South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.13, No.2, 2017, pp. 63-74

PREOPEN SETS IN BISPACES

Amar Kumar Banerjee[1] and Pratap Kumar Saha[2]

[1]Department of Mathematics,
The University of Burdwan, Burdwan-713104, West Bengal, INDIA.
E-mail: akbanerjee@math.buruniv.ac.in, akbanerjee1971@gmail.com

[2]Department of Mathematics,
Behala College, Kolkata-700060, West Bengal, INDIA.
E-mail: pratapsaha2@gmail.com

(Received: June 28, 2017)

Abstract: The notion of preopen sets and precontinuity in a topological space was introduced by Mashhour et. al in 1982 [13]. Later the same was studied in a bitopological space in [7] and [9]. Here we have studied the idea of pairwise preopen sets (semi preopen) and pairwise precontinuity (semi precontinuity) in a more general structure of a bispace and investigate how far several results as valid in a bitopological space are affected in a bispace.

Keywords and Phrases: Bispace, preopen sets, pairwise preopen sets, semi preopen sets, pairwise semi preopen sets, precontinuity, semi precontinuity.

2010 Mathematics Subject Classification: 54A05, 54E55, 54E99.

1. Introduction

The notion of topological space was generalized to a bitopological space by J. C. Kelly [8] in 1963. Levine [12] introduced the idea of semi open sets and semi continuity and Mashhour et al. [13] introduced the concept of preopen sets and precontinuity in a topological space. Many works of generalization on bitopological spaces have been seen in [5], [14], [15] etc. Jelic [6] generalized the idea of preopen sets and precontinuity in bitopological space. Later Khedr et al.[9] and A.Kar et al. [7] further studied the same in a bitopological space.

The idea of a topological space was generalized to a σ-space (or simply space) by A. D. Alexandroff [1] in 1940 weakening the union requirements where only contable union of open sets were taken to be open. Later Lahiri and Das [11] gave the idea of a bispace generalizing the notion of bitopological spaces. The concept of semi

open sets and quasi open sets in the setting of more general structure of a bispace were studied in [2], [3]. In this paper we wish to study the idea of preopen sets, precontinuity, semi preopen sets and semi precontinuity in more general structure of a bispace. We have also investigated here how far several results as valid in bitopological space are affected in a bispace.

2. Preliminaries

Definition 2.1 [1]: A set X is called an Alexandroff space or simply a space if in it is chosen a system $\mathcal{F}$ of subsets satisfying the following axioms:

1. The intersection of a countable number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.
2. The union of a finite number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.
3. The void set ϕ is a set in $\mathcal{F}$.
4. The whole set X is a set in $\mathcal{F}$.

Sets of $\mathcal{F}$ are called closed sets. Their complementary sets are called open. It is clear that instead of closed sets in the definition of the space, one may put open sets with subject to the conditions of countable summability, finite intersectibility and the condition that X and void set ϕ should be open. The collection of all such open sets will sometimes be denoted by τ and the space by (X, τ). Note that a topological space is a space but in general τ is not a topology as can be easily seen by taking $X = R$, the set of real numbers and τ as the collection of all F_σ-sets in R.

Definition 2.2 [1]: To every set M of a space (X, τ) we correlate its closure $\overline{M}$, the intersection of all closed sets containing M. The closure of a set M will be denoted by τclM or simply clM when there is no confusion about τ.

Generally the closure of a set in a space may not be a closed set. The definition of limit point of a set is parallel as in the case of a topological space.
From the axioms, it easily follows that
1) $\overline{M \cup N} = \overline{M} \cup \overline{N}$; 2) $M \subset \overline{M}$ 3) $\bar{M} = \overline{\overline{M}}$; 4) $\overline{\phi} = \phi$; 5) $\overline{A} = A \cup A'$
where A' denotes the set of all limit points of A.

Definition 2.3 [10]: The interior of a set M in a space (X, τ) is defined as the union of all open sets contained in M and is denoted by $\tau - intM$ or $intM$ when there is no confusion about τ.

Definition 2.4 [8]: A set X on which are defined two arbitrary topologies is called a bitopological space and in denoted by (X, P, Q).

Definition 2.5 [11]: Let X be a nonempty set. If τ_1 and τ_2 be two collections of subsets of X such that (X, τ_1) and (X, τ_2) are two spaces, then X is called a bispace and is denoted by (X, τ_1, τ_2).

Definition 2.6 [13]: Let (X, τ) be a topological space. A subset A of X is said to be preopen if $A \subset int(clA)$.

Definition 2.7 [9]: A subset A of bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j) preopen (or briefly (i, j) preopen) if there exists $U \in \tau_i$ such that $A \subset U \subset \tau_j cl(A)$ or equivalently $A \subset \tau_i int(\tau_j cl(A))$.
A subset A is said to be pairwise preopen if it is (i, j) preopen for $i, j = 1, 2; i \neq j$.

