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FIXED POINT THEOREMS FOR RATIONAL CONTRACTION MAPPING IN CONE b-METRIC SPACES

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Abstract: In this paper, we prove some fixed point theorems for contraction mappings in cone b-metric spaces. Dass-Gupta [6], Jaggi-Dass [16], Jaggi [17], M.S. Khan [18] and others proved theorems for different spaces using rational function and contraction conditions. Inspiring from above results, we proved theorems in cone b-metric spaces.

Keywords and Phrases: Fixed point, Cone Metric Spaces, Complete Cone Metric Spaces, Cone b-Metric Spaces.

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1. Introduction

Banach's contraction principle is one of the pivotal results in functional analysis. Let (X, d) be a complete metric space and mapping $T : X \to X$ is such that

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X \quad \text{where} \quad k \in [01)$$

Then T has a fixed point $x' \in X$.
Fixed point theory plays a basic role in application of many branches of mathematics. Finding a fixed point of contractive mapping becomes the center of strong research activity. There are many works about the fixed point of contractive maps

(see, for example, [1,2]). In [2] Polish Mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922. In [3], Bakhtin introduced b-metric spaces as a generalized of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. A lucid survey shows that there are many generalizations of metric spaces. One of them is b-metric space. The concept of b-metric space was introduced by Czerwik [11, 12]. Using this idea, he proved Banach's fixed point theorem in b-metric spaces. Later on, many researchers including Aydi [8], Bota [9], Chug [10], Shi [20], Du [13], Kir [19], Huang and Zhang [4] introduced cone metric spaces as generalized of metric spaces, replacing the real number by an ordered Banach spaces and define cone metric spaces. Moreover, they proved some fixed point theorems for contraction mapping that expanded certain results of fixed point in metric spaces. In [5], Hussain and Shah introduced cone b-metric spaces as a generalized of b-metric spaces and cone metric spaces. We prove some fixed point theorems in cone b-metric spaces on contraction mapping.
Before going to the main results, we define some definition, example and lemma required in sequel.

2. Preliminaries

Let E be real Banach space and P be a subset of E. Then P is called a cone if

(i) P is closed, nonempty, and satisfies $P \neq \{\theta\}$

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real number a,b,

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$, i.e., $P \cap (-P) = \theta$, where θ denote the zero element of E and intP the interior of P.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$. A cone P is a solid cone if $intP \neq \Phi$. We write $\| \, . \, \|$ as the norm on E. The cone P is called normal if there is a number $k > 0$ such that $\forall x, y \in E, \theta \leq x \leq y \; implies \; \| \, x \, \| \leq k \, \| \, y \, \|$. The least positive number k satisfying the above is called the normal constant of P. It is well known that $k \geq 1$.
In the following, we always suppose that E is a Banach space, P is cone in E with $intP \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1.[4] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ Satisfies:

$(CM1)$ $\theta \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$
$(CM2)$ $d(x,y) = d(y,x)$ for all $x,y \in X$,
$(CM3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.2. **[5]** Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to E$ Satisfies:
$(CBM1)\theta \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$
$(CBM2)d(x,y) = d(y,x)$ for all $x,y \in X$,
$(CBM3)d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.
Then d is called a cone b-metric on X and (X,d) is called a cone b-metric space.

Remark 2.3. The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric space generalized b-metric spaces and cone metric spaces.
Following is an example which shows that a cone b-metric spaces which are not cone metric spaces.

Example 2.4.[7] $E = R^2$, $P = \{(x,y) \in E : x,y \geq 0\} \subset E$, $X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|^p, \alpha|x-y|^p)$ where $\alpha \geq 0$ and $p > 1$ are two constant. Then (X,d) is a cone b-metric space but not a cone metric space. In fact, we only need to prove (iii) in Definition 2.2 as follows,
Let $x,y,z \in X$ set $u = x - z$, $v = z - y$, so $x - y = u + v$ from the inequality

$$(a+b)^p \leq (2\max\{a,b\})^p \leq 2^p(a^p + b^p) \quad \text{for all} \quad a,b \geq 0$$

We have

$$|x-y|^p = |u+v|^p \leq (|u| + |v|)^p \leq 2^p(|u|^p + |v|^p) = 2^p(|x-z|^p + |z-y|^p),$$

$$\Rightarrow d(x,y) \leq s[d(x,z) + d(z,y)] \text{ with } s = 2^p > 1$$

But $|x-y|^p \leq |x-z|^p + |z-y|^p$ is impossible for all $x > z > y$, indeed, taking account of the inequality.

$$(a+b)^p > a^p + b^p \quad \text{for all} \quad a,b > 0,$$

We arrive at

$$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p = |x-z|^p + |z-y|^p$$

for all $x > z > y$. Thus, (CM3) in Definition 2.1 is not satisfied ,i.e., (X,d) is not a cone metric space.

