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MULTIMODEL STRESS-STRENGTH UNDER PATHWAY MODELS

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Dedicated to Prof. A.M. Mathai on his 80^{th} ***birth anniversary***

Abstract: The reliability of a component or a system under a stress and strength situation is examined when both the stress and strength have distributions with several modes or when both the distributions are convex combinations of other densities. Pathway models are used for the individual components in stress and strength variables. Pathway model is a versatile model which can switch into three different functional forms through a pathway parameter q. When $q < 1$ the model is in a generalized type-1 beta family of functions. When $q \to 1$ it switches into a generalized gamma family of functions. When $q > 1$ the model is in a generalized type-2 beta family of functions. Under such a versatile model for each component in stress and strength, with different parameters, the reliability of a system is examined. Then special cases of the pathway models, in the independently distributed situations, are studied so that the reliability can be evaluated in explicit forms. Connection to fractional integral is also given.

Keywords: Reliability analysis, stress-strength models, pathway model.

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1. Introduction

In a physical system or in a component in the system let x represent stress and y represent strength then the reliability of the system, or component under consideration, is measured by the probability that $y > x$, that is, $Pr\{y > x\}$. This probability is studied by many authors under different models for x and y. For instance, see Awad et al. (1981) for bivariate exponential family, for exponential and other families see, for example, Church and Harris (1970), Downtown (1973), Govidarajulu (1967), Woodward and Kelley (1977) and Owen, Cresswell and Hanson (1977) for normal family, Gupta and Gupta (1990) for multivariate normal family, Kelley et al. (1976), Sathe and Shah (1981), Tong (1975, 1977) for exponential family, Constantine and Kerson (1986) for independent gamma random variables, Ahmad et al. (1997) and Surles and Padgett (1998, 2001) for Burr type X random variables, recently, Kundu and Gupta (2005) and Raqab and Kundu (2005) for generalized exponential distributions and Burr type X distributions respectively. A detailed treatment of the different stress strength models can be found in the monograph of Kotz, Lumelskii and Pensky (2003).

In a practical situation, stress on a component may be contributed by many factors and we may not expect that the underlying distribution is unimodel in nature. There may be many modes for this distribution or a convex combination of various densities may be a more appropriate model for stress. Let us consider a density of the following form for stress:

$$f(x) = p_1 f_1(x) + ... + p_k f_k(x) \tag{1.1}$$

for $p_j > 0, j = 1, ..., k, p_1 + ... + p_k = 1, f_j(x)$ is a density with $f_j(x) > 0$ for $0 \leq x < \infty$ and $f_j(x) = 0$ elsewhere. Let y have the density $g(y)$. Then the system is reliable if $y > x$. Let us start with independently distributed case. That is, we assume that x and y are independently distributed. The reliability of the system or for the component under consideration, is measured by the probability that y is greater than x, that is $Pr\{y > x\}$. This is given by the following:

$$Pr\{y > x\} = \int_{x=0}^{\infty} [\int_{y=x}^{\infty} g(y)\mathrm{d}y] f(x)\mathrm{d}x = \sum_{j=1}^{k} p_j \{\int_{x=0}^{\infty} f_j(x) [\int_{y=x}^{\infty} g(y)\mathrm{d}y]\mathrm{d}x.\} \tag{1.2}$$

It may be observed that if y also has a multimodel density then the procedure is exactly the same. If $g(y)$ is of the form

$$g(y) = r_1 g_1(y) + ... + r_m g_m, r_j > 0, j = 1, ..., m, r_1 + ... + r_m = 1,$$

for $g_j(y) > 0, 0 \leq y < \infty$ and $g_j(y) = 0$ elsewhere, $j = 1, ..., m$, then the reliability is given by

$$Pr\{y > x\} = \sum_{i=1}^{k} \sum_{j=1}^{m} p_i r_j \int_{x=0}^{\infty} f_i(x) [\int_{y=x}^{\infty} g_j(y)\mathrm{d}y]\mathrm{d}x. \tag{1.3}$$

