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ON MIXED TRILATERAL GENERATING RELATIONS OF LAGUERRE POLYNOMIALS

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Dedicated to Prof. A.M. Mathai on his 80^{th} ***birth anniversary***

Abstract: In this paper we adopt group theoretic method to obtain mixed trilateral generating relations by involving Laguerre polynomial with the use of linear partial differential operators from the existence of bilinear generating relations.

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1. Introduction

The Laguerre polynomials $L_n^\alpha(x)$ satisfy the following differential equation :

$$x\frac{d^2y}{dx^2} + (1+\alpha-x)\frac{dy}{dx} + ny = 0 \tag{1.1}$$

[cf. Majumdar [1]]

where n is not necessarily a non-negative integer.

The main object of the present paper is to derive some mixed trilateral generating relations by using the linear partial operators.

2. Main Results

Theorem 1 If there exists a bilinear generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) L_n^{(\alpha)}(u) w^n \tag{2.1}$$

then,

$$(1+wy)^{\alpha}\exp(-wxy-w)G[x(1+wy),u+w,wy]$$
$$=\sum_{n,p,q=0}^{\infty}\frac{a_n(n+1)_p(-1)^q w^{n+p+q}}{p!q!}y^{n+p}L_{n+p}^{(\alpha-p)}(x)L_n^{(\alpha+q)}(u) \tag{2.2}$$

where $a_n \neq 0$ is an arbitrary constant.

Importance of above theorem lies in the fact that all particular class of generating functions can be easily deduced by attributing different values to $'a_n\,'$.

Proof of the theorem

Let us assume the generating function

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)L_n^{(\alpha)}(u)w^n \tag{2.3}$$

Replacing w by wyt and multiplying both the sides by z^nv^n, we get

$$z^nv^nG(x,u,wyt)=z^nv^n\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)L_n^{(\alpha)}(u)(wyt)^n$$

or

$$z^nv^nG(x,u,wyt)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)y^nz^nL_n^{(\alpha)}(u)t^nv^nw^n \tag{2.4}$$

Now, we choose the following two partial differential operators R_1 and R_2 given by Majumdar [1] and McBride [2] respectively.

$$R_1=xyz\frac{\partial}{\partial x}-y^2z\frac{\partial}{\partial y}-(x-\alpha)yz \tag{2.5}$$

[cf. Majumdar [1]]

$$R_2=t\frac{\partial}{\partial u}-t \tag{2.6}$$

[cf. McBride [2]]

Such that

$$R_1\left[L_n^{(\alpha)}(x)y^nz^n\right]=(n+1)L_{n+1}^{(\alpha-1)}(x)y^{n+1}z^{n+1} \tag{2.7}$$

$$R_2\left[L_n^{(\alpha)}(u)t^nv^n\right]-L_n^{(\alpha+1)}(u)t^{n+1} \tag{2.8}$$

And also

$$e^{wR_1}F(x,y,z) = (1+wyz)^{\alpha}\exp(-wxyz)F\left[x(1+wyz), \frac{y}{1+wyz}, z\right] \tag{2.9}$$

[cf. Majumdar [1]]

$$e^{wR_2}F(u,t) = \exp(-wt)F[u+wt,t] \tag{2.10}$$

[cf. McBride [2]]

Now, we operating both the sides of (2.4) with $e^{wR_1}\ e^{wR_2}$, we obtain

$$e^{wR_1}e^{wR_2}[z^n v^n G(x,u,wyt)] = e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)y^n z^n L_n^{(\alpha)}(u)t^n v^n w^n \tag{2.11}$$

The left hand side of (2.11) becomes

$$(1+wyz)^{\alpha}\exp(-wxyz-wt)G[x(1+wyz), u+wt, wyt] \tag{2.12}$$

And the right hand side of (2.11) becomes

$$\sum_{n,p,q=0}^{\infty} \frac{a_n(n+1)_p(-1)^q w^{n+p+q}}{p!q!} y^{n+p}z^{n+p}t^{n+q}v^n L_{n+p}^{(\alpha-p)}(x)L_n^{(\alpha+q)}(u) \tag{2.13}$$

Now equating (2.12) and (2.13), and on putting $z=v=t=1$, the theorem is readily established.

3. Particular Case

(i) If we set $u=0$ in given theorem and proceeding as the proof of main theorem with operator R_1, we get

$$\begin{aligned}
&(1+wy)^{\alpha}\exp(-wxy)G[x(1+wy), wy] \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{a_n(n+1)_p w^{n+p}}{p!} L_{n+p}^{(\alpha-p)}(x)y^{n+p} \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{a_{n-p}(n-p+1)_p w^n}{p!} L_n^{(\alpha-p)}(x)y^n \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{a_{n-p}(n-p+1)_p w^n}{p!} L_{n+p}^{(\alpha-p)}(x)y^n
\end{aligned} \tag{3.1}$$

Which is given by Majumdar [3].

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