Example 2.5.[7] Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d : X \times X \to R^+$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = \{x_n\}$, $y = \{y_n\} \in l^p$. Then (x, d) is a b-metric space (see[5]). Put $E = l^1$, $p = \{\{x_n\} \in E : x_n \geq 0 \text{for all } n \geq 1\}$. Letting the mapping $d : X \times X \to E$ be defined by $d(x, y) = \left\{ \dfrac{d(x, y)}{2^n} \right\}_{n \geq 1}$, we conclude that (X, d) is a cone b-metric with coefficient $s = 2^{1/p} > 1$, but it is not metric space.

Example 2.6.[7] Let $X = \{1, 2, 3, 4\}$, $E = R^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$. Define $d : X \times X \to E$ by

$$d(x, y) = \begin{cases} (|x - y|^{-1}, |x - y|^{-1}), & \text{if } x \neq y \\ \theta & \text{if } x = y. \end{cases}$$

Then (X, d) is a cone b-metric space with the coefficient $s = 6/5$. But it is not a cone metric space since the triangle inequality is not satisfied .Indeed,

$$d(1, 2) > d(1, 4) + d(4, 2) \text{ and } \ d(3, 4) > d(3, 1) + d(1, 4).$$

Definition 2.7.[5] Let (X, d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Than

(i) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. We denote this by $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Lemma 2.8. Let (X, d) be a cone b-metric space and $\{x_n\} \in X$ such that

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1})$$

where $0 \leq k < 1$, then sequence $\{x_n\}$ is a Cauchy sequence.

3. Main Results. In 1975, B.K.Dass and S.Gupta [6] proved the following theorem:
If T be a self-mapping of a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$$

for all $x, y \in X, x \neq y$ for some $\alpha, \beta \in [0, 1]$ with $\alpha + \beta < 1$ and for some $x_0 \in X, \{T^n x_0\}$ has a convergent subsequence with limit z in X. Then z is the unique fixed point of T in X.
Now we generalized this theorem in cone b-metric space as.

Theorem 3.1. Let (X, d) be a complete cone b-metric space with constant coefficients $s \geq 1$ and let $a_i \geq 0$ $(i = 1, 2, 3, ...)$ and $T : X \times X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)] \frac{[d(x, y) + d(y, Ty)]}{[d(x, Ty)]}$$

$$+\alpha_3 [d(x, Ty) + d(y, Tx)] \frac{[d(x, y) + d(y, Ty) + d(x, Ty)]^2}{[d(x, Ty)]^2}$$

For all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 > 0$, $s\alpha_1 + \dfrac{(1+s)}{s}\alpha_2 + \dfrac{(1+s)^2}{s^2}\alpha_3 \leq 1$.
Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a arbitrary point. Suppose there is an point $x_1 \in X$ such that $Tx_0 = x_1$. By Induction a sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1} (n \geq 0)$. If for some $n, x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and x_n respectively in theorem (3.1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{[d(x_{n-1}, x_{n+1})]}$$

$$+\alpha_3 [d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})] \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})]^2}{[d(x_{n-1}, x_{n+1})]^2}$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]} + \alpha_3 [d$$

$$(x_{n-1},x_{n+1})+d(x_n,x_{n+1})]\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+s\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}]^2}{s^2[d(x_{n-1},x_n)+d(x_n,x_{n+1})]^2}$$

$$\leq \alpha_1 d(x_{n-1},x_n)+\alpha_2\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{s}+\alpha_3\frac{(1+s)^2}{s^2}[d(x_{n-1},x_{n+1})+d(x_n,$$

$$x_{n+1})]\left[1-\frac{\alpha_2}{s}-\alpha_3\frac{(1+s)^2}{s^2}\right]d(x_n,x_{n+1})\leq\left(\alpha_1+\frac{\alpha_2}{s}+\alpha_3\frac{(1+s)^2}{s^2}\right)d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1})\leq\frac{\left[\alpha_1+\frac{\alpha_2}{s}+\alpha_3\frac{(1+s)^2}{s^2}\right]}{\left[1-\frac{\alpha_2}{s}-\alpha_3\frac{(1+s)^2}{s^2}\right]}d(x_{n-1},x_n)$$

$$\left(s\alpha_1+\frac{(1+s)}{s}\alpha_2+\frac{(1+s)^2}{s^2}\alpha_3\right)\leq 1$$

$$\Rightarrow\frac{\left[\alpha_1+\frac{\alpha_2}{s}+\alpha_3\frac{(1+s)^2}{s^2}\right]}{\left[1-\frac{\alpha_2}{s}-\alpha_3\frac{(1+s)^2}{s^2}\right]}\leq\frac{1}{s}=k \text{ implies } \ d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)$$