Our aim here is to consider (1.2) under general pathway models for $f_j(x)$ as well as for $g(y)$ so that a wide variety of models in current use will be covered in our discussion. The original pathway model of Mathai (2005) is for the real rectangular matrix-variate case. This was extended to the complex domain in Mathai and Provost (2006). The pathway model for the real scalar positive variable case is the following:

$$P_1(x) = c_1 x^{\gamma}[1 - a(1-q)x^{\delta}]^{\frac{\eta}{1-q}}, q < 1 \tag{1.4}$$

for $\eta > 0, \gamma > -1, a > 0, \delta > 0, 1 - a(1-q)x^{\delta} > 0$ or $0 \le x \le [a(1-q)]^{-\frac{1}{\delta}}$ and $P_1(x) = 0$, otherwise. Observe that (1.4) is a generalized type-1 beta model. The standard type-1 beta model, uniform density and special models appearing in reliability analysis for the case $\gamma = \delta - 1$ are all special cases here. If $q > 1$ then write $1 - q = -(q-1), q > 1$ then the model in (1.4) switches into the model

$$P_2(x) = c_2 x^{\gamma}[1 + a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}}, q > 1 \tag{1.5}$$

for $x \ge 0, \delta > 0, a > 0, \eta > 0, \gamma > -1$ and $P_2(x) = 0$ otherwise. Observe that (1.5) can be taken as a generalized type-2 beta family of functions. Standard type-2 beta density, F-density, folded Student-t, Cauchy etc are special cases in (1.5). The exponentiated case, that is, $x = \mathrm{e}^{-cy}, c > 0$, leads to generalized logistic, logistic etc and a limiting form giving Fermi-Dirac density also. A limiting form in (1.4) gives Bose-Einstein density also. When $q \to 1_-$ in (1.4) and $q \to 1_+$ in (1.5) the models in (1.4) and (1.5) go to the model

$$P_3(x) = c_3 x^{\gamma}\mathrm{e}^{-a\eta x^{\delta}}, a > 0, \delta > 0, \eta > 0, x \ge 0 \tag{1.6}$$

and $P_3(x) = 0$ elsewhere. Note that (1.6) is the generalized gamma density where the particular cases include the standard gamma density, chisquare density Maxwell-Boltzmann density, Raleigh density, exponential density, Weibull density etc as special cases. Thus, we can go from $P_1(x)$ to $P_2(x)$ and $P_3(x)$ or from $P_2(x)$ to $P_1(x)$ and $P_3(x)$. All the three models are contained in $P_1(x)$ or in $P_2(x)$. Note that $P_1(x), P_2(x), P_3(x)$ can act as mathematical models or statistical models. If they are statistical densities then c_1, c_2, c_3 are the normalizing constants there. These will contain gamma functions. Our interest here is to consider a special case to avoid the gamma functions.

If $\gamma = \delta - 1$ then the normalizing constants reduce to simple forms. Then the densities are the following:

$$P_4(x) = a\delta(\eta + 1 - q)x^{\delta-1}[1 - a(1-q)x^{\delta}]^{\frac{\eta}{1-q}}, q < 1 \tag{1.7}$$

for $a > 0, \delta > 0, \eta > 0, 1 - a(1-q)x^{\delta} > 0$ and $P_4(x) = 0$ elsewhere.

$$P_5(x) = a\delta(\eta + 1 - q)[1 + a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}}, q > 1 \tag{1.8}$$

for $a > 0, \delta > 0, x \geq 0, \eta > 0, \eta + 1 - q > 0, 1 < q < \eta + 1$ and zero elsewhere.

$$P_6(x) = a\delta\eta x^{\delta-1}\mathrm{e}^{-a\eta x^{\delta}}, a > 0, \eta > 0, \delta > 0, x \geq 0 \tag{1.9}$$

zero elsewhere. We will take these simpler forms in (1.7) to (1.9) for our discussion from here onward. This is done for convenience only in order to avoid gamma functions appearing from the beginning steps. The technique to be introduced here will work for the general case in (1.5) to (1.7) also.

