

A NOTE ON SOME NEW INTERVAL-VALUED OPEN SETS IN TOPOLOGICAL SPACES

P. Sugirtha Nisha and I. Rajasekaran

PG and Research Department of Mathematics,
Tirunelveli Dakshina Mara Nadar Sangam College,
T. Kallikulam - 627113, Tirunelveli District, Tamil Nadu, INDIA
E-mail : sugirthanisha27@gmail.com, sekarmelakkal@gmail.com

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Abstract: The purpose of this paper is to introduce and study several new classes of interval-valued open sets in topological spaces. In particular, the notions of interval-valued $\hat{S}$ -open sets, interval-valued semi-regular sets, and interval-valued $\mathcal{Q}$ -closed sets are defined and investigated. Fundamental properties of these newly defined sets are examined and their relationships with existing types of open and closed sets are discussed. Characterizations and basic results concerning these sets are established. Several implications between interval-valued $\hat{S}$ -open sets, interval-valued semi-regular sets, and interval-valued $\mathcal{Q}$ -closed sets are obtained. The behavior of these sets under set-theoretic operations such as union and intersection is analyzed. Moreover, examples are provided to illustrate the concepts and to show the independence of certain properties. The results obtained extend and generalize some known results in the literature. These investigations contribute to the development of generalized open set theory in topology. The study may also provide a foundation for further research in related topological structures.

Keywords and Phrases: $\mathcal{IV}\hat{S}$ -open, $\mathcal{IV}s$ -regular and $\mathcal{IV}\mathcal{Q}$ -closed.

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1. Introduction

The notion of interval sets, originally introduced by Yao [6], offers a useful approach for representing and approximating uncertain or complex concepts. Subsequently, Kim et al. [4] described this framework using the term interval-valued

sets, interpreting it both as an extension of classical set theory and as a particular case of interval-valued fuzzy sets. In their study, they examined various forms of interval-valued neighborhoods and organized them into two main categories, while also exploring the associated interval-valued closure and interior operators. In more recent work, Cheong et al. [1] proposed the concept of interval-valued relations and studied their properties from a category-theoretic perspective, thereby extending the applicability of interval-valued structures in mathematical theory as well as computational contexts. Additionally, Rajasekaran et al. [5] introduced several types of kernel sets and investigated regular closed sets and generalized closed sets together with their fundamental characteristics.

Topology plays a fundamental role in modern mathematics by providing a framework to study continuity, convergence, and related structures through open and closed sets. Over the years, many generalized forms of open and closed sets have been introduced to extend classical topological concepts. These generalizations help in understanding finer structures and relationships within topological spaces. In this direction, several researchers have proposed different types of generalized open sets and investigated their properties. Motivated by these developments, this paper introduces some new classes of $\mathcal{IV}$ -open sets. In particular, the notions of $\mathcal{IV}\hat{S}$ -open sets, $\mathcal{IV}S$ -regular sets, and $\mathcal{IV}Q$ -closed sets are defined and studied. The basic properties and characterizations of these sets are investigated in detail. Relationships between these newly defined sets and other existing generalized open and closed sets are also discussed. Examples are provided to clarify the concepts and support the theoretical results. These results contribute to the ongoing development of generalized topological structures.

2. Preliminaries

Throughout this paper $(X, \mathcal{T}^X)$ (or X) represent interval-valued topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space (X, τ) , $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ and $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ denote the interval-valued closure of $[\mathcal{M}^-, \mathcal{M}^+]$ and the interval-valued interior of $[\mathcal{M}^-, \mathcal{M}^+]$ respectively. We recall the following definitions which are useful in the sequel.

For an interval-valued set $[\mathcal{M}^-, \mathcal{M}^+]$ in X , the symbols $\mathcal{M}^-$ and $\mathcal{M}^+$ are ordinary (crisp) subsets of X satisfying $\mathcal{M}^- \subseteq \mathcal{M}^+$, and *not* interval-valued sets themselves. Here $\mathcal{M}^-$ is called the *lower bound* and $\mathcal{M}^+$ the *upper bound* of the interval-valued set $[\mathcal{M}^-, \mathcal{M}^+]$. An element $x \in X$ is regarded as a *definite member* of $[\mathcal{M}^-, \mathcal{M}^+]$ if $x \in \mathcal{M}^-$, and as a *possible member* if $x \in \mathcal{M}^+ \setminus \mathcal{M}^-$; elements outside $\mathcal{M}^+$ are definitely not members. Thus $[\mathcal{M}^-, \mathcal{M}^+]$ should be read as the collection of all ordinary subsets $\mathcal{S}$ of X lying between $\mathcal{M}^-$ and $\mathcal{M}^+$, i.e., $\mathcal{M}^- \subseteq$

$\mathcal{S} \subseteq \mathcal{M}^+$, in accordance with Definition 2.1. The capital letter used (e.g., $\mathcal{M}, \mathcal{S}, \mathcal{K}, \mathcal{C}, \mathcal{H}, \mathcal{N}, \mathcal{A}, \mathcal{B}, \mathcal{D}$) is merely a label for the particular interval-valued set under discussion, and the corresponding superscripted $-$ and $+$ always denote its lower and upper bound sets, respectively, in the sense described above.

Definition 2.1. [6] Let X be a non-empty set. Then, the form $[\mathcal{M}^-, \mathcal{M}^+] = \{[\mathcal{S} \subset X : \mathcal{M}^- \subset \mathcal{S} \subset \mathcal{M}^+]\}$ is called an interval-valued set (briefly, $\mathcal{IVS}$) in X , if $\mathcal{M}^-, \mathcal{M}^+ \subset X$ and $\mathcal{M}^- \subset \mathcal{M}^+$.

In this case, $\mathcal{M}^-$ [resp. $\mathcal{M}^+$] represent the set of minimum [resp. maximum] memberships of elements of X to $[\mathcal{M}^-, \mathcal{M}^+]$.

Here, $[\emptyset, \emptyset]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and denoted by $\tilde{\emptyset}$ [resp. $\tilde{X}$]. The set of all $\mathcal{IVS}$ s in X is denoted by $\mathcal{IVS}(X)$. An interval-valued topological space is denoted by $\mathcal{IVTS}$ or $(X, \mathcal{T}^X)$.