Proceeding in the same manner up to n iteration, we have

$$d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)\leq k^2d(x_{n-2},x_{n-1})\leq ...\leq k^nd(x_0,x_1)$$

For $n\geq 1$ and letting $m>n$, we have

$$d(x_n,x_{n+m})\leq s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+m})]$$

$$\leq sd(x_n,x_{n+1})+s^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+m})]$$

$$\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3[d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+m})]$$

$$\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{n+3})+s^4d(x_{n+3},x_{n+4})$$

$$+...+s^{m-1}d(x_{n+m-2},x_{n+m-1})+s^md(x_{n+m-1},x_{n+m})$$

$$\leq sk^nd(x_0,x_1)+s^2k^{n+1}d(x_0,x_1)+s^3k^{n+2}d(x_0,x_1)+s^4k^{n+3}d(x_0,x_1)$$

$$+...+s^{m-1}k^{n+m-2}d(x_0,x_1)+s^mk^{n+m-1}d(x_0,x_1)$$

$$d(x_n,x_{n+m})\leq sk^n(1+sk+(sk)^2+(sk)^3+...+(sk)^{m-1})d(x_0,x_1)$$

$$d(x_n,x_{n+m})\leq sk^n\frac{1-(sk)^m}{1-sk}d(x_0,x_1)$$

$$d(x_n, x_{n+m}) \leq sk^n \frac{1}{1-sk} d(x_0, x_1)$$

Since $k \in \left[0, \frac{1}{s}\right) \Rightarrow sk^n \frac{1}{1-sk} d(x_0, x_1) \to \theta$ as $n \to \infty \forall m \in N^+$. Therefore $\{x_n\}$ is a Cauchy sequence in X.
Therefore sequence $\{x_n\}$ converse to a point $x^* \in X$. Now we prove that x^* is the unique fixed point of T.

$$d(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \leq s[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]$$

$$\leq sd(x^*, Tx_n) + sd(Tx_n, Tx^*)$$

$$\leq s\{d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2 [d(x_n, Tx_n) + d(x^*, Tx^*)] \frac{[d(x_n, Tx_n) + d(x^*, Tx^*)]}{s[d(x_n, Tx_n) + d(x^*, Tx^*)]}$$

$$+\alpha_3 [d(x_n, Tx^*) + d(x^*, Tx^*)] \frac{[d(x_n, Tx_n) + d(x^*, Tx^*) + s\{d(x_n, Tx_n) + d(x^*, Tx^*)\}]^2}{s^2 [d(x_n, Tx_n) + d(x^*, Tx^*)]^2}$$

$$\leq s\{d(x^*, Tx_n) + \alpha_1 d(x_n, x^*) + \alpha_2 \frac{d(x_n, x^*) + d(x^*, Tx^*)}{s} + \alpha_3 (1+s)^2$$

$$\frac{d(x_n, x^*) + d(x^*, Tx^*)}{s^2} \left(1 - \frac{\alpha_2}{s} - \frac{\alpha_3 (1+s)^2}{s^2}\right) \leq s\{d(x^*, x_{n+1}) +$$

$$\left(\alpha_1 + \alpha_2 \frac{1}{s} + \frac{\alpha_3 (1+s)^2}{s^2}\right) [d(x^*, x_{n+1}) + d(x_n, x_{n+1})]\}$$

$$\left(1 - \alpha_2 - \frac{\alpha_3 (1+s)^2}{s}\right) d(x^*, Tx^*) \leq s \left(1 + \alpha_1 + \alpha_2 \frac{1}{s} + \frac{\alpha_3 (1+s)^2}{s^2}\right) d(x^*, x_{n+1})$$

$$+s \left(\alpha_1 + \alpha_2 \frac{1}{s} + \frac{\alpha_3 (1+s)^2}{s^2}\right) d(x_n, x_{n+1})$$

$$(s - s\alpha_2 - \alpha_3 (1+s)^2) d(x^*, Tx^*) \leq (s^2 + s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2) d(x^*, x_{n+1})$$

$$+(s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2) d(x_n, x_{n+1})$$

$$d(x^*, Tx^*) \leq \frac{(s^2 + s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2)}{(s - s\alpha_2 - \alpha_3 (1+s)^2)} d(x^*, x_{n+1})$$

$$+\frac{(s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2)}{(s - s\alpha_2 - \alpha_3 (1+s)^2)} d(x_n, x_{n+1})$$