2. Stress-Strength Model

Let $f_j(x)$ of (1.1) have a pathway density of the type in (1.8) with the parameters $a_j, \delta_j, \eta_j, q_j$ or with the density

$$f_j(x) = a_j\delta_j(\eta_j + 1 - q_j)x^{\delta_j - 1}[1 + a_j(q_j - 1)x^{\delta_j}]^{-\frac{\eta_j}{q_j - 1}} \tag{2.1}$$

and let $g(y)$ have a pathway model of the type in (1.8) with the parameters a, δ, η, q or with the density in (1.8). Then the reliability is the following:

$$\begin{aligned} Pr\{y > x\} &= \sum_{j=1}^{k} p_j Pr\{y > x \text{ in } f_j\} \\ &= \sum_{j=1}^{k} p_j \int_{x=0}^{\infty} a_j\delta_j(\eta_j + 1 - q_j)x^{\delta_j - 1}[1 + a_j(q_j - 1)x^{\delta_j}]^{-\frac{\eta_j}{q_j - 1}} \\ &\times[\int_{y=x}^{\infty} a\delta(\eta + 1 - q)y^{\delta - 1}[1 + a(q-1)y^{\delta}]^{-\frac{\eta}{q-1}}\mathrm{d}y]\mathrm{d}x. \end{aligned} \tag{2.2}$$

Consider the evaluation of the integral

$$\begin{aligned} I_j &= \int_{x=0}^{\infty} a_j\delta_j(\eta_j + 1 - q_j)x^{\delta_j - 1}[1 + a_j(q_j - 1)x^{\delta_j}]^{-\frac{\eta_j}{q_j - 1}} \\ &\times[\int_{y=x}^{\infty} a\delta(\eta + 1 - q)y^{\delta - 1}[1 + a(q-1)y^{\delta}]^{-\frac{\eta}{q-1}}\mathrm{d}y]\mathrm{d}x. \end{aligned} \tag{2.3}$$

Straight integration by putting $u = y^{\delta}, a(q-1)u = v$ gives

$$\int_{y=x}^{\infty} a\delta(\eta + 1 - q)y^{\delta - 1}[1 + a(q-1)y^{\delta}]^{-\frac{\eta}{q-1}}\mathrm{d}y = [1 + a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}+1}. \tag{2.4}$$

Then put $z = a_j(q_j-1)x^{\delta_j}$, to obtain

$$I_j = a_j\delta_j(\eta_j+1-q_j)\int_{x=0}^{\infty} x^{\delta_j-1}[1+a_j(q_j-1)x^{\delta_j}]^{-\frac{\eta_j}{q_j-1}}$$
$$\times[1+a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}+1}\mathrm{d}x$$
$$= \frac{(\eta_j+1-q_j)}{(q_j-1)}\int_{x=0}^{\infty}(1+z)^{-\frac{\eta_j}{q_j-1}}[1+\frac{a(q-1)}{[a_j(q_j-1)]^{\frac{\delta}{\delta_j}}}z^{\frac{\delta}{\delta_j}}]^{-\frac{\eta}{q-1}+1}\mathrm{d}z. \tag{2.5}$$

The integral in (2.5) can be evaluated by observing that (2.5) has the following structure:

$$I_j = \frac{(\eta_j+1-q_j)}{(q_j-1)}\int_{v=0}^{\infty} vK_2(v)K_1(uv)\mathrm{d}v \tag{2.6}$$

and in this case, by using Mellin convolution of a ratio, the Mellin transform of I_j with Mellin parameter s, denoted by $M_{I_j}(s) = \frac{(\eta_j+1-q_j)}{(q_j-1)}M_{K_1}(s)M_{K_2}(2-s)$ where $M_{K_j}(s)$ is the Mellin transform of $K_j, j=1,2$. Take

$$u = \frac{[a(q-1)]^{\frac{\delta_j}{\delta}}}{a_j(q_j-1)}, K_1(x_1) = [1+x_1^{\frac{\delta}{\delta_j}}]^{-\frac{\eta}{q-1}+1}, K_2(x_2) = \frac{1}{x_2}[1+x_2]^{-\frac{\eta_j}{q_j-1}}.$$