3. Preopen sets:

Throughout our discussion, (X, τ_1, τ_2) or simply X stands for a bispace, R stands for the set of real numbers , Q stands for the set of rational numbers, P for the set of irrational numbers, N for the set of natural numbers and sets are always subsets of X unless otherwise stated.

In a topological space the following conditions (I) and (II) are equivalent.
Condition (I): $A \subset int(cl(A))$ and
Condition (II): there exists an open set U such that $A \subset U \subset cl(A)$.

But in a space (Alexandroff space) these two condition are not equivalent. In fact condition (I) is weaker than the condition (II) as shown in the example 3.1. In view of above we take condition (II) to define preopen sets in a space.

Definition 3.1: Let (X, τ) be a space. A subset A of X is said to be preopen if there exists an open set U such that $A \subset U \subset cl(A)$ and A is said to be weakly preopen if $A \subset int(cl(A))$.

Note that if A is open then $intA = A$. So $A = intA \subset int(clA)$ and hence A is weakly preopen. Also if A is open then the condition (II) holds if we take $U = A$.

So every open set is preopen also. But converse may not be true as shown in the following example.

Example 3.1: Let $X = [1, 2]$ and $\tau = \{X, \phi, F_i\}$ where F_i's are the countable subsets of irrational in $[1, 2]$. Let $A = [1, 2] - Q$. Then $cl(A) = X$. So A is weakly preopen, since condition (I) holds. Also condition (II) holds if we take $U = X$. Hence A is preopen also. But A is not open. Next, let us consider $B = ([1, 2] - Q) - \{\sqrt{2}\}$. Then $cl(B) = X - \{\sqrt{2}\}$ and $int(cl(B)) = B$. Therefore, $B \subset int(cl(B))$. Again, there does not exists any τ open set U such that $B \subset U \subset cl(B)$. So B is weakly preopen but not preopen.

In the definition of preopen sets in a bitopological space (X, τ_1, τ_2) Khedr et al.[9] used the following condition C_1 where as Kar et al. [7] used the condition C_2.
C_1 : there exists a $U \in \tau_i$ such that $A \subset U \subset \tau_j cl(A)$.
C_2 : $A \subset int_i(cl_j A)$
It can be checked that the conditions C_1 and C_2 are equivalent in a bitopological

space. But in case of a bispace (X, τ_1, τ_2) it can be easily checked that the condition C_1 implies the condition C_2. But converse may not be true as shown in the following example.

Example 3.2: Let $X = [0, 2]$ and $\tau_1 = \{X, \phi, F_i\}$, $\tau_2 = \{X, \phi, G_i\}$ where F_i's and G_i's are the countable subsets of $[0, 1] - Q$ and $[1, 2] - Q$ respectively. Then clearly (X, τ_1, τ_2) is a bispace which is not a bitopological space. Let $A = [0, 1] - Q$ then $\tau_2 clA = [0, 1] \cup ([1, 2] \cap Q) \neq X$. Since τ_1 open sets other than X are countable and A contain uncountable number of irrational numbers, there does not exists any τ_1 open set U such that $A \subset U \subset \tau_2 cl(A)$ holds. But $\tau_1 int(\tau_2 clA) = \tau_1 int([0, 1] \cup ([1, 2] \cap Q)) = [0, 1] - Q = A$. Therefore, $A \subset \tau_1 int(\tau_2 clA)$.

In view of above discussion, we give the definition of pairwise preopenness in a bispace as follows:

Definition 3.2(cf.[9]): Let (X, τ_1, τ_2) be a bispace and A be a subset of X then A is said to be (τ_i, τ_j) preopen (or briefly (i, j) preopen) if there exists $U \in \tau_i$ such that $A \subset U \subset \tau_j clA$, $i, j = 1, 2; i \neq j$.

A is called pairwise preopen if A is both (τ_1, τ_2) and (τ_2, τ_1) preopen.

Note 3.1: In [7], it is shown that pairwise preopenness in a bitopological space (X, τ_1, τ_2) does not imply the preopenness in the individual topological spaces (X, τ_1) and (X, τ_2). The same assertion is valid for pairwise preopen sets in a bispace as shown in the following example.