Definition 2.2. [4] A non-empty set X with $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ are $\mathcal{IVS}$ respectively. Then

1. $[\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+] \iff \mathcal{M}^- \subset \mathcal{S}^- \text{ and } \mathcal{M}^+ \subset \mathcal{S}^+$
2. $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \iff [\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+] \text{ and } [\mathcal{S}^-, \mathcal{S}^+] \subset [\mathcal{M}^-, \mathcal{M}^+]$
3. $([\mathcal{M}^-, \mathcal{M}^+])^c = [(\mathcal{M}^+)^c, (\mathcal{M}^-)^c]$
4. $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cup \mathcal{S}^-, \mathcal{M}^+ \cup \mathcal{S}^+]$
5. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cap \mathcal{S}^-, \mathcal{M}^+ \cap \mathcal{S}^+]$

Definition 2.3. [4] Let X be a non-empty set and let τ be a non-empty family of $\mathcal{IVS}$ s on X . Then τ is called an interval-valued topology (briefly, $\mathcal{IVT}$) on X if it satisfies the following axioms:

1. $[\emptyset, \emptyset], [X, X] \in \tau$,
2. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] \in \tau$ for any $[\mathcal{M}^-, \mathcal{M}^+], [\mathcal{S}^-, \mathcal{S}^+] \in \tau$.
3. $\bigcup_{j \in J} \mathcal{S}_j \in \tau$ for any family $(\mathcal{S}_j)_{j \in J}$ of members of τ .

In this case, $(X, \mathcal{T}^X)$, where $\mathcal{T}^X = \tau$, is called an interval-valued topological space (briefly, $\mathcal{IVTS}$) and each member of τ is called an interval-valued open set (briefly, $\mathcal{IVOs}$) in X . An $\mathcal{IVS}$ $[\mathcal{S}^-, \mathcal{S}^+]$ is called a $\mathcal{IV}$ -closed set (briefly, $\mathcal{IVC}[\mathcal{S}^-, \mathcal{S}^+]$) in X .

Definition 2.4. [4] If $(X, \mathcal{T}^X)$ is an interval-valued topological space with respect to X and if $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \tilde{X}$, then

1. the an interval-valued interior of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the union of all an interval-valued open subsets of $[\mathcal{M}^-, \mathcal{M}^+]$ and it is denoted by $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$.

That is, $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ is the largest an interval-valued open subset of $[\mathcal{M}^-, \mathcal{M}^+]$.

2. the an interval-valued closure of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the intersection of all an interval-valued closed sets containing $[\mathcal{M}^-, \mathcal{M}^+]$ and it is denoted by $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$.

That is, $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ is the smallest an interval-valued closed set containing $[\mathcal{M}^-, \mathcal{M}^+]$.

Definition 2.5. [3] A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is called an ;

1. interval-valued pre-open (resp. $\mathcal{IVp}$ -open) if $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))$.
2. interval-valued semi-open (resp. $\mathcal{IVs}$ -open) if $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$.

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.6. [2] A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is called interval-valued dense (resp. $\mathcal{IV}$ -dense) if $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = [X, X]$.

3. On some new interval-valued open sets

Definition 3.1. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is said to be an

1. interval-valued α -open (briefly, $\mathcal{IV}\alpha$ -open) if

$$[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))).$$
2. interval-valued β -open (briefly, $\mathcal{IV}\beta$ -open) if

$$[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))).$$
3. interval-valued $\hat{\mathcal{S}}$ -open (briefly, $\mathcal{IV}\hat{\mathcal{S}}$ -open) if

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$
4. interval-valued t -set (briefly, $\mathcal{IV}t$ -set) if

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]).$$

Example 3.2. Let $X = \{a, b, c, d\}$ and $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\{a, b\}, \{a, b, d\}]\}$ be an interval-valued topology on X and $(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{c\}, \{c, d\}]\}$.

1. Since $[\{a, b\}, \{a, b, d\}]$ is already $\mathcal{IV}$ -open, $\mathcal{IVI}([\{a, b\}, \{a, b, d\}]) = [\{a, b\}, \{a, b, d\}]$. The only $\mathcal{IV}$ -closed set containing $[\{a, b\}, \{a, b, d\}]$ is $\tilde{X}$, so $\mathcal{IVC}([\{a, b\}, \{a, b, d\}]) = \tilde{X}$, and hence $\mathcal{IVI}(\mathcal{IVC}([\{a, b\}, \{a, b, d\}])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$. Since

$$[\{a, b\}, \{a, b, d\}] \subseteq \tilde{X} = \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\{a, b\}, \{a, b, d\}]))),$$

$[\{a, b\}, \{a, b, d\}]$ is $\mathcal{IV}\alpha$ -open set.

2. Since $\{a, c\} \not\subseteq \{c\}$, the only $\mathcal{IV}$ -closed set containing $[\{a, c\}, \{a, c, d\}]$ is $\tilde{X}$; thus $\mathcal{IVC}([\{a, c\}, \{a, c, d\}]) = \tilde{X}$. Then $\mathcal{IVI}(\mathcal{IVC}([\{a, c\}, \{a, c, d\}])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$, and so $\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\{a, c\}, \{a, c, d\}]))) = \mathcal{IVC}(\tilde{X}) = \tilde{X}$. Since

$$[\{a, c\}, \{a, c, d\}] \subseteq \tilde{X} = \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\{a, c\}, \{a, c, d\}]))),$$

$[\{a, c\}, \{a, c, d\}]$ is an $\mathcal{IV}\beta$ -open set.

3. Since the only $\mathcal{IV}$ -closed set containing $[\{a, b, c\}, X]$ is $\tilde{X}$, we have $\mathcal{IVC}([\{a, b, c\}, X]) = \tilde{X}$, so $\mathcal{IVI}(\mathcal{IVC}([\{a, b, c\}, X])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$.
On the other hand, $[\{a, b\}, \{a, b, d\}] \subseteq [\{a, b, c\}, X]$ while $\tilde{X} \not\subseteq [\{a, b, c\}, X]$, so $\mathcal{IVI}([\{a, b, c\}, X]) = [\{a, b\}, \{a, b, d\}]$. From part (1), $\mathcal{IVC}([\{a, b\}, \{a, b, d\}]) = \tilde{X}$, hence $\mathcal{IVC}(\mathcal{IVI}([\{a, b, c\}, X])) = \tilde{X}$.

Therefore

$$\mathcal{IVI}(\mathcal{IVC}([\{a, b, c\}, X])) = \tilde{X} = \mathcal{IVC}(\mathcal{IVI}([\{a, b, c\}, X])),$$

so $[\{a, b, c\}, X]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open set.

4. Since $[\{c\}, \{c, d\}]$ is already $\mathcal{IV}$ -closed, $\mathcal{IVC}([\{c\}, \{c, d\}]) = [\{c\}, \{c, d\}]$. Since $\{a, b\} \not\subseteq \{c\}$, no nonempty $\mathcal{IV}$ -open set is contained in $[\{c\}, \{c, d\}]$, so $\mathcal{IVI}([\{c\}, \{c, d\}]) = \tilde{\emptyset}$. Then

$$\mathcal{IVI}(\mathcal{IVC}([\{c\}, \{c, d\}])) = \mathcal{IVI}([\{c\}, \{c, d\}]) = \tilde{\emptyset} = \mathcal{IVI}([\{c\}, \{c, d\}]),$$

so $[\{c\}, \{c, d\}]$ is $\mathcal{IV}t$ -set.