Then

$$\int_v vK_2(v)K_1(uv)\mathrm{d}v = \int_v (1+v)^{-\frac{\eta_j}{q_j-1}}$$
$$\times [1+\frac{a(q-1)}{[a_j(q_j-1)]^{\frac{\delta}{\delta_j}}}v^{\frac{\delta}{\delta_j}}]^{-\frac{\eta}{q-1}+1}\mathrm{d}v$$

which is the integral in (2.5) to be evaluated. Hence we will use the relation

$$M_{I_j}(s) = \frac{(\eta_j+1-q_j)}{(q_j-1)}M_{K_1}(s)M_{k_2}(2-s) \tag{2.7}$$

where

$$\begin{aligned} M_{K_1}(s) &= \int_0^{\infty} x_1^{s-1}[1+x_1^{\frac{\delta}{\delta_j}}]^{-\frac{\eta}{q-1}+1}\mathrm{d}x_1 \\ &= \frac{\delta_j}{\delta}\frac{\Gamma(\frac{\delta_j}{\delta}s)\Gamma(\frac{\eta}{q-1}-1-\frac{\delta_j}{\delta}s)}{\Gamma(\frac{\eta}{q-1}-1)}, \Re(s)>0, \Re(\frac{\eta}{q-1}-1-\frac{\delta_j}{\delta}s)>0. \\ M_{K_2}(2-s) &= \int_0^{\infty} x_2^{-s+1}K_2(x_2)\mathrm{d}x_2 = \int_0^{\infty} x_2^{-s}(1+x_2)^{-\frac{\eta}{q-1}}\mathrm{d}x_2 \\ &= \frac{\Gamma(1-s)\Gamma(\frac{\eta_j}{q_j-1}-1+s)}{\Gamma(\frac{\eta_j}{q_j-1}-1)}, \Re(s)<1, \Re(\frac{\eta_j}{q_j-1}-1-s)>0. \end{aligned}$$

Then

$$M_{K_1}(s)M_{K_2}(2-s)=\frac{\delta_j}{\delta}\frac{\Gamma(1-s)\Gamma(\frac{\eta}{q-1}-1-\frac{\delta_j}{\delta}s)\Gamma(\frac{\delta_j}{\delta}s)\Gamma(\frac{\eta_j}{q_j-1}-1+s)}{\Gamma(\frac{\eta_j}{q_j-1}-1)\Gamma(\frac{\eta}{q-1}-1)}. \quad (2.8)$$

That is,

$$\begin{aligned} I_j &= \frac{\delta_j}{\delta}\frac{(\eta_j+1-q_j)}{(q_j-1)}\frac{1}{\Gamma(\frac{\eta_j}{q_j-1}-1)\Gamma(\frac{\eta}{q-1}-1)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{\delta_j}{\delta}s)\Gamma(\frac{\eta_j}{q_j-1}-1+s) \\ &\quad \times\Gamma(1-s)\Gamma(\frac{\eta}{q-1}-1-\frac{\delta_j}{\delta}s)\{\frac{[a(q-1)]^{\frac{\delta_j}{\delta}}}{a_j(q_j-1)}\}^{-s}\mathrm{d}s, i=\sqrt{-1}, \quad (2.9) \end{aligned}$$

$$=\frac{\delta_j}{\delta}\frac{(\eta_j+1-q_j)}{(q_j-1)}\frac{1}{\Gamma(\frac{\eta_j}{q_j-1}-1)\Gamma(\frac{\eta}{q-1}-1)}H_{2,2}^{2,2}\left[\frac{[a(q-1)]^{\frac{\delta_j}{\delta}}}{a_j(q_j-1)}\Big|_{(0,\frac{\delta_j}{\delta}),(\frac{\eta_j}{q_j-1}-1,1)}^{(0,1),(2-\frac{\eta}{q-1},\frac{\delta_j}{\delta})}\right] \quad (2.10)$$

for $[a(q-1)]^{\frac{\delta_j}{\delta}} < a_j(q_j-1)$, where $H(\cdot)$ is the H-function. For the theory and applications of the H-function, see for example, Mathai et al. (2010). Computations of the H-function can be carried out by using MATHEMATICA programs.