Example 3.3: Let $X = [1, 3], \tau_1 = \{X, \phi, G_i \cup \{\frac{5}{2}\}\}$ and $\tau_2 = \{X, \phi, F_i \cup \{\frac{3}{2}\}\}$ where G_i's and F_i's are countable subsets of irrational numbers in $[1, \sqrt{3}]$ and $[\sqrt{3}, 3]$ respectively. Then (X, τ_1, τ_2) is a bispace which is not a bitopological space. Let $A = \{\sqrt{3}\}$, then $\tau_1 clA = (X - P_1) - \{\frac{5}{2}\}$ where P_1 is set of all irrational numbers in $[1, \sqrt{3})$ and $\tau_2 clA = (X - P_2) - \{\frac{3}{2}\}$ where P_2 is set of all irrational numbers in $(\sqrt{3}, 3]$. Then $U = \{\sqrt{3}, \frac{5}{2}\}$ is a τ_1 open set and $V = \{\frac{3}{2}, \sqrt{3}\}$ is a τ_2 open set containing A and $A \subset U \subset \tau_2 clA$ and $A \subset V \subset \tau_1 clA$. But there does not exists any τ_1 open set U such that $A \subset U \subset \tau_1 clA$ holds and also there does not exists any τ_2 open set V such that $A \subset V \subset \tau_2 clA$ holds. So A is pairwise preopen but not individually preopen.

Note 3.2: Clearly every τ_i open set is (τ_i, τ_j) preopen but converse may not be true which is shown in the following example:

Example 3.4: Let $X = [0, 3]$, $\tau_1 = \{X, \phi, F_i \cup \{\sqrt{2}\}\}$ and $\tau_2 = \{X, \phi, G_i\}$ where F_i's are the countable subsets of rational number in $[0, 1]$ and G_i's are countable subsets of irrational number in $[2, 3]$. Then (X, τ_1, τ_2) is a bispace which is not a bitopo-

logical space. Let $A = \{0, 1\}$ which is not τ_1 open. Clearly $\tau_2 clA = [0, 2] \cup ([2, 3] \cap Q)$ and also the set $U = \{0, 1\} \cup \{\sqrt{2}\} \in \tau_1$. So $A \subset U \subset \tau_2 clA$. Thus A is (τ_1, τ_2) preopen but not τ_1 open.

Definition 3.3[3]: Let (X, τ_1, τ_2) be a bispace. Then a subset A of X is said to be τ_i semi open with respect to τ_j if and only if there exists a τ_i open set O such that $O \subset A \subset \tau_j clO$ $i, j = 1, 2; i \neq j$.

Definition 3.4 (cf.[9]): A subset A of a bispace (X, τ_1, τ_2) is said to be (τ_i, τ_j) semi preopen (or briefly (i, j) semi preopen) if there exists an (i, j) preopen set U such that $U \subset A \subset \tau_j clU$, $i, j = 1, 2; i \neq j$.

A is called pairwise semi preopen if A is both (τ_1, τ_2) and (τ_2, τ_1) semi preopen.

Clearly every τ_i semi open set with respect to τ_j is (i, j) semi preopen and similarly every (i, j) preopen set is (i, j) semi preopen but converse may not be true as shown in the following example.

Example 3.5: Example of a (i, j) semi preopen set which is not (i, j) preopen and τ_i semi open w.r. to τ_j.

Let X, τ_1 and τ_2 be as in example 3.4. Let $B = \{0, 1\} \cup \{\sqrt{3}\}$. Then B is not (τ_1, τ_2) preopen. But $A = \{0, 1\}$ is (τ_1, τ_2) preopen [by example 3.4] and $A \subset B \subset \tau_2 clA$. So B is (τ_1, τ_2) semi preopen. Again, B does not contain any τ_1 open set. So B can not be τ_1 semi open w.r. to τ_2.

Theorem 3.1: Let A be a subset in a bispace (X, τ_1, τ_2). (a) If there exists an (i, j) preopen set U such that $A \subset U \subset \tau_j clA$ then A is (i, j) preopen. (b) If there exists an (i, j) semi preopen set U such that $U \subset A \subset \tau_j clU$, then A is (i, j) semi preopen.

Proof: (a): Since U is (i, j) preopen, there exists a τ_i open set G such that $U \subset G \subset \tau_j clU$. So $A \subset U \subset G \subset \tau_j clU \subset \tau_j cl(\tau_j clA) = \tau_j clA$. This implies A is (i, j) preopen.
(b): Next, since U is (i, j) semi preopen there exists (i, j) preopen set V such that $V \subset U \subset \tau_j clV$. Therefore, $V \subset U \subset A \subset \tau_j clU \subset \tau_j cl(\tau_j clV) = \tau_j clV$. This implies A is (i, j) semi preopen.

Theorem 3.2: In a bispace (X, τ_1, τ_2). If $A \subset X$ is (i, j) preopen then for every τ_j closed set G containing A, $A \subset \tau_i intG, i, j = 1, 2; i \neq j$.

Proof: Let A be (i, j) preopen. Then there exists an $U \in \tau_i$ such that $A \subset U \subset \tau_j clA$. Let G be any τ_j closed set containing A. Then $\tau_j clA \subset G$ and so $A \subset U \subset \tau_i int(\tau_j clA) \subset \tau_i intG$.