Proposition 3.3. In an interval-valued topological space $(X, \mathcal{T}^X)$,

1. Each $\mathcal{IV}$ -open set is also $\mathcal{IV}\beta$ -open.
2. Every $\mathcal{IV}s$ -open set is contained in the class of $\mathcal{IV}\beta$ -open sets.
3. Any $\mathcal{IV}p$ -open set can be regarded as $\mathcal{IV}\beta$ -open.
4. Every $\mathcal{IV}r$ -open set belongs to the family of an $\mathcal{IV}\beta$ -open sets.
5. Each $\mathcal{IV}\alpha$ -open set is likewise a $\mathcal{IV}\beta$ -open.

Proof.

1. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -open,
 $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) = [\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$. Then
 $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$. There-
 fore $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$. Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is a
 $\mathcal{IV}\beta$ -open.
2. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open,
 then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$.
 Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open
3. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}p$ -open,
 then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$.
 Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open
4. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}r$ -open,
 then $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$.
 Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open
5. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\alpha$ -open,
 $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$, $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$.
 Therefore $\mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))) \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq$
 $\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$. Hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open.

Proposition 3.4. For a subset of an interval-valued topological space $(X, \mathcal{T}^X)$, the following properties hold:

1. Each $\mathcal{IV}\alpha$ -open set is also $\mathcal{IV}\hat{S}$ -open.
2. Any $\mathcal{IV}t$ -set belongs to the class of $\mathcal{IV}\hat{S}$ -open sets.

Proof.

1. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\alpha$ -open set,

$$[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Then we obtain $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ and

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

2. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}t$ -set, $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Then we obtain $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Example 3.5. Let $X = \{a, b, c\}$ and $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\{a\}, \{a, b\}]\}$ be an interval-valued topology.

Then $(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{c\}, \{b, c\}]\}$, since $([\{a\}, \{a, b\}])^c = [(\{a, b\})^c, (\{a\})^c] = [\{c\}, \{b, c\}]$.

1. Since $[\{c\}, \{b, c\}]$ is itself $\mathcal{IV}$ -closed, $\mathcal{IVC}([\{c\}, \{b, c\}]) = [\{c\}, \{b, c\}]$. As $\{a\} \not\subseteq \{c\}$, the only $\mathcal{IV}$ -open subset of $[\{c\}, \{b, c\}]$ is $\tilde{\emptyset}$, so $\mathcal{IVI}([\{c\}, \{b, c\}]) = \tilde{\emptyset}$. Then

$$\mathcal{IVI}(\mathcal{IVC}([\{c\}, \{b, c\}])) = \mathcal{IVI}([\{c\}, \{b, c\}]) = \tilde{\emptyset}$$

and

$$\mathcal{IVC}(\mathcal{IVI}([\{c\}, \{b, c\}])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $\tilde{\emptyset} \subseteq \tilde{\emptyset}$, we obtain

$$\mathcal{IVI}(\mathcal{IVC}([\{c\}, \{b, c\}])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\{c\}, \{b, c\}])),$$

so $[\{c\}, \{b, c\}]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

However, for $\mathcal{IV}\alpha$ -openness we require

$$[\{c\}, \{b, c\}] \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\{c\}, \{b, c\}]))).$$

Since $\mathcal{IVI}([\{c\}, \{b, c\}]) = \tilde{\emptyset}$, we get

$$\mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\{c\}, \{b, c\}]))) = \mathcal{IVI}(\mathcal{IVC}(\tilde{\emptyset})) = \mathcal{IVI}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $[\{c\}, \{b, c\}] \not\subseteq \tilde{\emptyset}$, $[\{c\}, \{b, c\}]$ is *not* $\mathcal{IV}\alpha$ -open.

2. Since $[\{a\}, \{a, b\}]$ is $\mathcal{IV}$ -open, $\mathcal{IVI}([\{a\}, \{a, b\}]) = [\{a\}, \{a, b\}]$. The only $\mathcal{IV}$ -closed set containing $[\{a\}, \{a, b\}]$ is $\tilde{X}$ (since $\{a\} \not\subseteq \{c\}$), so $\mathcal{IVC}([\{a\}, \{a, b\}]) = \tilde{X}$. Then

$$\mathcal{IVI}(\mathcal{IVC}([\{a\}, \{a, b\}])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$$

and

$$\mathcal{IVC}(\mathcal{IVI}([\{a\}, \{a, b\}])) = \mathcal{IVC}([\{a\}, \{a, b\}]) = \tilde{X}.$$

Since $\tilde{X} \subseteq \tilde{X}$, we have

$$\mathcal{IVI}(\mathcal{IVC}([\{a\}, \{a, b\}])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\{a\}, \{a, b\}])),$$

so $[\{a\}, \{a, b\}]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

However, for $[\{a\}, \{a, b\}]$ to be an $\mathcal{IV}t$ -set we would need $\mathcal{IVI}(\mathcal{IVC}([\{a\}, \{a, b\}])) = \mathcal{IVI}([\{a\}, \{a, b\}])$, i.e.

$$\tilde{X} = [\{a\}, \{a, b\}],$$

which is false. Hence $[\{a\}, \{a, b\}]$ is *not* an $\mathcal{IV}t$ -set.

Proposition 3.6. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is $\mathcal{IV}s$ -open $\iff \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$.

Proof. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open set, then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ and $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$.

But $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$. Hence

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Conversely, let the condition hold. We have $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ and $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open.

Theorem 3.7. For a subset of an interval-valued topological space $(X, \mathcal{T}^X)$, the following properties hold:

1. Each $\mathcal{IV}$ -open set is $\mathcal{IV}s$ -open.
2. Every $\mathcal{IV}\alpha$ -open set is also $\mathcal{IV}s$ -open.

Proof.

1. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -open, then

$$[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open.

2. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\alpha$ -open set, then

$$[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}s$ -open.

Proposition 3.8. *Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a subset of an interval-valued topological space $(X, \mathcal{T}^X)$. Then $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\beta$ -closed $\iff \mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) = \mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])$.*

Proof. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\beta$ -closed set, $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ and then we obtain $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])$.

But $\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])))$. Thus we have $\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])))$.

Conversely, let the condition hold.

We have $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) = \mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{TV}\beta$ -closed.

Theorem 3.9. *For a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$, the following properties are equivalent:*

1. $[\mathcal{M}^-, \mathcal{M}^+]$ is a $\mathcal{TV}s$ -closed.
2. $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\beta$ -closed and $\mathcal{TV}\hat{\mathcal{S}}$ -closed.