Since $\{x_n\}$ is a Cauchy sequence, therefore for any $c \in intP$, we select an$n_i \in N$ for all $n \geq n_i$ such that

$$d(x^*, Tx_n) << \frac{(s^2 + s^2\alpha_1 + s\alpha_2 + \alpha_3(1+s)^2)c}{2(s - s\alpha_2 - \alpha_3(1+s)^2)}$$

and

$$d(x_n, x^*) << \frac{(s^2\alpha_1 + s\alpha_2 + \alpha_3(1+s)^2)c}{2(s - s\alpha_2 - \alpha_3(1+s)^2)}$$

Thus, for any $c \in intP, d(x^*, Tx^*) << c \forall n \geq n_i$. Therefor x^* is fixed point of T i.e. $x^* = Tx^*$.

Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$. Now from theorem (3.1), we have

$$d(x^*, x') = d(Tx^*, Tx'') \leq \alpha_1 d(x^*, x'')$$

$$+\alpha_2[d(x^*, Tx^{*'}) + d(x', Tx'')]\frac{[d(x^*, x'') + d(x', Tx')]}{[d(x^*, Tx'')]}$$

$$+\alpha_3[d(x^*, Tx'') + d(x', Tx^{*'})]\frac{[d(x^*, x'') + d(x', Tx'') + d(x^*, Tx'')]}{[d(x^*, Tx'')]}$$

$$d(x^*, x'') \leq \alpha_1 d(x^*, x'') + \alpha_2[d(x^*, x^{*'}) + d(x', x'')]\frac{[d(x^*, x'') + d(x', x'')]}{[d(x^*, x'')]}$$

$$+\alpha_3[d(x^*, x'') + d(x', x^{*'})]\frac{[d(x^*, x'') + d(x', x'') + d(x^*, x'')]}{[d(x^*, x'')]}$$

$$\leq \alpha_1 d(x^*, x') + 2\alpha_3 d(x^*, x')$$

$$\leq (\alpha_1 + 2\alpha_3)d(x^*, x')$$

This is a contradiction. Hence$x^* = x'$ i.e. x^* is a unique fixed point of the mapping T.

This completes the proof.

Theorem 3.2. Let (X, d) be a complete cone b-metric space with constant coefficient $s \geq 1$ and let $\alpha_i \geq 0 (i = 1, 2, 3)$ and $T : X \times X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 \frac{(d(x, Ty))^2}{d(x, y) + d(y, Ty)} + \alpha_3 \frac{(d(y, Ty))^2}{d(x, y) + d(x, Ty)}$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 > 0$, $s\alpha_1 + s^2(1+s)\alpha_2 + s^3(2+s)\alpha_2 \leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a arbitrary point. Suppose there is an point $x_1 \in X$ such that $Tx_0 = x_1$. By Induction a sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1}(n \geq 0)$. If for some $n, x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and x_n respectively in theorem (3.2), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_n, Tx_n)}$$

$$+\alpha_3 \frac{(d(x_n, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_n)}$$

$$d(x_n, x_{n+1}) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}$$

$$+\alpha_3 \frac{(d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}$$

$$d(x_n, x_{n+1}) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{s^2(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}$$

$$+\alpha_3 \frac{(d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})}$$

$$d(x_n, x_{n+1}) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 s^2\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$

$$+\alpha_3 s^2\{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})\}$$

$$d(x_n, x_{n+1}) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 s^2\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$

$$+\alpha_3 s^2\{d(x_{n-1}, x_n) + s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}$$

$$(1 - \alpha_2 s^2 - \alpha_3 s^3) d(x_n, x_{n+1}) \leq (\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2(1 + s)) d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2(1 + s))}{(1 - \alpha_2 s^2 - \alpha_3 s^3)} d(x_{n-1}, x_n)$$

Since $s\alpha_1 + s^2(1 + s)\alpha_2 + s^3(2 + s)\alpha_2 \leq 1 \Rightarrow \frac{(\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2(1 + s))}{(1 - \alpha_2 s^2 - \alpha_3 s^3)} \leq \frac{1}{s} = k$ implies $d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n)$
Proceeding in the same manner up to n iteration, we have