Note 3.3: The converse of the theorem 3.2 is true in a bitopological space as seen

in theorem 3.8 of [7]. But unlikely in case of a bispace the converse may not be true as shown in the following example.

Example 3.6: Taking X, τ_1, τ_2 and A as in example 3.2. Then every τ_2 closed set G contains A and $\tau_1 intG = A$. So $A \subset \tau_1 intG$. But A is not (τ_1, τ_2) preopen.

Theorem 3.3: Countable union of (i, j) preopen [(i, j) semi preopen] sets is (i, j) preopen [(i, j) semi preopen].

Proof: Let $\{A_k : k = 1, 2, 3...\}$ be a countable collection of (i, j) preopen sets in the bispace (X, τ_1, τ_2). Let $A = \bigcup_{i=1}^{\infty} A_k$. Since each A_k is (i, j) preopen there exists τ_i open set U_k such that $A_k \subset U_k \subset \tau_j cl(A_k)$. Since $A_k \subset A$ for each k, $\tau_j cl(A_k) \subset \tau_j cl(A)$ for each k. So $A_k \subset U_k \subset \tau_j cl(A)$ for each k. This implies that $\bigcup_{i=1}^{\infty} A_k \subset \bigcup_{i=1}^{\infty} U_k \subset \tau_j cl(A)$ i.e., $A \subset U \subset \tau_j cl(A)$ where $U = \bigcup_{i=1}^{\infty} U_k$ is τ_i open.

The proof for the case that A is (i, j) semi preopen is similar.

Arbitrary union of (i, j) preopen set may not be (i, j) preopen as shown in the following examples.

Example 3.7: Take X, τ_1 and τ_2 as in example 3.2. Let $A_s = \{s\}$ where $s \in [0, 1] - Q$. Then A_s is τ_1 open and hence (τ_1, τ_2) preopen. But $\cup\{A_s : s \in [0, 1] - Q\} = [0, 1] - Q$ is not (τ_1, τ_2) preopen as shown in example 3.2.

Example 3.8: Let $X = [0, 3]$, $\tau_1 = \{X, \phi, F_i \cup \{\frac{3}{2}\}\}$ where F_i's are the countable subset of $[0, 1] - Q$ and $\tau_2 = \{X, \phi, G_i\}$ where G_i's are the countable subset of $[2, 3] - Q$. Obviously (X, τ_1, τ_2) is a bispace but not a bitopological space. Now consider a set $A_s = \{s\}$ where $s \in [0, 1] - Q$. Then A_s is not τ_1 open. Now $\tau_2 cl(A_s) = [0, 2] \cup ([2, 3] \cap Q)$. Also $A_s \cup \{\frac{3}{2}\}$ is a τ_1 open set and $A_s \subset A_s \cup \{\frac{3}{2}\} \subset \tau_2 cl(A_s)$. Therefore, A_s is (τ_1, τ_2) preopen but not τ_1 open. But $\cup\{A_s : s \in [0, 1] - Q\} = [0, 1] - Q$ is not (τ_1, τ_2) preopen, since $\tau_2 cl([0, 1] - Q) = [0, 2] \cup ([2, 3] \cap Q) \neq X$ and the τ_1 open set containing $[0, 1] - Q$ is only the set X.

Remark 3.1: In example 3.3 of [9] it is shown that intersection of two (i, j) preopen [(i, j) semi preopen] set in a bitopological space may not be (i, j) preopen [(i, j) semi preopen]. So the same assertion is true in a bispace.

Lemma 3.1: In a bispace (X, τ_1, τ_2), if $A \subset X$ and $B \in \tau_j$ then $\tau_j cl(A) \cap B \subset \tau_j cl(A \cap B)$, for $j = 1, 2$.

Proof: Let $x \in \tau_j cl(A) \cap B$ then $x \in B$ and $x \in \tau_j cl(A)$ where $\tau_j cl(A) = A \cup A^{\prime j}$, $A^{\prime j}$ being the set of all τ_j limit points of A. So $x \in A \cup A^{\prime j}$.
Case I : Now if $x \in A$ then $x \in A \cap B$, so $x \in \tau_j cl(A \cap B)$.

Case II : If $x \in A'^{j}$, then x is a τ_j limit point of A. Let V be any τ_j open set containing x then $V \cap B$ is also τ_j open set containing x. Since x is a τ_j limit point of A, $(V \cap B) \cap (A - \{x\}) \neq \phi$ that is $V \cap ((A \cap B) - \{x\}) \neq \phi$. So x is a τ_j limit point of $A \cap B$. Hence $x \in \tau_j cl(A \cap B)$. Hence the result is proved.

Theorem 3.4: In a bispace (X, τ_1, τ_2), if A is (i, j) preopen [(i, j) semi preopen] and $B \in \tau_1 \cap \tau_2$ then $A \cap B$ is (i, j) preopen [(i, j) semi preopen], $i \neq j, i, j = 1, 2$.