When $\delta=\delta_j, j=1,...,k$ then one can express the Mellin-Barnes integral in (2.9) in terms of hypergeometric series. Then the Mellin-Barnes representation in (2.9) becomes

$$\begin{aligned} I_j &= \frac{(\eta_j+1-q_j)}{(q_j-1)}\frac{1}{\Gamma(\frac{\eta_j}{q_j-1}-1)\Gamma(\frac{\eta}{q-1}-1)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(\frac{\eta_j}{q_j-1}-1+s) \\ &\times \Gamma(1-s)\Gamma(\frac{\eta}{q-1}-1-s)u^{-s}\mathrm{d}s, u=\frac{a(q-1)}{a_j(q_j-1)}. \end{aligned}$$

Consider the case $\frac{\eta_j}{q_j-1}-1\neq 0,1,2,...$ so that the poles of $\Gamma(s)\Gamma(\frac{\eta_j}{q_j-1}-1+s)$ are simple. The poles of $\Gamma(s)$ are at $s=-\nu, \nu=0,1,2,....$ The sum of the residues at $s=-\nu$ is

$$\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\Gamma(\frac{\eta_j}{q_j-1}-1-\nu)\Gamma(1+\nu)\Gamma(\frac{\eta}{q-1}-1+\nu)u^\nu.$$

But

$$\Gamma(\frac{\eta_j}{q_j-1}-1-\nu)=\frac{\Gamma(\frac{\eta_j}{q_j-1}-1)}{(-1)^\nu(-\frac{\eta_j}{q_j-1}+2)_\nu},\Gamma(\frac{\eta}{q-1}-1+\nu)=\Gamma(\frac{\eta}{q-1}-1)(\frac{\eta}{q-1}-1)_\nu$$

and $\Gamma(1+\nu)=(1)_\nu$. Then the sum of the residues is the following, observing that $\Gamma(\frac{\eta_j}{q_j-1}-1)\Gamma(\frac{\eta}{q-1}-1)$ is canceled:

$$
{}_2F_1\left(1,\frac{\eta}{q-1}-1;-\frac{\eta_j}{q_j-1}+2;\frac{a(q-1)}{a_j(q_j-1)}\right)
$$

for $a(q-1)<a_j(q_j-1)$. Note that our starting assumptions are $\eta+1-q>0, \eta_j+1-q_j>0, \eta_j>0, \eta>0, 1<q_j<\eta_j+1, j=1,...,k$. The poles of $\Gamma(\frac{\eta_j}{q_j-1}-1+s)$ are at $s=1-\frac{\eta_j}{q_j-1}-\nu$ or at $-s=\frac{\eta_j}{q_j-1}-1+\nu, \nu=0,1,....$ The sum of the residues at $s=1-\frac{\eta_j}{q_j-1}-\nu, \nu=0,1,...$ is

$$
\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\Gamma(1-\frac{\eta_j}{q_j-1}-\nu)\Gamma(\frac{\eta_j}{q_j-1}+\nu)
$$

$$
\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2+\nu)u^{\frac{\eta_j}{q_j-1}-1+\nu}; u=\frac{a(q-1)}{a_j(q_j-1)}.
$$

But

$$
\Gamma(1-\frac{\eta_j}{q_j-1}-\nu)=\frac{\Gamma(1-\frac{\eta_j}{q_j-1})}{(-1)^\nu(\frac{\eta_j}{q_j-1})_\nu}.
$$

Then the sum of the residues is

$$
\begin{aligned}
&u^{\frac{\eta_j}{q_j-1}-1}\Gamma(1-\frac{\eta_j}{q_j-1})\frac{\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)}{\Gamma(\frac{\eta}{q-1}-1)}\\
&\quad\times\sum_{\nu=0}^{\infty}\frac{(\frac{\eta_j}{q_j-1})_\nu(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)_\nu}{(\frac{\eta_j}{q_j-1})_\nu}\frac{u^\nu}{\nu!}\\
&\quad=u^{\frac{\eta_j}{q_j-1}-1}\Gamma(1-\frac{\eta_j}{q_j-1})\frac{\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)}{\Gamma(\frac{\eta}{q-1}-1)}(1-u)^{-(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)}
\end{aligned}
$$

for $0<u<1$. Hence

$$
\begin{aligned}
I_j=\frac{(\eta_j+1-q_j)}{(q_j-1)}\Big\{&{}_2F_1\left(1,\frac{\eta}{q-1}-1;-\frac{\eta_j}{q_j-1}+2;u\right)\\
&+u^{\frac{\eta_j}{q_j-1}-1}\frac{\Gamma(1-\frac{\eta_j}{q_j-1})\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)}{\Gamma(\frac{\eta}{q-1}-1)}(1-u)^{-(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)}\Big\}
\end{aligned}
\tag{2.11}
$$

for $0<u<1$ or for $a(q-1)<a_j(q_j-1)$, and $\frac{\eta_j}{q_j-1}-1\neq 0,1,2,...,$ $\delta_j=\delta$, $1<q_j<\eta_j+1$, $1<q<\eta+1$, $j=1,...,k$.