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq ... \leq k^n d(x_0, x_1)$$

for $n \geq 1$ and letting $m > n$, we have

$$d(x_n, x_{n+m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})]$$

$$\leq sd(x_n, x_{n+1}) + s^2 d[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m})]$$

$$\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+m})]$$

$$\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4})$$

$$+ ... + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^m d(x_{n+m-1}, x_{n+m})$$

$$\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + s^3 k^{n+2} d(x_0, x_1) + s^4 k^{n+3} d(x_0, x_1)$$

$$+ ... + s^{m-1} k^{n+m-2} d(x_0, x_1) + s^m k^{n+m-1} d(x_0, x_1)$$

$$d(x_n, x_{n+m}) \leq sk^n (1 + sk + (sk)^2 + (sk)^3 + ... + (sk)^{m-1}) d(x_0, x_1)$$

$$d(x_n, x_{n+m}) \leq sk^n \frac{1 - (sk)^m}{1 - sk} d(x_0, x_1)$$

$$d(x_n, x_{n+m}) \leq sk^n \frac{1}{1 - sk} d(x_0, x_1)$$

Since $k \in \left[0, \frac{1}{s}\right) \Rightarrow sk^n \frac{1}{1 - sk} d(x_0, x_1) r \to \theta$ as $n \to \infty \forall m \in N^+$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Therefore sequence $\{x_n\}$ converses to a point $x^* \in X$. Now we prove that x^* is the unique fixed point of T.

$$d(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \leq s[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]$$

$$\leq sd(x^*, Tx_n) + sd(Tx_n, Tx^*)$$

$$\leq s\{d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2 \frac{[d(x_n, Tx^*)]^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3 \frac{[d(x^*, Tx^*)]^2}{d(x_n, x^*) + d(x_n, Tx^*)}\}$$

$$\leq s\{d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2 \frac{s^2[d(x_n, x^*) + d(x^*, Tx^*)]^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3 \frac{s^2[d(x^*, Tx^*)]^2}{d(x^*, Tx^*)}\}$$

$$\leq s\{d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2 s^2[d(x_n, x^*) + d(x^*, Tx^*)] + \alpha_3 s^2 d(x^*, Tx^*)\}$$

$$(1 - \alpha_2 s^2 - \alpha_3 s^2) d(x^*, Tx^*) \leq s^2 d(x^*, x_{n+1}) + s(\alpha_1 + \alpha_2 s^2) d(x^*, x_n)$$

$$d(x^*, Tx^*) \leq \frac{s^2}{(1 - \alpha_2 s^2 - \alpha_3 s^2)} d(x^*, x_{n+1}) + \frac{s(\alpha_1 + \alpha_2 s^2)}{(1 - \alpha_2 s^2 - \alpha_3 s^2)} d(x^*, x_n)$$

Since $\{x_n\}$ is a Cauchy sequence, therefore for any $c \in intP$, we select an $n_i \in N$ for all $n \geq n_i$ such that

$$d(x^*, Tx_n) << \frac{s^2 c}{2(1 - \alpha_2 s^2 - \alpha_3 s^2)}$$

and

$$d(x_n, x^*) << \frac{s(\alpha_1 + \alpha_2 s^2)}{2(1 - \alpha_2 s^2 - \alpha_3 s^2)}$$

Thus, for any $c \in intP, d(x^*, Tx^*) << c \forall n \geq n_i$. Therefore x^* is fixed point of T i.e. $x^* = Tx^*$

Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$. Now from theorem (3.2), we have

$$d(x^*, x') = d(Tx^*, Tx'') \leq \alpha_1 d(x^*, x'') + \alpha_2 \frac{[d(x^*, Tx'')]^2}{[d(x^*, x'') + d(x', Tx'')]}$$

$$+\alpha_3 \frac{[d(x', Tx'')]^2}{[d(x^*, x'') + d(x^*, Tx'')]}$$

$$d(x^*, x') \leq \alpha_1 d(x^*, x'') + \alpha_2 \frac{[d(x^*, x'')]^2}{[d(x^*, x'') + d(x', x'')]} + \alpha_3 \frac{[d(x', x'')]^2}{[d(x^*, x'') + d(x^*, x'')]}$$

$$d(x^*, x') \leq \alpha_1 d(x^*, x'') + \alpha_2 d(x^*, x'')$$

$$d(x^*, x') \leq (\alpha_1 + \alpha_2) d(x^*, x'')$$

This is a contradiction. Hence $x^* = x'$ i.e. x^* is a unique fixed point of the mapping T.

This completes the proof.

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