South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.12, No.1 2016, pp. 87-94

PRODUCT FORMULAS FOR MOCK THETA FUNCTIONS

K.B. Chand, G.S. Pant and V.P. Pande*
Department of Mathematics,
S.S.J. College, Almora (U.K.) India
*Prof & Head, Department of Mathematics,
S.S.J. College, Almora (U.K.) India
E-mail:- krishnabdrchand@gmail.com, gspant2070@rediffmail.com

Abstract: In this paper, we have established an identity by making use of the Bailey's transform and used it for obtaining the product formulas of mock theta functions.

Keywords and Phrases: Mock theta function, identity, Bailey transform, product formula.

2000 AMS Subject Classification: Primary 05A30, 05A19, Secondary 82B23, 17B67, 33D90.

1. Introduction Notations and Definitions

In his last letter to G.H. Hardy, Ramanujan mentioned 20 mock theta functions out of which seven are of order three, ten are of order five and three are of order seven. The main aim of this paper is to establish product formulas for these mock theta functions. Taking two at a time one can establish 190 product formulas. We shall here give few examples only.

As usual, for a and q complex numbers and $|q| < 1$, we define q-rising factorial as,

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad n \geq 1, \tag{1.1}$$

$$(a;q)_0 = 1 \tag{1.2}$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r). \tag{1.3}$$

Following are the lists of mock theta functions and corresponding partial mock theta functions of different orders.

(a)

Mock theta functions of order three | **Partial mock theta functions of order three**

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}, \qquad f_m(q)=\sum_{n=0}^{m}\frac{q^{n^2}}{(-q;q)_n^2}. \tag{1.4}$$

$$\Phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}, \qquad \Phi_m(q)=\sum_{n=0}^{m}\frac{q^{n^2}}{(-q^2;q^2)_n}. \tag{1.5}$$

$$\Psi(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{(q;q^2)_{n+1}}, \qquad \Psi_m(q)=\sum_{n=0}^{m}\frac{q^{(n+1)^2}}{(q;q^2)_{n+1}}. \tag{1.6}$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-\omega q,-\omega^2 q;q)_n}, \qquad \chi_m(q)=\sum_{n=0}^{m}\frac{q^{n^2}}{(-\omega q,-\omega^2 q;q)_n}. \tag{1.7}$$

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \qquad \omega_m(q)=\sum_{n=0}^{m}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}. \tag{1.8}$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(\omega q,\omega^2 q;q^2)_{n+1}}, \qquad \rho_m(q)=\sum_{n=0}^{m}\frac{q^{2n(n+1)}}{(\omega q,\omega^2 q;q^2)_{n+1}}. \tag{1.9}$$

$$\nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \qquad \nu_m(q)=\sum_{n=0}^{m}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}. \tag{1.10}$$

where $\omega=e^{2\pi i/3}$ (cube root of unity)

(b)

Mock theta functions of order five | **Partial mock theta functions of order five**

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}, \qquad f_{0,m}(q)=\sum_{n=0}^{m}\frac{q^{n^2}}{(-q;q)_n}. \tag{1.11}$$

$$\Psi_0(q)=\sum_{n=0}^{\infty}(-q;q)_n q^{(n+1)(n+2)/2}, \qquad \Psi_{0,m}(q)=\sum_{n=0}^{m}(-q;q)_n q^{(n+1)(n+2)/2}. \tag{1.12}$$

$$\Phi_0(q)=\sum_{n=0}^{\infty}(-q;q^2)_n q^{n^2}, \qquad \Phi_{0,m}(q)=\sum_{n=0}^{m}(-q;q^2)_n q^{n^2}. \tag{1.13}$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}, \qquad \chi_{0,m}(q) = \sum_{n=0}^{m} \frac{q^n}{(q^{n+1};q)_n}. \tag{1.14}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \qquad F_{0,m}(q) = \sum_{n=0}^{m} \frac{q^{2n^2}}{(q;q^2)_n}. \tag{1.15}$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \qquad f_{1,m}(q) = \sum_{n=0}^{m} \frac{q^{n(n+1)}}{(-q;q)_n}. \tag{1.16}$$

$$\Phi_1(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{(n+1)^2}, \qquad \Phi_{1,m}(q) = \sum_{n=0}^{m} (-q;q^2)_n q^{(n+1)^2}. \tag{1.17}$$

$$\Psi_1(q) = \sum_{n=0}^{\infty} (-q;q)_n q^{n(n+1)/2}, \qquad \Psi_{1,m}(q) = \sum_{n=0}^{m} (-q;q)_n q^{n(n+1)/2}. \tag{1.18}$$

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}, \qquad \chi_{1,m}(q) = \sum_{n=0}^{m} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{1.19}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \qquad F_{1,m}(q) = \sum_{n=0}^{m} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}. \tag{1.20}$$

(c)

Mock theta functions of order seven **Partial mock theta functions of order seven**