Proof. (1) $\implies$ (2) : Let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{TV}s$ -closed. By Proposition 3.3(2), $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\beta$ -closed. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}s$ -closed, $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ and $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])$. It gives that

$$\mathcal{TV}C(\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Thus

$$\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TV}C(\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))$$

and so $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\hat{\mathcal{S}}$ -closed.

(2) $\implies$ (1) : Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\hat{\mathcal{S}}$ -closed,

$$\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))$$

and

$$\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))).$$

Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}\beta$ -closed, $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$.
Then $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ and so $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{TV}S$ -closed.

Remark 3.10. *There is no implication between $\mathcal{TV}\beta$ -open sets and $\mathcal{TV}\hat{S}$ -closed sets; hence, these notions are independent.*

Example 3.11. Let $X = \{a, b, c, d\}$ and $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\{a, b\}, \{a, b, c\}]\}$ be an interval-valued topology. Then $(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{d\}, \{c, d\}]\}$, since

$$([\{a, b\}, \{a, b, c\}])^c = [(\{a, b, c\})^c, (\{a, b\})^c] = [\{d\}, \{c, d\}].$$

1. Let $[\mathcal{M}^-, \mathcal{M}^+] = [\{a, b, c\}, \{a, b, c\}]$.

$\hat{S}$ -closedness : $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))$.

Since $\{a, b, c\} \not\subseteq \{d\}$, the set $[\{d\}, \{c, d\}]$ does not contain $[\mathcal{M}^-, \mathcal{M}^+]$, so the only $\mathcal{TV}$ -closed superset of $[\mathcal{M}^-, \mathcal{M}^+]$ is $\tilde{X}$; thus $\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{X}$, and hence $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{TVI}(\tilde{X}) = \tilde{X}$.

Also, $[\{a, b\}, \{a, b, c\}] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ (since $\{a, b\} \subseteq \{a, b, c\}$ and $\{a, b, c\} \subseteq \{a, b, c\}$), so $\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]) = [\{a, b\}, \{a, b, c\}]$.

Since $\{a, b\} \not\subseteq \{d\}$, the smallest $\mathcal{TV}$ -closed set containing $[\{a, b\}, \{a, b, c\}]$ is again $\tilde{X}$, so $\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+])) = \tilde{X}$.

Thus $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) = \tilde{X} = \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))$, so $[\mathcal{M}^-, \mathcal{M}^+]$ satisfies $\mathcal{TVI}(\mathcal{TV}C([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))$, and $[\{a, b, c\}, \{a, b, c\}]$ is $\mathcal{TV}\hat{S}$ -closed.

Failure of β -closedness : $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$.

From above, $\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]) = [\{a, b\}, \{a, b, c\}]$ and $\mathcal{TV}C([\{a, b\}, \{a, b, c\}]) = \tilde{X}$, so

$$\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{M}^-, \mathcal{M}^+]))) = \mathcal{TVI}(\tilde{X}) = \tilde{X}.$$

Since $\tilde{X} \not\subseteq [\mathcal{M}^-, \mathcal{M}^+]$ (as $[\mathcal{M}^-, \mathcal{M}^+] \neq \tilde{X}$), the inclusion fails, so $[\{a, b, c\}, \{a, b, c\}]$ is *not* $\mathcal{TV}\beta$ -closed.

2. Let $[\mathcal{C}^-, \mathcal{C}^+] = [\{d\}, \{b, c, d\}]$.

β -closedness : $\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{C}^-, \mathcal{C}^+]))) \subseteq [\mathcal{C}^-, \mathcal{C}^+]$.

Since $\{a, b\} \not\subseteq \{d\}$, the only $\mathcal{TV}$ -open subset of $[\mathcal{C}^-, \mathcal{C}^+]$ is $\tilde{\emptyset}$, so $\mathcal{TVI}([\mathcal{C}^-, \mathcal{C}^+]) = \tilde{\emptyset}$. Then $\mathcal{TV}C(\mathcal{TVI}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{TV}C(\tilde{\emptyset}) = \tilde{\emptyset}$, and so

$$\mathcal{TVI}(\mathcal{TV}C(\mathcal{TVI}([\mathcal{C}^-, \mathcal{C}^+]))) = \mathcal{TVI}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $\tilde{\emptyset} \subseteq [\mathcal{C}^-, \mathcal{C}^+]$ trivially, $[\{d\}, \{b, c, d\}]$ is $\mathcal{TV}\beta$ -closed.

Failure of $\hat{S}$ -closedness : $\mathcal{TVI}(\mathcal{TV}C([\mathcal{C}^-, \mathcal{C}^+])) \subseteq \mathcal{TV}C(\mathcal{TVI}([\mathcal{C}^-, \mathcal{C}^+]))$.

Since $\{b, c, d\} \not\subseteq \{c, d\}$, the set $[\{d\}, \{c, d\}]$ does not contain B , so the only $\mathcal{IV}$ -closed superset of $[\mathcal{C}^-, \mathcal{C}^+]$; thus $\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]) = \tilde{X}$, and hence

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}.$$

From above, $\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+]) = \tilde{\emptyset}$, so $\mathcal{IVC}(\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}$.

Since $\tilde{X} \not\subseteq \tilde{\emptyset}$, the inclusion fails, so $[\{d\}, \{b, c, d\}]$ is not $\mathcal{IV}\hat{\mathcal{S}}$ -closed.

Theorem 3.12. *Let $(X, \mathcal{T}^X)$ be an interval-valued topological space. Then a subset of $\tilde{X}$ is $\mathcal{IV}\alpha$ -open $\iff$ it is both $\mathcal{IV}\hat{\mathcal{S}}$ -open and $\mathcal{IV}p$ -open.*

Proof. ($\implies$) : Let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IV}\alpha$ -open set.

Then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])))$.

It implies that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ and

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open set. On the other hand, since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\alpha$ -open set, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}p$ -open set.

($\impliedby$) : Let $[\mathcal{M}^-, \mathcal{M}^+]$ be both $\mathcal{IV}\hat{\mathcal{S}}$ -open and $\mathcal{IV}p$ -open. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open, we have $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ and hence $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])))$. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}p$ -open, we have $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))$. Therefore we obtain that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])))$ which proves that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\alpha$ -open set.

Remark 3.13. *There is no implication between $\mathcal{IV}\hat{\mathcal{S}}$ -open sets and $\mathcal{IV}p$ -closed sets; hence, these notions are independent.*

Example 3.14. Referring to Example 3.11,

1. Let $[\mathcal{C}^-, \mathcal{C}^+] = [\{d\}, \{d\}]$.

$\hat{\mathcal{S}}$ -openness : $\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+]))$.