Proof: Since A is (i, j) preopen there exists τ_i open set U such that $A \subset U \subset \tau_j cl(A)$. Now $A \cap B \subset U \cap B \subset \tau_j cl(A) \cap B \subset \tau_j cl(A \cap B)$, since B is τ_j open. Again $U \cap B$ is τ_i open, since B is τ_i open also. Therefore, $A \cap B$ is (i, j) preopen. Next, let A be (i, j) semi preopen set. So there exists a (i, j) preopen set O such that $O \subset A \subset \tau_j cl(O)$. So $O \cap B \subset A \cap B \subset \tau_j cl(O) \cap B \subset \tau_j cl(O \cap B)$, since B is τ_j open. Since by first part of the theorem, $O \cap B$ is (i, j) preopen, $A \cap B$ is (i, j) semi preopen.

Note 3.4: Let $Y \subset X$ and let $\tau_{i/Y} = \{U \cap Y : U \in \tau_i\}; i = 1, 2$. Then $(Y, \tau_{1/Y}, \tau_{2/Y})$ is a bispace called subbispace of (X, τ_1, τ_2). As in the case of a topological space, it can be checked that if $A \subset Y \subset X$ then $\tau_{i/Y} cl(A) = \tau_i cl(A) \cap Y$.

Theorem 3.5: If $A \subset Y \subset X$ in a bispace (X, τ_1, τ_2) and if A is (τ_i, τ_j) preopen [(τ_i, τ_j) semi preopen] in X then A is $(\tau_{i/Y}, \tau_{j/Y})$ preopen [$(\tau_{i/Y}, \tau_{j/Y})$ semi preopen] in $(Y, \tau_{i/Y}, \tau_{j/Y})$ $i, j = 1, 2, i \neq j$.If, in addition, $Y \in \tau_i$, then the converse holds.

Proof: Let A be (τ_i, τ_j) preopen in X then there exists an $U \in \tau_i$ such that $A \subset U \subset \tau_j cl(A)$. Therefore, $A \subset U \cap Y \subset \tau_j cl(A) \cap Y = \tau_{j/Y} cl(A)$. Since $U \cap Y$ is $\tau_{i/Y}$ open, A is $(\tau_{i/Y}, \tau_{j/Y})$ preopen. Conversely let A be $(\tau_{i/Y}, \tau_{j/Y})$ preopen. So there exists $\tau_{i/Y}$ open set U such that $A \subset U \subset \tau_{j/Y} cl(A)$. Since Y is τ_i open, U is τ_i open. Also $\tau_{j/Y} cl(A) = \tau_j cl(A) \cap Y \subset \tau_j cl(A)$. So $A \subset U \subset \tau_j cl(A)$ where U is τ_i open. So A is (τ_i, τ_j) preopen. The proof for the case of semi preopenness is similar.

Definition 3.5 (cf. [9]): A subset $A \subset X$ is said to be (τ_i, τ_j) or (i, j) preclosed [resp. (τ_i, τ_j) or (i, j) semi preclosed] if its complement is (τ_i, τ_j) or (i, j) preopen [resp. (τ_i, τ_j) or (i, j) semi preopen]. A is called pairwise preclosed [resp. pairwise semi preclosed] if its complement is pairwise preopen [resp. pairwise semi preopen].

Definition 3.6 (cf. [9]): The intersection of all (i, j) preclosed [resp. (i, j) semi preclosed] sets containing A in (X, τ_1, τ_2) is called (i, j) preclosure [resp. (i, j) semi preclosure] of A and is denoted by $(i, j)pcl(A)$ or $(\tau_i, \tau_j)pcl(A)$ [resp. $(i, j)spcl(A)$ or $(\tau_i, \tau_j)spcl(A)$], $i, j = 1, 2; i \neq j$.

Theorem 3.6: Let A and B be subsets of X in a bispace (X, τ_1, τ_2) and let $x \in X$, then
(i) $x \in (i,j)pcl(A)$ if and only if $A \cap U \neq \phi$ for every (i,j) preopen set U containing $x, i, j = 1, 2; i \neq j$.
(ii) if $A \subset B$ then $(i,j)pcl(A) \subset (i,j)pcl(B)$.

Proof: (i) Let U be a (i,j) preopen set containing x such that $A \cap U = \phi$. This implies that $A \subset X - U$, where $X - U$ is (i,j) preclosed. So $(i,j)pcl(A) \subset X - U$. Since $x \notin X - U$, $x \notin (i,j)pcl(A)$. Therefore, if $x \in (i,j)pcl(A)$ then every (i,j) preopen set U intersects A. Conversely, let every (i,j) preopen set containing x intersects A and let $x \notin (i,j)pcl(A)$. Then there exists a (i,j) preclosed set F containing A such that $x \notin F$. So $X - F$ is an (i,j) preopen set containing x such that $(X - F) \cap A = \phi$ which is a contradiction. Hence $x \in (i,j)pcl(A)$.
The proof of (ii) is straight forward and so is omitted.