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{(q;q)_n q^{n^2}}{(q;q)_{2n}}, \qquad \mathcal{F}_{0,m}(q) = \sum_{n=0}^{m} \frac{(q;q)_n q^{n^2}}{(q;q)_{2n}} \tag{1.21}$$

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{(q;q)_n q^{(n+1)^2}}{(q;q)_{2n+1}}, \qquad \mathcal{F}_{1,m}(q) = \sum_{n=0}^{m} \frac{(q;q)_n q^{(n+1)^2}}{(q;q)_{2n+1}} \tag{1.22}$$

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{(q;q)_n q^{n(n+1)}}{(q;q)_{2n+1}}, \qquad \mathcal{F}_{2,m}(q) = \sum_{n=0}^{m} \frac{(q;q)_n q^{n(n+1)}}{(q;q)_{2n+1}} \tag{1.23}$$

2. An Identity

In this section we establish an identity which will be used in next section. The Bailey transform is stated as,

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{2.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.3}$$

[Slater 2; (2.4.1)-(2.4.3), p. 59]

where α_r, u_r, v_r and δ_r are rational functions of r alone.
For $u_r = v_r = 1$, (2.1), (2.2) and (2.3) takes the following form,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r \tag{2.4}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r = \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \tag{2.5}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r, \tag{2.6}$$

which on simplification gives

$$\sum_{r=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \alpha_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r, \tag{2.7}$$

provided the infinite series involved are convergent. We shall use (2.7) in order to establish the product formulas.

3. Product Formulae for Mock Theta Functions

(a) Choosing $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$, $\delta_n = \dfrac{q^{n^2}}{(-q^2;q^2)_n}$ in (2.7) making use of (1.4) and (1.5) we have,

$$f(q)\Phi(q) + \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_n^2 (-q^2;q^2)_n}$$

$$= \sum_{n=0}^{\infty} \Phi_n(q) \frac{q^{n^2}}{(-q;q)_n^2} + \sum_{n=0}^{\infty} f_n(q) \frac{q^{n^2}}{(-q^2;q^2)_n}. \tag{3.1}$$

(3.1) is the product formula involving two mock theta functions of order three.

(b) Taking $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$ and $\delta_n = \dfrac{q^{n^2}}{(-q;q)_n}$ in (2.7) and using (1.4) (1.11) we obtain following result,

$$f(q)f_0(q) + \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_n^3}$$

$$= \sum_{n=0}^{\infty} f_{0,n}(q) \frac{q^{n^2}}{(-q;q)_n^2} + \sum_{n=0}^{\infty} f_n(q) \frac{q^{n^2}}{(-q;q)_n}. \tag{3.2}$$

(3.2) is a product formula involving a mock theta function of order three and a mock theta function five of first group.

(c) Putting $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$ and $\delta_n = (-q;q)_n q^{(n+1)(n+2)/2}$ in (2.7) and using (1.4) and (1.12) we obtain,

$$f(q)\Psi_0(q) + \sum_{n=0}^{\infty} \frac{q^{\frac{3}{2}n^2+\frac{3}{2}n+1}}{(-q;q)_n}$$

$$= \sum_{n=0}^{\infty} \Psi_{0,n}(q) \frac{q^{n^2}}{(-q;q)_n^2} + \sum_{n=0}^{\infty} f_n(q)(-q;q)_n q^{(n+1)(n+2)/2}, \tag{3.3}$$

which is similar to (3.2).

(d) Taking $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$ and $\delta_n = (-q;q)_n q^{n(n+1)/2}$ in (2.7) and using (1.4) and (1.18) we have,

$$f(q)\Psi_1(q) + \sum_{n=0}^{\infty} \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(-q;q)_n}$$

$$= \sum_{n=0}^{\infty} \Psi_{1,n}(q) \frac{q^{n^2}}{(-q;q)_n^2} + \sum_{n=0}^{\infty} f_n(q)(-q;q)_n q^{n(n+1)/2}. \tag{3.4}$$

(3.4) gives the product formula involving a mock theta function of order three and a mock theta function of order five of second group.

(e) Taking $\alpha_n = (-q;q)_n q^{(n^2+3n+2)/2}$, $\delta_n = (-q;q)_n q^{(n^2+n)/2}$ in (2.7) and making

use of (1.12) and (1.18) we find,

$$\Psi_0(q)\Psi_1(q) + \sum_{n=0}^{\infty}(-q;q)_n^2 q^{(n^2+2n+1)}$$

$$= \sum_{n=0}^{\infty}\Psi_{1,n}(q)(-q;q)_n q^{\frac{1}{2}(n^2+3n+2)} + \sum_{n=0}^{\infty}\Psi_{0,n}(q)(-q;q)_n q^{\frac{1}{2}(n^2+n)}. \tag{3.5}$$

(3.5) is a product formula involving two mock theta functions of order five of different groups.