Since $\{d\} \subseteq \{d\}$ and $\{d\} \subseteq \{c, d\}$, we have $[\mathcal{C}^-, \mathcal{C}^+] \subseteq [\{d\}, \{c, d\}]$, and this is the smallest $\mathcal{IV}$ -closed set containing $[\mathcal{C}^-, \mathcal{C}^+]$ (as $\tilde{X}$ is larger). Thus $\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]) = [\{d\}, \{c, d\}]$.

Since $\{a, b\} \not\subseteq \{d\}$, the set $[\{a, b\}, \{a, b, c\}]$ is not contained in $[\{d\}, \{c, d\}]$, so the only $\mathcal{IV}$ -open subset of $[\{d\}, \{c, d\}]$ is $\tilde{\emptyset}$. Hence

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{IVI}([\{d\}, \{c, d\}]) = \tilde{\emptyset}.$$

Also, since $\{a, b\} \not\subseteq \{d\}$, the only $\mathcal{IV}$ -open subset of $[\mathcal{C}^-, \mathcal{C}^+]$ is $\tilde{\emptyset}$, so $\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+]) = \tilde{\emptyset}$, and hence

$$\mathcal{IVC}(\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $\tilde{\emptyset} \subseteq \tilde{\emptyset}$, we obtain $\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+]))$, so $[\{d\}, \{d\}]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Failure of p -openness : $[\mathcal{C}^-, \mathcal{C}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]))$.

From above, $\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]) = [\{d\}, \{c, d\}]$ and $\mathcal{IVI}([\{d\}, \{c, d\}]) = \tilde{\emptyset}$. Since $[\mathcal{C}^-, \mathcal{C}^+] = [\{d\}, \{d\}] \not\subseteq \tilde{\emptyset}$, the inclusion fails, so $[\{d\}, \{d\}]$ is not $\mathcal{IV}p$ -open.

2. Let $[\mathcal{D}^-, \mathcal{D}^+] = [\{a\}, \{a, b, c\}]$.

p -openness : $[\mathcal{D}^-, \mathcal{D}^+] \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{D}^-, \mathcal{D}^+]))$.

Since $\{a, b, c\} \not\subseteq \{c, d\}$, the set $[\{d\}, \{c, d\}]$ does not contain $[\mathcal{D}^-, \mathcal{D}^+]$, so the only $\mathcal{IV}$ -closed superset of $[\mathcal{D}^-, \mathcal{D}^+]$; thus $\mathcal{IVC}([\mathcal{D}^-, \mathcal{D}^+]) = \tilde{X}$, and hence

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{D}^-, \mathcal{D}^+])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}.$$

Since $[\mathcal{D}^-, \mathcal{D}^+] \subseteq \tilde{X}$ trivially, $[\{a\}, \{a, b, c\}]$ is $\mathcal{IV}p$ -open.

Failure of $\hat{\mathcal{S}}$ -openness. We check the condition $\mathcal{IVI}(\mathcal{IVC}([\mathcal{D}^-, \mathcal{D}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{D}^-, \mathcal{D}^+]))$.

From above, $\mathcal{IVI}(\mathcal{IVC}([\mathcal{D}^-, \mathcal{D}^+])) = \tilde{X}$.

Since $\{a, b\} \not\subseteq \{a\}$, the set $[\{a, b\}, \{a, b, c\}]$ is not contained in $[\mathcal{D}^-, \mathcal{D}^+]$, so the only $\mathcal{IV}$ -open subset of $[\mathcal{D}^-, \mathcal{D}^+]$ is $\tilde{\emptyset}$; thus $\mathcal{IVI}([\mathcal{D}^-, \mathcal{D}^+]) = \tilde{\emptyset}$, and hence

$$\mathcal{IVC}(\mathcal{IVI}([\mathcal{D}^-, \mathcal{D}^+])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $\tilde{X} \not\subseteq \tilde{\emptyset}$, the inclusion fails, so $[\{a\}, \{a, b, c\}]$ is *not* $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Proposition 3.15. Let $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{K}^-, \mathcal{K}^+]$ be subsets of an interval-valued topological space $(X, \mathcal{T}^X)$.

If $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ and $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open, then $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Proof. Suppose that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ and $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open in $\tilde{X}$. Then, we have $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Since $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{K}^-, \mathcal{K}^+]))$ and $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{K}^-, \mathcal{K}^+]))$.

Since $[\mathcal{K}^-, \mathcal{K}^+] \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$, we have

$$\mathcal{IVC}([\mathcal{K}^-, \mathcal{K}^+]) \subseteq \mathcal{IVC}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$$

and

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{K}^-, \mathcal{K}^+])) \subseteq \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])).$$

Therefore we obtain that

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{K}^-, \mathcal{K}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{K}^-, \mathcal{K}^+])).$$

This shows that $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open set.

Corollary 3.16. *Let $(X, \mathcal{T}^X)$ be an interval-valued topological space. If $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \tilde{X}$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open and $\mathcal{IV}$ -dense in $\mathcal{T}^X$, then every subset of $\tilde{X}$ containing $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.*

Proof. Let $[\mathcal{N}^-, \mathcal{N}^+] \subseteq \tilde{X}$ be such that

$$[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{N}^-, \mathcal{N}^+].$$

Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -dense in $\mathcal{T}^X$, we have $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{X}$.

Thus, $\tilde{X} \subseteq \mathcal{IVC}([\mathcal{N}^-, \mathcal{N}^+])$, which implies $\mathcal{IVC}([\mathcal{N}^-, \mathcal{N}^+]) = \tilde{X}$.

Hence, $[\mathcal{N}^-, \mathcal{N}^+]$ is also $\mathcal{IV}$ -dense in $\mathcal{T}^X$.

Now, since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open and is contained in $[\mathcal{N}^-, \mathcal{N}^+]$, and $\mathcal{IV}\hat{\mathcal{S}}$ -openness is preserved under supersets of $\mathcal{IV}$ -dense $\mathcal{IV}\hat{\mathcal{S}}$ -open sets, it follows that $[\mathcal{N}^-, \mathcal{N}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Therefore, every superset of $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\hat{\mathcal{S}}$ -open.

Proposition 3.17. *In an interval-valued topological space $(X, \mathcal{T}^X)$, every $\mathcal{IV}$ -closed set is $\mathcal{IV}t$ -set.*

Proof. Let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IV}$ -closed set. Then $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ and we have $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ which proves that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}t$ -set.

Example 3.18. Referring to Example 3.5, recall $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\{a\}, \{a, b\}]\}$ and $(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{c\}, \{b, c\}]\}$. Let $[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{b\}]$.