Theorem 3.7: Let A and B be subsets of X in a bispace (X, τ_1, τ_2) and let $x \in X$, then

(i) $x \in (i,j)spcl(A)$ if and only if $A \cap U \neq \phi$ for every (i,j) semi preopen set U containing $x, i, j = 1, 2; i \neq j$.

(ii) if $A \subset B$ then $(i,j)spcl(A) \subset (i,j)spcl(B)$.

Proof : Proof is similar as the proof of theorem 3.6.

4. Pairwise precontinuity :
Definition 4.1 [2]: A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise continuous or simply continuous if the induced functions $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ both are continuous.

Definition 4.2 (cf. [9]): A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise open or simply open if the induced functions $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ both are open.

Definition 4.3 (cf. [9]): A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise precontinuous if the inverse image of each σ_i open set of Y is (i,j) preopen in X where $i \neq j; i, j = 1, 2$.

Clearly every pairwise continuous function is pairwise precontinuous. But converse may not be true even in the case of a bitopological space as seen in example 4.2[9].

Note 4.1: If (X, τ) and (Y, σ) be two σ-spaces and $f : X \to Y$ is a continuous function then as in the case of a topological space the condition $f(\tau cl(A)) \subset \sigma cl(f(A))$ holds good. But converse of this result may not hold as shown in the following

example although the converse also holds in case of topological spaces.

Example 4.1: Let $X = [0,1]$ and $Y = [1,2]$ and let $\tau = \{\phi, X, F_i\}$ and $\sigma = \{\phi, X, G_i\}$ where F_i's and G_i's are countable subsets of irrational numbers in X and Y respectively. Then (X,τ) and (Y,σ) are two spaces but not topological spaces. Now consider a function $f : (X,\tau) \to (Y,\sigma)$ such that
$f(x) = \sqrt{2}$ if x is irrational
$= \frac{3}{2}$ if x is rational,
Now $\{\sqrt{2}\} \in \sigma$ and $f^{-1}(\{\sqrt{2}\}) = [0,1] - Q$ which is not τ open. So f is not continuous. Let A be any subset of X. Then the following cases may arise.
Case I: A contains only rational numbers. Then $\tau cl(A) = [0,1] \cap Q$ and $f(\tau cl(A)) = \{\frac{3}{2}\}$. Also $f(A) = \{\frac{3}{2}\}$ and $\sigma cl(f(A)) = [1,2] \cap Q$. So $f(\tau cl(A)) \subset \sigma cl(f(A))$.
Case II: A contains only irrational numbers. Then $\tau cl(A) = ([0,1] \cap Q) \cup A$ and $f(\tau cl(A)) = \{\frac{3}{2}, \sqrt{2}\}$, $f(A) = \{\sqrt{2}\}$ and $\sigma cl(f(A)) = ([1,2] \cap Q) \cup \{\sqrt{2}\}$. So $f(\tau cl(A)) \subset \sigma cl(f(A))$.
Case III: A contains rational and irrational numbers of X. Then let $A = A_1 \cup A_2$ where A_1 contains only rational points of A and A_2 contains only irrational points of A. So $\tau cl(A) = \tau cl(A_1 \cup A_2) = \tau cl(A_1) \cup \tau cl(A_2)$. Now $\tau cl(A_1) = [0,1] \cap Q$ and $\tau cl(A_2) = ([0,1] \cap Q) \cup A_2$. Therefore, $\tau cl(A) = ([0,1] \cap Q) \cup A_2$. Again, $f(\tau cl(A)) = \{\frac{3}{2}, \sqrt{2}\}$; $f(A) = \{\frac{3}{2}, \sqrt{2}\}$ and $\sigma cl(f(A)) = ([1,2] \cap Q) \cup \{\sqrt{2}\}$. So $f(\tau cl(A)) \subset \sigma cl(f(A))$.

Theorem 4.1: Let a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be pairwise continuous and pairwise open. If A is a (i,j) preopen[(i,j) semi preopen] subset of X, then $f(A)$ is (i,j) preopen[(i,j) semi preopen] in Y.

Proof: The proof is similar as in the proof of theorem 4.1[9] and so is omitted.

Theorem 4.2: Let a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be pairwise precontinuous and pairwise open. If A is a (i,j) preopen[(i,j) semi preopen] subset of Y, then $f^{-1}(A)$ is (i,j) preopen [(i,j) semi preopen] in X.

Proof: The proof is similar as in the proof of theorem 4.2[9] and so is omitted.

Theorem 4.3: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is pairwise precontinuous if and only if the inverse image of each σ_i closed set of Y is (i,j) preclosed in X.