(f) For $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n}$, $\delta_n = (-q;q)_n q^{(n^2+3n+2)/2}$, (2.7) on an appeal of (1.11) and (1.12) yields,

$$f_0(q)\Psi_0(q) + q\sum_{n=0}^{\infty} q^{\frac{3}{2}n(n+1)}$$

$$= \sum_{n=0}^{\infty}\Psi_{0,n}(q)\frac{q^{n^2}}{(-q;q)_n} + \sum_{n=0}^{\infty} f_{0,n}(q)(-q;q)_n q^{(n+1)(n+2)/2}. \tag{3.6}$$

This is a product formula involving mock theta functions of order five of first group.

(g) Taking $\alpha_n = (-q;q)_n q^{n(n+1)/2}$, $\delta_n = \dfrac{q^{n(n+1)}}{(-q;q)_n}$ in (2.7) and using (1.16) and (1.18) we obtain,

$$f_1(q)\Psi_1(q) + \sum_{n=0}^{\infty} q^{\frac{3}{2}n(n+1)}$$

$$= \sum_{n=0}^{\infty}\Psi_{1,n}(q)\frac{q^{n(n+1)}}{(-q;q)_n} + \sum_{n=0}^{\infty} f_{1,n}(q)(-q;q)_n q^{n(n+1)/2}. \tag{3.7}$$

It is the product formula of mock theta functions of order five of second group.

(h) Choosing $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$ and $\delta_n = \dfrac{(q;q)_n q^{n^2}}{(q;q)_{2n}}$ in (2.7) and appealing (1.4) and (1.21) we get,

$$f(q)\mathcal{F}_1(q) + \sum_{n=0}^{\infty}\frac{(q;q)_n q^{2n^2}}{(-q;q)_n^2(q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty}\mathcal{F}_{1,n}(q)\frac{q^{n^2}}{(-q;q)_n^2} + \sum_{n=0}^{\infty} f_n(q)\frac{(q;q)_n q^{n^2}}{(q;q)_{2n}}, \tag{3.8}$$

which provides the product formula of a mock theta function of order three and a mock theta function of order seven.

(i) Taking $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n}$ and $\delta_n = \dfrac{(q;q)_n q^{n^2}}{(q;q)_{2n}}$ in (2.7) and using (1.11) and (1.21) we obtain,

$$f_0(q)\mathcal{F}_1(q) + \sum_{n=0}^{\infty} \frac{(q;q)_n q^{2n^2}}{(-q;q)_n (q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty} \mathcal{F}_{1,n}(q) \frac{q^{n^2}}{(-q;q)_n} + \sum_{n=0}^{\infty} f_{0,n}(q) \frac{(q;q)_n q^{n^2}}{(q;q)_{2n}}. \tag{3.9}$$

This is a product formula involving mock theta function of order five of first group and a mock theta function of order seven.

(j) Taking $\alpha_n = \dfrac{q^{n(n+1)}}{(-q;q)_n}$, $\delta_n = \dfrac{(q;q)_n q^{n^2}}{(q;q)_{2n}}$ in (2.7) and making use of (1.16) and (1.21) we have,

$$f_1(q)\mathcal{F}_1(q) + \sum_{n=0}^{\infty} \frac{(q;q)_n q^{n(2n+1)}}{(-q;q)_n (q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty} \mathcal{F}_{1,n}(q) \frac{q^{n(n+1)}}{(-q;q)_n} + \sum_{n=0}^{\infty} f_{1,n}(q) \frac{(q;q)_n q^{n^2}}{(q;q)_{2n}}. \tag{3.10}$$

(3.10) provides a product formula of mock theta functions of order five of second group and order seven.

(k) Lastly taking $\alpha_n = \dfrac{(q;q)_n q^{n^2}}{(q;q)_{2n}}$, $\delta_n = \dfrac{(q;q)_n q^{(n+1)^2}}{(q;q)_{2n+1}}$ in (2.7) and applying (1.21) and (1.22) we get,

$$\mathcal{F}_1(q)\mathcal{F}_2(q) + q \sum_{n=0}^{\infty} \frac{(q;q)_n^2 q^{2n(n+1)}}{(q;q)_{2n} (q;q)_{2n+1}}$$

$$= \sum_{n=0}^{\infty} \mathcal{F}_{2,n}(q) \frac{(q;q)_n q^{n^2}}{(q;q)_{2n}} + \sum_{n=0}^{\infty} \mathcal{F}_{1,n}(q) \frac{(q;q)_n q^{(n+1)^2}}{(q;q)_{2n+1}}. \tag{3.11}$$

(3.11) is a product formula of two mock theta functions of order seven.

For detail about mock theta functions one is referred Agarwal [1], Zagier, D. [3].

References

[1] Agarwal, R.P., Resonance of Ramanujan's Mathematics, Vol. II, New Age International (P) Limited, New Delhi (1996).

[2] Slater, L.J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge (1966).

[3] Zogier, D., Ramanujan's mock theta functions and their applications, Séminaire BOURBAKI, 60 éin anneé, 2006-2007, n^0 986-01-20.