Continuation part

Consider the integrand

$$\Gamma(s)\Gamma(\frac{\eta_j}{q_j-1}-1+s)\Gamma(1-s)\Gamma(\frac{\eta}{q-1}-1-s)u^{-s}; u=\frac{a(q-1)}{a_j(q_j-1)}.$$

The poles of $\Gamma(1-s)$ are at $s=1+\nu, \nu=0,1,1...$ and the poles of $\Gamma(\frac{\eta}{q-1}-1-s)$ are at $s=\frac{\eta}{q-1}-1+\nu, \nu=0,1,....$ Hence, if $\frac{\eta}{q-1}-1\neq 0,1,2,...$ then the poles of $\Gamma(1-s)\Gamma(\frac{\eta}{q-1}-1-s)$ are simple. In this case the sum of the residues at $s=1+\nu$ is

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(1+\nu)\Gamma(\frac{\eta_j}{q_j-1}+\nu)\Gamma(\frac{\eta}{q-1}-2-\nu)u^{-1-\nu}$$

$$=\Gamma(\frac{\eta_j}{q_j-1})\Gamma(\frac{\eta}{q-1}-2)\frac{(1)_{\nu}(\frac{\eta_j}{q_j-1})_{\nu}}{(3-\frac{\eta}{q-1})_{\nu}}\frac{u^{-1-\nu}}{\nu!}$$

$$=\Gamma(\frac{\eta_j}{q_j-1})\Gamma(\frac{\eta}{q-1}-2)u^{-1}{}_2F_1(1,\frac{\eta_j}{q_j-1};3-\frac{\eta}{q-1};\frac{1}{u}), u=\frac{a(q-1)}{a_j(q_j-1)},$$

$\delta_j=\delta, j=1,...,k.$

The sum of the residues at the poles $s=\frac{\eta}{q-1}-1+\nu$ is

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{\eta}{q-1}-1+\nu)\Gamma(\frac{\eta_j}{q_j-1}+\frac{\eta}{q-1}-2+\nu)\Gamma(2-\frac{\eta}{q-1}-\nu)u^{-(\frac{\eta}{q-1}-1)-\nu}$$

$$=\Gamma(\frac{\eta}{q-1}-1)\Gamma(\frac{\eta_j}{q_j-1}+\frac{\eta}{q-1}-2)\Gamma(2-\frac{\eta}{q-1})$$

$$\times\sum_{\nu=0}^{\infty}\frac{(\frac{\eta}{q-1}-1)_{\nu}(\frac{\eta_j}{q_j-1}+\frac{\eta}{q-1}-2)_{\nu}}{(\frac{\eta}{q-1}-1)_{\nu}}\frac{u^{-\frac{\eta}{q-1}+1-\nu}}{\nu!}$$

$$=\Gamma(\frac{\eta}{q-1}-1)\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)\Gamma(2-\frac{\eta}{q-1})$$

$$\times[1-\frac{1}{u}]^{-(\frac{\eta_j}{q_j-1}+\frac{\eta}{q-1}-2)}u^{-\frac{\eta}{q-1}+1}.$$

Hence for $u>1$ we have

$$I_j=\frac{(\eta_j+1-q_j)}{(q_j-1)}\{\frac{\Gamma(\frac{\eta}{q-1}-2)}{\Gamma(\frac{\eta}{q-1}-1)}u^{-1}{}_2F_1\left(1,\frac{\eta_j}{q_j-1};3-\frac{\eta}{q-1};\frac{1}{u}\right)$$

$$+\frac{\Gamma(\frac{\eta}{q-1}+\frac{\eta_j}{q_j-1}-2)\Gamma(2-\frac{\eta}{q-1})}{\Gamma(\frac{\eta_j}{q_j-1}-1)}u^{-\frac{\eta}{q-1}+1}[1-\frac{1}{u}]^{-(\frac{\eta_j}{q_j-1}+\frac{\eta}{q-1}-2)}\} \qquad (2.12)$$