Proof: Let f be pairwise precontinuous and let A be σ_i closed set in Y. Therefore $Y - A$ is σ_i open and so $f^{-1}(Y - A)$ is (i,j) preopen in X which implies that $X - f^{-1}(A)$ is (i,j) preopen in X. Hence $f^{-1}(A)$ is (i,j) preclosed. Conversely, let A be a σ_i open set in Y. So $Y - A$ is σ_i closed set in Y. Then $f^{-1}(Y - A)$ is (i,j) preclosed set in X. Therefore, $X - f^{-1}(A)$ is (i,j) preclosed in X. So,

$f^{-1}(A)$ is (i,j) preopen in X. Thus f is (i,j) precontinuous and hence pairwise precontinuous.

Theorem 4.4: Let a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be pairwise precontinuous. Then the following properties hold:
(i) for each $x \in X$ and each $V \in \sigma_i$ containing $f(x)$, there exists an (i,j) preopen set U of X containing x such that $f(U) \subset V$.
(ii) $f((i,j)pcl(A)) \subset \sigma_i cl(f(A))$ for every subset A of X.
(iii) $(i,j)pclf^{-1}(B) \subset f^{-1}(\sigma_i cl(B))$ for every subset B of Y.

Proof: (i) Let $V \in \sigma_i$ containing $f(x)$. So $x \in f^{-1}(V)$. Since f is pairwise precontinuous, $f^{-1}(V)$ is (i,j) preopen set containing x. Let $U = f^{-1}(V)$ then $f(U) = V \subset V$.
(ii) Let $p \in f((i,j)pcl(A))$. Clearly $A \subset (i,j)pcl(A)$. Now if $p \in f(A)$ then $p \in \sigma_i cl(f(A))$. So let $p \notin f(A)$ and $p = f(q)$ where $q \in (i,j)pcl(A)$ but $q \notin A$. Let U be any σ_i open set containing p. Then $f^{-1}(U)$ is (i,j) preopen in X containing q. Since $q \in (i,j)pcl(A)$, $A \cap f^{-1}(U) \neq \phi$ [by theorem 3.6(i)]. Let $z \in A \cap f^{-1}(U)$. Then $f(z) \in f(A) \cap U$. But $f(z) \neq p$, since $p \notin f(A)$. So U intersect $f(A)$ in some point $f(z)$ other than p. Then $p \in \sigma_i cl(f(A))$. Therefore, $f((i,j)pcl(A)) \subset \sigma_i cl(f(A))$.
(iii) Let B be any subset of Y. Let $f^{-1}(B) = A$. Then by (ii), $f((i,j)pcl(A)) \subset \sigma_i cl(f(A)) = \sigma_i clf(f^{-1}(B)) = \sigma_i cl(B \cap f(X)) \subset \sigma_i cl(B)$. So $(i,j)pcl(A) \subset f^{-1}(\sigma_i cl(B))$ i.e., $(i,j)pclf^{-1}(B) \subset f^{-1}(\sigma_i cl(B))$.

Note 4.2: In all cases converse is true in the case of bitopological space as seen in theorem 4.3[9].

Theorem 4.5: If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise precontinuous and $A \in \tau_1 \cap \tau_2$. Then $f : (A, \tau_{1/A}, \tau_{2/A}) \to (Y, \sigma_1, \sigma_2)$ is pairwise precontinuous.

Proof: Let U be a σ_i open set in Y. Then $f^{-1}(U)$ is (τ_i, τ_j) preopen in X. So by theorem 3.4, $A \cap f^{-1}(U)$ is (τ_i, τ_j) preopen. Since $A \cap f^{-1}(U) \subset A$, by theorem 3.5, $A \cap f^{-1}(U)$ is $(\tau_{i/A}, \tau_{j/A})$ preopen in $(A, \tau_{1/A}, \tau_{2/A})$. Hence $f : (A, \tau_{1/A}, \tau_{2/A}) \to (Y, \sigma_1, \sigma_2)$ is $(\tau_{i/A}, \tau_{j/A})$ precontinuous and hence pairwise precontinuous.

Theorem 4.6: Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be a (i,j) precontinuous mapping such that the following condition 'C' is satisfied.

C : $f(\tau_j clf^{-1}(U^*)) = U^*$ for every τ_i^* open set U^*.

Then if $\{x_\alpha : \alpha \in D\}$ is a net converging to x in (X, τ_i) then $\{f(x_\alpha) : \alpha \in D\}$ converges to $f(x)$ in (X^*, τ_i^*).