Since $\{a\} \not\subseteq \emptyset$, the set $[\{a\}, \{a, b\}]$ is not contained in $[\emptyset, \{b\}]$, so the only $\mathcal{IV}$ -open subset of $[\emptyset, \{b\}]$ is $\tilde{\emptyset}$. Hence

$$\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{\emptyset}.$$

Also, since $\emptyset \subseteq \{c\}$ and $\{b\} \subseteq \{b, c\}$, we have $[\emptyset, \{b\}] \subseteq [\{c\}, \{b, c\}]$, and this is the smallest $\mathcal{IV}$ -closed set containing $[\emptyset, \{b\}]$ (as $\tilde{X}$ is a larger closed set). Thus

$$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = [\{c\}, \{b, c\}].$$

Since $\{a\} \not\subseteq \{c\}$, the set $[\{a\}, \{a, b\}]$ is not contained in $[\{c\}, \{b, c\}]$, so the only $\mathcal{IV}$ -open subset of $[\{c\}, \{b, c\}]$ is $\tilde{\emptyset}$. Therefore

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\{c\}, \{b, c\}]) = \tilde{\emptyset}.$$

Since $[\emptyset, \{b\}] \notin (\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{c\}, \{b, c\}]\}$, the set $[\mathcal{M}^-, \mathcal{M}^+]$ is not $\mathcal{IV}$ -closed.

However, we have shown

$$\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \tilde{\emptyset} = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]),$$

which is exactly the $\mathcal{IVt}$ -set condition.

Hence, $[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{b\}]$ is an $\mathcal{IVt}$ -set but not $\mathcal{IV}$ -closed set.

Theorem 3.19. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is $\mathcal{IVs}$ -closed $\iff [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVt}$ -set.

Proof. Let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IVs}$ -closed set in $\tilde{X}$. Then $\tilde{X} - [\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -open. By Proposition 3.6, we have $\mathcal{IVC}(\tilde{X} - [\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}(\tilde{X} - [\mathcal{M}^-, \mathcal{M}^+]))$.

It follows that $\tilde{X} - \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\tilde{X} - \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \tilde{X} - \mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))$. Thus, $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ and hence $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVt}$ -set in $\tilde{X}$.

Conversely, let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IVt}$ -set.

Then $\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -closed.

4. On interval valued semi-regular and interval-valued $\mathcal{Q}$ -closed sets

Definition 4.1. A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of an interval-valued topological space $(X, \mathcal{T}^X)$ is called an

1. interval-valued $\mathcal{Q}$ -closed (briefly, $\mathcal{IVQ}$ -closed) if

$$[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])).$$

The complement of $\mathcal{IVQ}$ -open set is called $\mathcal{IVQ}$ -closed set.

2. interval-valued semi-regular (briefly, $\mathcal{IVs}$ -regular) if $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -open and a $\mathcal{IVt}$ -set.

Example 4.2. $X = \{a, b\}$, $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\{a\}, X], [\emptyset, \{b\}]\}$

1. Let $[\mathcal{M}^-, \mathcal{M}^+] = [\{a\}, X]$.

Then $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) = \tilde{X}$ and $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) = \mathcal{IVC}(\tilde{X}) = \tilde{X}$. Hence, $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$, so $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVQ}$ -closed set.

2. Let $[\mathcal{S}^-, \mathcal{S}^+] = [\emptyset, \{b\}]$. Then $\mathcal{IVI}([\mathcal{S}^-, \mathcal{S}^+]) = [\mathcal{S}^-, \mathcal{S}^+]$. Also,

$$\mathcal{IVC}([\mathcal{S}^-, \mathcal{S}^+]) = \tilde{X} \Rightarrow \mathcal{IVI}(\mathcal{IVC}([\mathcal{S}^-, \mathcal{S}^+])) = \tilde{X}.$$

Thus, $\mathcal{IVI}(\mathcal{IVC}([\mathcal{S}^-, \mathcal{S}^+])) = \mathcal{IVI}([\mathcal{S}^-, \mathcal{S}^+])$, and $[\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IVt}$ -set. Further, since $[\mathcal{S}^-, \mathcal{S}^+] \in \mathcal{T}^X$, it is $\mathcal{IVs}$ -open.

Hence, $[\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IVs}$ -regular set.

Theorem 4.3. Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a subset of an interval-valued topological space $\mathcal{T}^X$. Then $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -regular $\iff [\mathcal{M}^-, \mathcal{M}^+]$ is both $\mathcal{IV}\beta$ -open and $\mathcal{IVs}$ -closed.

Proof. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -regular, then $[\mathcal{M}^-, \mathcal{M}^+]$ is both $\mathcal{IVs}$ -open and $\mathcal{IVt}$ -set. Since every $\mathcal{IVs}$ -open set is $\mathcal{IV}\beta$ -open, $[\mathcal{M}^-, \mathcal{M}^+]$ is both $\mathcal{IV}\beta$ -open and $\mathcal{IVt}$ -set. By Theorem 3.19, we obtain the result.

Conversely, let $[\mathcal{M}^-, \mathcal{M}^+]$ be $\mathcal{IVs}$ -closed and $\mathcal{IV}\beta$ -open. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -closed, by Theorem 3.19 $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVt}$ -set. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open, $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Therefore $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -open. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is both $\mathcal{IVs}$ -open and $\mathcal{IVt}$ -set, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVs}$ -regular.

Remark 4.4. There is no implication between $\mathcal{IV}\beta$ -open sets and $\mathcal{IVs}$ -closed sets; hence, these notions are independent.

Example 4.5. Let $X = \{a, b, c\}$ and $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\emptyset, \{a\}], [\{a\}, \{a, b\}], [\emptyset, \{a, b\}]\}$.

Computing complements,

$$[\emptyset, \{a\}]^c = [\{b, c\}, X], \quad [\{a\}, \{a, b\}]^c = [\{c\}, \{b, c\}], \quad [\emptyset, \{a, b\}]^c = [\{c\}, X],$$

so

$$(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{b, c\}, X], [\{c\}, \{b, c\}], [\{c\}, X]\}.$$

1. Let $[\mathcal{A}^-, \mathcal{A}^+] = [\{a, b\}, \{a, b\}]$.

β -openness : $[\mathcal{A}^-, \mathcal{A}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+])))$.

Since $\{a, b\} \not\subseteq \{c\}$ and $\{a, b\} \not\subseteq \{b, c\}$, none of $[\{b, c\}, X]$, $[\{c\}, \{b, c\}]$, $[\{c\}, X]$ contains $[\mathcal{A}^-, \mathcal{A}^+]$, so the only $\mathcal{IV}$ -closed superset of $[\mathcal{A}^-, \mathcal{A}^+]$ is $\tilde{X}$; thus $\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+]) = \tilde{X}$.