for $u = \frac{a(q-1)}{a_j(q_j-1)} > 1, \frac{\eta}{q-1} - 1 \neq 0, 1, ..., \eta_j > 0, \eta > 0, 1 < q_j < \eta_j + 1, 1 < q < \eta + 1, \delta_j = \delta, j = 1, ..., k$. Hence the reliability is given by the following:

$$Pr\{y > x\} = \sum_{j=1}^{k} p_j \begin{cases} I_j^{(1)} \text{ for } a(q-1) < a_j(q_j-1), j = 1, ..., k \\ I_j^{(2)} \text{ for } a(q-1) > a_j(q_j-1), j = 1, ..., k \end{cases}$$

where $I_j^{(1)}$ is given in (2.11) and $I_j^{(2)}$ is given in (2.12).

2.1 One factor of type-1 beta form

Stress is supposed to be a finite range behavior. After a certain threshold the system breaks down. Hence a type-1 beta form may be appropriate for stress. In this case let the stress follow a pathway model with $q_j < 1$. Then I_j of (2.5) reduces to the following form:

$$I_j = a_j\delta_j(\eta_j+1-q_j)\int_{x=0}^{a} x^{\delta_j-1}[1-a_j(1-q_j)x^{\delta_j}]^{\frac{\eta_j}{1-q_j}}[1+a(q-1)x^{\delta}]^{-\frac{\eta}{q-1}+1}\mathrm{d}x \quad (2.13)$$

where $a = [a_j(1-q_j)]^{-\frac{1}{\delta_j}}$. Put $u = a_j(1-q_j)x^{\delta_j}$. Then

$$I_j = \frac{(\eta_j+1-q_j)}{(1-q_j)}\int_{u=0}^{1}(1-u)^{\frac{\eta_j}{1-q_j}}[1+\frac{a(q-1)}{[a_j(1-q_j)]^{\frac{\delta}{\delta_j}}}u^{\frac{\delta}{\delta_j}}]^{-\frac{\eta}{q-1}+1}\mathrm{d}u.$$

Expand the second factor for $b < 1$ where $b = \frac{a(q-1)}{[a_j(1-q_j)]^{\frac{\delta}{\delta_j}}}$. Therefore

$$[1+bu^{\frac{\delta}{\delta_j}}]^{-(\frac{\eta}{q-1}-1)} = \sum_{k=0}^{\infty}\frac{(-b)^k}{k!}(\frac{\eta}{q-1}-1)_k u^{\frac{\delta}{\delta_j}k}$$

That is,

$$I_j = \frac{(\eta_j+1-q_j)}{(1-q_j)}\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}(\frac{\eta}{q-1}-1)_k\int_0^1 u^{\frac{\delta}{\delta_j}k}(1-u)^{\frac{\eta_j}{1-q_j}}\mathrm{d}u.$$

But

$$\int_0^1 u^{\frac{\delta}{\delta_j}k}(1-u)^{\frac{\eta_j}{1-q_j}}\mathrm{d}u = \frac{\Gamma(1+\frac{\delta}{\delta_j}k)\Gamma(\frac{\eta_j}{1-q_j}+1)}{\Gamma(2+\frac{\eta_j}{1-q_j}+\frac{\delta}{\delta_j}k)} \text{ for general } \delta_j, \delta$$

$$= \frac{\Gamma(1+k)\Gamma(\frac{\eta_j}{1-q_j}+1)}{\Gamma(2+\frac{\eta_j}{1-q_j}+k)} = \frac{(1)_k\Gamma(\frac{\eta_j}{1-q_j}+1)}{\Gamma(2+\frac{\eta_j}{1-q_j})(2+\frac{\eta_j}{1-q_j})_k} \text{ for } \delta_j = \delta.$$

Hence

$$I_j = \frac{(\eta_j+1-q_j)}{(1-q_j)}\Gamma(\frac{\eta_j}{1-q_j}+1)\sum_{k=0}^{\infty}\frac{(-b)^k}{k!}(\frac{\eta}{q-1}-1)_k\frac{\Gamma(1+\frac{\delta}{\delta_j}k)}{\Gamma(2+\frac{\eta_j}{1-q_j}+\frac{\delta}{\delta_j}k)}. \quad (2.14)$$

