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On certain Transformation Formula of Ordinary Hypergeometric Series using laplace transform

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Abstract:

This is a brief survey of theory of hypergeometric series with particular emphasis on establishing some transformation formula in hypergeometric series using Laplace transform and Pochamar's integral.

Keywords and Phrases: Hypergeometric series, summation formula and Laplace transform.

Notations and Known results

An infinite series of the form $\sum_{n=0}^{\infty} \frac{(a_n)(b_n)}{(c_n)(1_n)} z^n$ is known as ordinary hypergeometric series. It is denoted as ${}_2F_1[a, b; c; z]$. So,

$$ {}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \tag{1}$$

Where a,b and c are real or complex parameters and z is an argument with $|z| < 1$ and

$$(a_n) = a(a + 1)(a + 2)...(a + n - 1).$$

The Pochamer integral is given by

$$ {}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt \tag{2}$$

For $Re(b) > 0$ and $Re(c - a - b) > 0$.
The integral (2) gives

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{3}$$

This is known as gauss Summation formula. The restriction $Re(b) > 0$ is not necessary for this formula to hold. Gauss [1813] used one of his contiguous relation to prove (3) is hold if $Re(c - a - b) > 0$.

$$ {}_2F_1[-n, b; c; 1] = \frac{(c - b)_n}{(c)_n}. \tag{4}$$

This is simply an alternative form of the discrete binomial formula.

$$(a+b)_n = \sum_{k=0}^{n}(n,k)(a)_k(b)_{n-k} \tag{5}$$

Which is attributed to Vander monde [1772] alone but it was discovered long by Chinese Mathematician Chu [1303].
Kummar established the following transformation formula

$${}_2F_1[a,b;1+a-b;-1] = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a/2-b)\Gamma(1+a)}, \quad \text{provided } Re(1-2b)>0 \tag{6}$$

$$L\{f(t)\} = \int_0^\infty e^{-pt}f(t)dt \tag{7}$$

is known as laplace transformation where p is a parameter and f is continuous function on $[0,\infty)$.
Main Result:

$$f(p) = \frac{\Gamma(b)\Gamma(1+a-2b)}{\Gamma(1+a-b)}\ {}_2F_1[a,b;1+a-b;1], \tag{8}$$

where $f(p) = L(e^s-1)^{b-1}(2e^s-1)^{-a}$ and p = -b.
Proof:
Replacing z by z-1 in (2) we have

$${}_2F_1[a,b;c;z-1] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+t-tz)^{-a}dt. \tag{9}$$

Putting $t = \dfrac{s}{s+1}$, $dt = \dfrac{1}{(s+1)^2}ds$ and s varies from 0 to ∞, we have

$${}_2F_1[a,b;c;z-1] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty s^{b-1}(s+1)^{a-c}(2s+1-sz)^{-a}ds. \tag{10}$$

Putting z=0 in (10) we have

$${}_2F_1[a,b;c;-1] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty s^{b-1}(s+1)^{a-c}(2s+1)^{-a}ds. \tag{11}$$

Replacing s by e^s-1 in the above integral we obtain

$${}_2F_1[a,b;c;-1] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty (e^s-1)^{b-1}e^{s(a-c+1)}(2e^s-1)^{-a}ds. \tag{12}$$

Putting $c = 1+a-b$ in above integral we have

$$\begin{aligned} &{}_2F_1[a,b;1+a-b;-1] \\ &= \frac{\Gamma(1+a-b)}{\Gamma(b)\Gamma(1+a-2b)}\int_0^\infty e^{-s(-b)}(e^s-1)^{b-1}(2e^s-1)^{-a}ds, \end{aligned} \tag{13}$$

Putting $p = -b$ in above integral, we get

$$ {}_2F_1[a, b; 1 + a - b; -1]$$

$$= \frac{\Gamma(1 + a - b)}{\Gamma(b)\Gamma(1 + a - 2b)} \int_0^\infty e^{-ps}(e^s - 1)^{b-1}(2e^s - 1)^{-a} ds,$$

$$= \frac{\Gamma(1 + a - b)}{\Gamma(b)\Gamma(1 + a - 2b)} \int_0^\infty e^{-ps} f(s) ds,$$

where $f(s) = (2e^s - 1)^{-a}(e^s - 1)^{b-1}$

$$= \frac{\Gamma(1 + a - b)}{\Gamma(b)\Gamma(1 + a - 2b)} L[f(s)].$$

This gives

$$ {}_2F_1[a, b; 1 + a - b; -1] = \frac{\Gamma(1 + a - b)}{\Gamma(b)\Gamma(1 + a - 2b)} f(p), \tag{14}$$

where $f(p) = L[f(s)]$.
This gives,

$$f(p) = \frac{\Gamma(b)\Gamma(1 + a - 2b)}{\Gamma(1 + a - b)} \ {}_2F_1[a, b; 1 + a - b; -1], \quad \text{provided } Re(c - a - b) > 0 \tag{15}$$

Deduction:

Using Kummer's theorem (6) in (15) we have

$$f(p) = \frac{\Gamma(b)\Gamma(1 + a - 2b)\Gamma(1 + a/2)}{\Gamma(1 + a/2 - b)\Gamma(1 + a)}. \tag{16}$$

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