Proof: Let U^* be a τ_i^* open set in X^* containing $f(x)$. So $x \in f^{-1}(U^*)$. Since

f is (i,j) precontinuous, $f^{-1}(U^*)$ is (i,j) preopen. So there exists a τ_i open set U such that $x \in f^{-1}(U^*) \subset U \subset \tau_j cl f^{-1}(U^*)$. Since $\{x_\alpha : \alpha \in D\}$ converges to x in (X, τ_i) there exists $\alpha_o \in D$ such that $x_\alpha \in U$, for all $\alpha \geq \alpha_o$. So $f(x_\alpha) \in f(U) \subset f(\tau_j cl f^{-1}(U^*)) = U^*$ [by the condition C]. Hence $\{f(x_\alpha) : \alpha \in D\}$ is eventually in every open set U^* in (X^*, τ_i^*). So $\{f(x_\alpha) : \alpha \in D\}$ converges to $f(x)$ in (X^*, τ_i^*).

5. Pairwise semi precontinuity (sp-continuity):

Definition 5.1 (cf. [9]): A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise semi precontinuous or pairwise sp-continuous (resp. pairwise semi continuous) if the inverse image of each σ_i open set of Y is (i,j) semi preopen (resp. (i,j) semi open) in X where $i \neq j, i, j = 1, 2$.

Remark 5.1: As in the case of bitopological space [9], it can be easily checked that pairwise continuity implies pairwise semi continuity and pairwise precontinuity. Also pairwise semi continuity implies pairwise sp-continuity and pairwise precontinuity implies pairwise sp-continuity. However the reverse implications are not true in a bispace even in case of a bitopological space also which is seen in [9].

Theorem 5.1: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is pairwise sp-continuous if and only if the inverse image of each σ_i closed set of Y is (i,j) semi preclosed in X.

Proof : Proof is similar to the proof of theorem 4.3.

Theorem 5.2: Let a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be pairwise sp-continuous. Then the following properties hold.

(i) for each $x \in X$ and each $V \in \sigma_i$ containing $f(x)$, there exists an (i,j) semi preopen set U of X containing x such that $f(U) \subset V$.
(ii) $f((i,j)spcl(A)) \subset \sigma_i cl(f(A))$ for every subset A of X.
(iii) $(i,j)spcl f^{-1}(B) \subset f^{-1}(\sigma_i cl(B))$ for every subset B of Y.
Proof: Proof is similar to the proof of theorem 4.4.

Theorem 5.3: If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise sp-continuous and $A \in \tau_1 \cap \tau_2$. Then $f : (A, \tau_{1/A}, \tau_{2/A}) \to (Y, \sigma_1, \sigma_2)$ is pairwise sp-continuous.

Proof : Proof is similar to the proof of theorem 4.5.

References

[1] Alexandroff A. D., Additive set functions in abstract spaces, (a) Mat. Sb. (N. S), 8: 50 (1940), 307-348.(English,Russian Summary). (b) ibid, 9: 51(1941), 563-628, (English,Russian Summary).

[2] Banerjee A. K. and Saha P. K., Bispace Group, Int. J. of Math. Sci. and Engg. Appl.(IJMSEA); Vol.5 No V(2011) pp. 41-47.

[3] Banerjee A. K. and Saha P. K., Semi open sets in bispaces, CUBO A Mathematical Journal; Vol. 17, No. 01,(2015),99-106.

[4] Banerjee A. K. and Saha P. K., Quasi open sets in bispaces, General Math. Notes, Vol. 30, No. 2, (2015), pp. 1-9.

[5] Bose S., Semi open sets, Semi continuity and Semi open mapping in bitopological spaces, Bull. Cal. Math. Soc. 73, (1981), 237-246.

[6] Jelic M., A decomposition of pairwise continuity, J. Inst. Math. Comput. Sci. Math. Soc., 3, (1990), 25-29.

[7] Kar A. and Bhattacharyya P., Bitopological preopen sets, Precontinuity and Preopen Mappings, Indian J. of Math., 34 ,(1992), 295-309.

[8] Kelley J.C., Bitopological spaces, Proc. London. Math. Soc., 13(1963), 71-89.

[9] Khedr F. H. and Areefi S. M., Precontinuity and semi precontinuity in bitopological spaces, Indian J. of Pure appl. Math., 23(9),(1992), 625-633.

[10] Lahiri B.K. and Das P., Semi Open set in a space, Sains malaysiana 24(4), (1995), 1-11.

[11] Lahiri B.K. and Das P., Certain bitopological concept in a Bispace, Soochow J. of Math. Vol. 27. No.2,(2001), pp.175-185.

[12] Levine N., Semi Open sets and Semi Continuity in topological spaces, Amer. Math. Monthly 70(1963), 36-41.

[13] Mashhour A. S., Monsef M. E. Abd. El. and Deeb S. N. E., On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt 53,(1982), 47-53.

[14] Pervin W.J., Connectedness in Bitopological spaces, Ind. Math. 29, (1967), 369-372.

[15] Reilly I. L., On bitopological separation properties, Nanta Math., 5 (1972), 14-25.