Then $\mathcal{IVI}(\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$, and $\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+]))) = \mathcal{IVC}(\tilde{X}) = \tilde{X}$. Since $[\mathcal{A}^-, \mathcal{A}^+] \subseteq \tilde{X}$ trivially, $[\{a, b\}, \{a, b\}]$ is $\mathcal{IV}\beta$ -open.

Failure of s -closedness : $\mathcal{IVI}(\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+])) \subseteq [\mathcal{A}^-, \mathcal{A}^+]$.

From above, $\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+]) = \tilde{X}$, so $\mathcal{IVI}(\mathcal{IVC}([\mathcal{A}^-, \mathcal{A}^+])) = \mathcal{IVI}(\tilde{X}) = \tilde{X}$. Since $\tilde{X} \not\subseteq [\mathcal{A}^-, \mathcal{A}^+]$ (as $[\mathcal{A}^-, \mathcal{A}^+] \neq \tilde{X}$), the inclusion fails, so $[\{a, b\}, \{a, b\}]$ is not $\mathcal{IVs}$ -closed.

2. Let $[\mathcal{B}^-, \mathcal{B}^+] = [\{c\}, \{b, c\}]$.

s-closedness : $\mathcal{IVI}(\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+])) \subseteq [\mathcal{B}^-, \mathcal{B}^+]$.

Since $[\mathcal{B}^-, \mathcal{B}^+] = [\{c\}, \{b, c\}]$ is itself $\mathcal{IV}$ -closed (it belongs to $(\mathcal{T}^X)^c$),

$$\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) = [\mathcal{B}^-, \mathcal{B}^+].$$

To find $\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+])$: none of $[\emptyset, \{a\}]$, $[\{a\}, \{a, b\}]$, $[\emptyset, \{a, b\}]$ is contained in $[\mathcal{B}^-, \mathcal{B}^+]$, since each requires $\{a\} \subseteq \{b, c\}$ or $\{a\} \subseteq \{c\}$, both false. So the only $\mathcal{IV}$ -open subset of $[\mathcal{B}^-, \mathcal{B}^+]$ is $\tilde{\emptyset}$, giving $\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+]) = \tilde{\emptyset}$.

Hence

$$\mathcal{IVI}(\mathcal{IVC}(B)) = \mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+]) = \tilde{\emptyset} \subseteq B,$$

so $[\{c\}, \{b, c\}]$ is $\mathcal{IV}$ s-closed.

Failure of β -openness : $[\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+])))$.

From above, $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) = [\mathcal{B}^-, \mathcal{B}^+]$ and $\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+]) = \tilde{\emptyset}$, so

$$\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $[\mathcal{B}^-, \mathcal{B}^+] \not\subseteq \tilde{\emptyset}$, the inclusion fails, so $[\{c\}, \{b, c\}]$ is *not* $\mathcal{IV}\beta$ -open.

Theorem 4.6. Let $(X, \mathcal{T}^X)$ be an interval-valued topological space and $[\mathcal{M}^-, \mathcal{M}^+]$ a subset of X . Then the following properties are equivalent.

1. $[\mathcal{M}^-, \mathcal{M}^+] \neq \tilde{\emptyset}$ is $\mathcal{IVQ}$ -closed.
2. There exists a non-empty $\mathcal{IV}$ -open set $[\mathcal{H}^-, \mathcal{H}^+]$ such that $[\mathcal{H}^-, \mathcal{H}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+])$.
3. There exists a non-empty $\mathcal{IV}$ -open set $[\mathcal{H}^-, \mathcal{H}^+]$ such that $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{H}^-, \mathcal{H}^+] \cup (\mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) - [\mathcal{H}^-, \mathcal{H}^+])$.

Proof. (1) $\implies$ (2): Suppose $[\mathcal{M}^-, \mathcal{M}^+] \neq \tilde{\emptyset}$, is $\mathcal{IVQ}$ -closed set. Then $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Let $[\mathcal{H}^-, \mathcal{H}^+] = \mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$. $[\mathcal{H}^-, \mathcal{H}^+]$ is the required $\mathcal{IV}$ -open set such that $[\mathcal{H}^-, \mathcal{H}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+])$.

(2) $\implies$ (3): Since $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) = [\mathcal{H}^-, \mathcal{H}^+] \cup (\mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) - [\mathcal{H}^-, \mathcal{H}^+])$ where $[\mathcal{H}^-, \mathcal{H}^+]$ is a nonempty $\mathcal{IV}$ -open set, (3) follows.

(3) $\implies$ (1): $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{H}^-, \mathcal{H}^+] \cup (\mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) - [\mathcal{H}^-, \mathcal{H}^+])$ implies that $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{H}^-, \mathcal{H}^+])) \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$, since $[\mathcal{H}^-, \mathcal{H}^+]$ is $\mathcal{IV}$ -open and $[\mathcal{H}^-, \mathcal{H}^+] \subseteq [\mathcal{M}^-, \mathcal{M}^+]$. Again $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{M}^-, \mathcal{M}^+]$ implies that $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}([\mathcal{H}^-, \mathcal{H}^+]) = [\mathcal{M}^-, \mathcal{M}^+]$. Therefore $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ which implies

that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVQ}$ -closed.

Theorem 4.7. *Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a subset of an interval-valued topological space $(X, \mathcal{T}^X)$. If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open, then $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ is $\mathcal{IVQ}$ -closed.*

Proof. Suppose $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open. Then $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$ and so $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]))) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ which implies that $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$.

Therefore $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ is $\mathcal{IVQ}$ -closed.

Theorem 4.8. *Let $[\mathcal{M}^-, \mathcal{M}^+]$ be a subset of an interval-valued topological space $(X, \mathcal{T}^X)$. Then the following properties are equivalent.*

1. $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVQ}$ -closed.
2. $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open and $\mathcal{IV}$ -closed.
3. $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open and $\mathcal{IV}$ -closed.

Proof. (1) $\implies$ (2): If $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVQ}$ -closed, then

$[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ and

$\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Since

$[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}s$ -open. Also,

$[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$ and so $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}$ -closed.

(2) $\implies$ (3): It follows from the fact that every $\mathcal{IV}s$ -open set is $\mathcal{IV}\beta$ -open.

(3) $\implies$ (1): Suppose $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\beta$ -open and $\mathcal{IV}$ -closed.

Then $[\mathcal{M}^-, \mathcal{M}^+] \subset \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])))$ and

$[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$. Now $\mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])) \subseteq \mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+]) = [\mathcal{M}^-, \mathcal{M}^+]$. Also, $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$. Therefore $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IVC}(\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+]))$ which implies that $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IVQ}$ -closed.

Remark 4.9.

1. The class of $\mathcal{IV}s$ -open sets is independent of the class of $\mathcal{IV}$ -closed sets; that is, neither of these notions implies the other.
2. Likewise, the class of $\mathcal{IV}\beta$ -open sets and the class of $\mathcal{IV}$ -closed sets are independent, since one does not necessarily lead to the other.

