, \tau')$ be continuous. Let $(T_U, I_U, F_U) \in \tau'$, property (5) implies $U : (Y, a_{\tau'}) \rightarrow (N_s, \tau_s)$ is an $\mathbb{A}$ -morphism. Also, for $(T_U, I_U, F_U) \in \tau'$, $f^{\leftarrow}(U) = (T_{U \circ f}, I_{U \circ f}, F_{U \circ f}) \in \tau$, again by property (5), $U \circ f : (X, a_\tau) \rightarrow (N_s, \tau_s)$ is an $\mathbb{A}$ -morphism. Since $\{f : (Y, s_{\tau'}) \rightarrow (N_s, \tau_s)|f \in \tau'\}$ is an initial family, f is an $\mathbb{A}$ -morphism. It is easy to verify that G is inverse to F .

Having identified the Sierpiński object and characterized $N\text{Top}$, we now determine the epireflective hull of the Sierpiński object in $N\text{Top}$.

Remark 5.4. [1] *Since $N\text{Top}$ is a topological construct, a morphism in $N\text{Top}$ is an epimorphism if and only if it is surjective. Moreover, a morphism is an extremal monomorphism if and only if it is an embedding. Therefore, the extremal subobjects in $N\text{Top}$ are precisely the subspaces.*

Since $N\text{Top}$ is a topological construct, it is an epi-co-wellpowered category [12]. Consequently, we have the following proposition.

Proposition 5.5. *Let $\mathbb{C}$ be a topological category and let $\mathcal{B}$ be a class of $\mathbb{C}$ -objects. Then a $\mathbb{C}$ -object B belongs to $EH(\mathcal{B})$, the epireflective hull of $\mathcal{B}$, if and only if B is an extremal subobject of a product of objects of $\mathcal{B}$.*

Theorem 5.6. *$N\text{Top}_0$ is the epireflective hull of N_s in $N\text{Top}$.*

Proof. Since (id_1, id_2, id_3) separates points of N_s , (N_s, τ_s) is T_0 . Then, by Proposition 5.5, it easily follows that $EH(N_s) \subseteq N\text{Top}_0$. Conversely, let $(X, \delta) \in N\text{Top}_0$. Define a mapping e from X to N_s^δ as follows:

$$e(x) = \prod_{U \in \delta} (T_U(x), I_U(x), F_U(x))$$

We need to prove that $e : (X, \delta) \rightarrow (N_s^\delta, \tau_s^\delta)$ is continuous. Consider the U -th projection map $P_U : N_s^\delta \rightarrow N_s$. Then, $\forall U \in \delta$ and $\forall x \in X$, $P_U \circ e(x) = (T_U(x), I_U(x), F_U(x)) = U$, hence e is continuous. The fact that e is injective follows since (X, δ) is T_0 ; for distinct $x, y \in X$, $\exists V \in \delta$ such that $T_V(x) \neq T_V(y)$ or $I_V(x) \neq I_V(y)$ or $F_V(x) \neq F_V(y)$, thus $e(x) \neq e(y)$. It remains to show that $e : X \rightarrow e(X)$ is a neutrosophic open map. We know that $\forall U \in \delta$, $P_U : N_s^\delta \rightarrow (N_s, \tau_s)$ is a continuous map. Then, by Proposition 5.1, $P_U \in N_s^\delta$.

For $U \in \delta$, consider the neutrosophic set $e^{\rightarrow}(U)$ on $e(X)$. For any $x \in X$,

$$\begin{aligned} e^{\rightarrow}(U)(e(x)) &= \left(\bigvee_{y \in e^{-1}(e(x))} T_U(y), \bigwedge_{y \in e^{-1}(e(x))} I_U(y), \bigwedge_{y \in e^{-1}(e(x))} F_U(y) \right) \\ &= (T_U(x), I_U(x), F_U(x)) = U(x) = P_U(e(x)). \end{aligned}$$

Thus we get $e^\rightarrow(U) = P_U \cap (1, 0, 0)_{e(X)}$, therefore $e^\rightarrow(U)$ is a neutrosophic open set on $e(X)$. Hence, e embeds (X, δ) into a product of neutrosophic Sierpiński spaces and thus $NTop_0 \subseteq EH(N_s)$.

Remark 5.7. *As a consequence of the previous theorem, $NTop_0$ is an epireflective subcategory of $NTop$. We now describe the epireflection of an object $(X, \tau) \in NTop$ in $NTop_0$.*

Define a relation $\sim$ on X by $x \sim z \iff T_U(x) = T_U(z), I_U(x) = I_U(z)$, and $F_U(x) = F_U(z)$ for every $U \in \tau$. It is readily verified that $\sim$ is an equivalence relation on X . Let $q_X : (X, \tau) \rightarrow (\bar{X}, \bar{\tau})$ be the corresponding quotient map, where $(\bar{X}, \bar{\tau})$ is the quotient neutrosophic topology. Then $(\bar{X}, \bar{\tau})$ is the epireflection of (X, τ) in $NTop_0$.

6. Conclusion

Motivated by the fundamental principle of independence among the three membership functions in neutrosophic theory, we introduced an alternative definition of the category $NSet$. We showed that $NSet$ is a topological and Cartesian closed category. Within the newly introduced category $NTop$, we identified two subcategories, namely the categories of intuitionistic-induced neutrosophic topologies and strong intuitionistic-induced neutrosophic topologies. We demonstrated that the category of intuitionistic fuzzy topological spaces can be identified with a subcategory of $NTop$ in a categorical sense. The category of stratified neutrosophic topological spaces was introduced and shown to be a coreflective subcategory of $NTop$.

We also identified the Sierpiński object in $NTop$ and proved that it serves as an open-set classifier. Furthermore, we obtained a characterization of $NTop$ in $Struct(SET)$ and showed that $NTop_0$ is the epireflective hull of the Sierpiński object in $NTop$. The coreflective hull of the Sierpiński object remains an open problem.

The study of $NTop$ from the viewpoint of a topological universe is a direction for future research. Further subcategories of $NTop$ and their categorical relationships with $NTop$ will also be investigated. Moreover, there is scope for exploring further connections between neutrosophic topology and existing fuzzy topological structures.

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