This, for $\delta_j=\delta$ reduces to the form

$$I_j = \frac{(\eta_j+1-q_j)}{(1-q_j)}\frac{\Gamma(\frac{\eta_j}{1-q_j}+1)}{\Gamma(2+\frac{\eta_j}{1-q_j})}{}_2F_1(\frac{\eta}{q-1}-1,1;2+\frac{\eta_j}{1-q_j};-b) \quad (2.15)$$

for $0<b<1, \delta_j=\delta$. The following table gives the reliability in (2.15) for $a(q-1)<a_j(1-q_j), a>0, a_j>0, q>1, q_j<1, j=1, \eta>0, \eta_j>0, 1<q<\eta+1$ and for $\delta_j=\delta, j=1$.

a	a_1	η	η_1	q	q_1	$I_1=\Pr\{y>x\}$
1	2	1	3	1.5	0.9	0.9316
6	9	0.5	9	1.3	0.3	0.9875
7	7	0.6	7	1.5	0.5	0.9883
4	5	0.7	6	1.6	0.6	0.9890
1.5	3	0.9	4	1.8	0.8	0.9895
2	4	0.8	5	1.7	0.7	0.9916
5	8	0.5	8	1.4	0.4	0.9934
7	10	0.4	10	1.3	0.7	0.9936
0.0001	6	0.6	2	1.9	0.9	1.0000 (*approx..*)

Table 1: Reliability $Pr\{y>x\}$

3. Connection to Fractional Integral

Let the stress in I_j be of a pathway model for $q_j<1, j=1$ and the strength a pathway model with $q<1$ or $q>1$ or $q\to 1$. Then I_j of (2.13) for $q>1$ reduces to the following form:

$$I_j = a_j\delta_j(\eta_j+1-q_j)\int_{t=0}^{z} t^{\delta_j-1}[1-a_j(1-q_j)t^{\delta_j}]^{\frac{\eta_j}{1-q_j}}[1+a(q-1)t^{\delta}]^{-\frac{\eta}{q-1}+1}\mathrm{d}t \quad (3.1)$$

where $z=[a_j(1-q_j)]^{-\frac{1}{\delta_j}}$. Take out $a_j(1-q_j)$ from the first factor in the integrand of (3.1). Then I_j reduces to the following, remembering that our notation here is $z=[a_j(1-q_j)]^{-\frac{1}{\delta_j}}$:

$$I_j = a_j\delta_j(\eta_j+1-q_j)[a_j(1-q_j)]^{\frac{\eta_j}{1-q_j}}\int_{t=0}^{z}[z^{\delta_j}-t^{\delta_j}]^{\frac{\eta_j}{1-q_j}}t^{\delta_j-1}$$

$$\times[1 + a(q-1)t^{\delta}]^{-\frac{\eta}{q-1}+1}\mathrm{d}t$$
$$= \frac{1}{\Gamma(\alpha)}\int_{t=0}^{z}[z^{\delta_j} - t^{\delta_j}]^{\alpha-1}f(t)\mathrm{d}t \tag{3.2}$$

where

$$\alpha = \frac{\eta_j}{1-q_j} + 1, f(t) = a_j\delta_j(\eta_j + 1 - q_j)[a_j(1-q_j)]^{\frac{\eta_j}{1-q_j}}$$
$$\times\Gamma(\frac{\eta_j}{1-q_j} + 1)t^{\delta_j - 1}[1 + a(q-1)t^{\delta}]^{-\frac{\eta}{q-1}+1}. \tag{3.3}$$

Note that (3.2) is Riemann-Liouville left sided factional integral of order α for $\delta_j = 1$ and (3.1) gives a generalized Riemann-Liouville fractional integral of the first kind, or left sided, of order α for the function $f(t)$ defined in (3.3), for the α also defined in (3.3). Observe that whenever the pathway model with pathway parameter $q_j < 1$ is involved we can convert the corresponding integral into a fractional integral, and thus a connection to fractional integral can be established. For the pathway fractional integral operator, see Seema S. Nair (2009, 2011). For a general definition of fractional integrals, in the scalar and real and complex matrix-variate cases, may be seen from Mathai (2014).

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