Example 4.10. In Example 4.5, recall $\mathcal{T}^X = \{\tilde{\emptyset}, \tilde{X}, [\emptyset, \{a\}], [\{a\}, \{a, b\}], [\emptyset, \{a, b\}]\}$ and $(\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{b, c\}, X], [\{c\}, \{b, c\}], [\{c\}, X]\}$.

1. Let $[\mathcal{C}^-, \mathcal{C}^+] = [\emptyset, \{a\}]$.

Since $[\mathcal{C}^-, \mathcal{C}^+]$ is itself $\mathcal{IV}$ -open, $\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+]) = [\mathcal{C}^-, \mathcal{C}^+]$.

To find $\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+])$: among the $\mathcal{IV}$ -closed sets, $[\{b, c\}, X]$ and $[\{c\}, X]$ both contain $[\mathcal{C}^-, \mathcal{C}^+]$ (since $\emptyset \subseteq \{b, c\}$, $\{a\} \subseteq X$, and $\emptyset \subseteq \{c\}$, $\{a\} \subseteq X$), while $[\{c\}, \{b, c\}]$ does not (as $\{a\} \not\subseteq \{b, c\}$). Since $[\{c\}, X] \subseteq [\{b, c\}, X]$ (as $\{c\} \subseteq \{b, c\}$ and $X \subseteq X$), the smallest closed superset of $[\mathcal{C}^-, \mathcal{C}^+]$ is

$$\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]) = [\{c\}, X].$$

s -openness :

$[\mathcal{C}^-, \mathcal{C}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{C}^-, \mathcal{C}^+])) = \mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]) = [\{c\}, X]$. Since $\emptyset \subseteq \{c\}$ and $\{a\} \subseteq X$, this holds, so $[\emptyset, \{a\}]$ is $\mathcal{IV}s$ -open.

β -openness :

$$[\mathcal{C}^-, \mathcal{C}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+]))).$$

To find $\mathcal{IVI}([\{c\}, X])$: among the open sets, $[\emptyset, \{a\}]$ and $[\emptyset, \{a, b\}]$ are both contained in $[\{c\}, X]$ (since $\emptyset \subseteq \{c\}$ and $\{a\} \subseteq X$, $\{a, b\} \subseteq X$), while $[\{a\}, \{a, b\}]$ is not (as $\{a\} \not\subseteq \{c\}$). Their union gives

$$\mathcal{IVI}([\{c\}, X]) = [\emptyset, \{a\}] \cup [\emptyset, \{a, b\}] = [\emptyset, \{a, b\}].$$

Next, $\mathcal{IVC}([\emptyset, \{a, b\}])$: $[\{c\}, \{b, c\}]$ fails since $\{a, b\} \not\subseteq \{b, c\}$; both $[\{b, c\}, X]$ and $[\{c\}, X]$ contain it, and $[\{c\}, X] \subseteq [\{b, c\}, X]$, so

$$\mathcal{IVC}([\emptyset, \{a, b\}]) = [\{c\}, X].$$

Hence $\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{C}^-, \mathcal{C}^+)))) = [\{c\}, X]$, and $[\mathcal{C}^-, \mathcal{C}^+] = [\emptyset, \{a\}] \subseteq [\{c\}, X]$ holds. Thus $[\emptyset, \{a\}]$ is $\mathcal{IV}\beta$ -open.

Not $\mathcal{IV}$ -closed :

Since $[\emptyset, \{a\}] \notin (\mathcal{T}^X)^c = \{\tilde{\emptyset}, \tilde{X}, [\{b, c\}, X], [\{c\}, \{b, c\}], [\{c\}, X]\}$, the set $[\emptyset, \{a\}]$ is not $\mathcal{IV}$ -closed.

2. Let $[\mathcal{B}^-, \mathcal{B}^+] = [\{c\}, \{b, c\}]$.

Since $[\mathcal{B}^-, \mathcal{B}^+]$ is itself $\mathcal{IV}$ -closed, $[\mathcal{B}^-, \mathcal{B}^+] \in (\mathcal{T}^X)^c$, so $[\{c\}, \{b, c\}]$ is $\mathcal{IV}$ -closed, and $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) = [\mathcal{B}^-, \mathcal{B}^+]$.

To find $\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+])$: checking each nontrivial open set, $[\emptyset, \{a\}]$ requires $\{a\} \subseteq \{b, c\}$ (false), $[\{a\}, \{a, b\}]$ requires $\{a\} \subseteq \{c\}$ (false), and $[\emptyset, \{a, b\}]$

requires $\{a, b\} \subseteq \{b, c\}$ (false, since $a \notin \{b, c\}$). Hence the only $\mathcal{IV}$ -open subset of $[\mathcal{B}^-, \mathcal{B}^+]$ is $\tilde{\emptyset}$, so

$$\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+]) = \tilde{\emptyset}.$$

Failure of s -openness : $[\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+])) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}$. Since $[\mathcal{B}^-, \mathcal{B}^+] \neq \tilde{\emptyset}$, this fails, so $[\{c\}, \{b, c\}]$ is *not* $\mathcal{IV}s$ -open.

Failure of β -openness : $[\mathcal{B}^-, \mathcal{B}^+] \subseteq \mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+))))$. Since $\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+]) = B$ and $\mathcal{IVI}([\mathcal{B}^-, \mathcal{B}^+]) = \tilde{\emptyset}$,

$$\mathcal{IVC}(\mathcal{IVI}(\mathcal{IVC}([\mathcal{B}^-, \mathcal{B}^+)))) = \mathcal{IVC}(\tilde{\emptyset}) = \tilde{\emptyset}.$$

Since $[\mathcal{B}^-, \mathcal{B}^+] \not\subseteq \tilde{\emptyset}$, this fails, so $[\{c\}, \{b, c\}]$ is *not* $\mathcal{IV}\beta$ -open.

5. Conclusion

In this paper, new classes of interval-valued open sets, namely $\mathcal{IV}\hat{S}$ -open sets, $\mathcal{IV}t$ -sets, $\mathcal{IV}s$ -regular sets, and $\mathcal{IVQ}$ -closed sets, have been introduced and studied. Their relationships with existing generalized open sets such as $\mathcal{IV}\alpha$ -open, $\mathcal{IV}\beta$ -open, $\mathcal{IV}p$ -open, $\mathcal{IV}s$ -open, and $\mathcal{IV}r$ -open sets have been established, along with several characterizations. Examples have been given to support the results and to show the independence of certain classes of sets. These results extend and generalize known results in the literature to interval-valued topological spaces and may pave the way for further study of related structures such as separation axioms and continuity in this setting.

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