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RAMANUJAN'S EISENSTEIN SERIES OF LEVEL 3 AND 6, ITS APPLICATION

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Abstract: In this paper, we give an elementary proof of Eisenstein series of level 3 and 6, recorded by Ramanujan. Further, as an application of these Eisenstein series identities, we establish certain incomplete elliptic integrals of the first kind, involving theta-functions, which are analogous to integrals recorded by Ramanujan in his Lost Notebook.

Keywords and Phrases: Dedekind eta-function, Theta-functions, Eisenstein series, Bilateral basic hypergeometric series, Incomplete elliptic integrals.

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1. Introduction

Let $P(q)$ denote the Eisenstein series of weight 2, defined by

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}. \tag{1.1}$$

For any positive integer n, let P_n be defined by

$$P_n := P(q^n).$$

In Chapter 21 of his Second Notebook [10], Ramanujan recorded many identities relating certain linear combination of Eisenstein series with theta-functions. For a proof of these identities one may refer [3, pp. 454−488]. In his famous paper [12], Ramanujan used some of these identities to obtain series for $1/\pi$. For more details one may refer [6, Chapter 5]. In Section 2 of this paper, we prove following equalities recorded by Ramanujan which involve P_1, P_2, P_3 and P_6:

$$-\frac{1}{2}[P_1 \ -3P_3] = \left\{1+6\left(\frac{q}{1-q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^5}{1-q^5}+\cdots\right)\right\}^2 \tag{1.2}$$

$$= \left\{\frac{f_2^6 f_3}{f_1^3 f_6^2}+3q\frac{f_1 f_6^6}{f_2^2 f_3^3}\right\}^2 \tag{1.3}$$

$$= \left\{\frac{f_1^{12}+27qf_3^{12}}{f_1^3 f_3^3}\right\}^{2/3}. \tag{1.4}$$

$$-\frac{1}{2}[P_2-3P_6] = \left\{\frac{f_2^{15}f_3^2f_{12}^2}{f_1^6f_4^6f_6^5}+3\frac{f_1^2f_4^2f_6^{15}}{f_2^5f_3^6f_{12}^6}\right\}^2 \tag{1.5}$$

$$= \frac{f_2^{10}f_6^{10}}{f_1^4f_3^4f_4^4f_{12}^4}-4q\frac{f_1^2f_3^2f_4^2f_{12}^2}{f_2^2f_6^2}, \tag{1.6}$$

where $f_n := \prod_{k=1}^{\infty}(1-q^{kn})$.

B. C. Berndt have proved (1.2)−(1.6) in [3, pp. 460−463], using the identity (2.3) of [3, p. 459]. Our proof depends on the following identity due to W. N. Bailey [2, p. 159, Eq. 5]:

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2}-\frac{bq^n}{(1-bq^n)^2}\right] = af_6^6\frac{f\left(-ab,\frac{-q}{ab}\right)f\left(\frac{-b}{a},\frac{-aq}{b}\right)}{f^2\left(-a,\frac{-q}{a}\right)f^2\left(-b,\frac{-q}{b}\right)}, \tag{1.7}$$

where $f(a,b)$ is the Ramanujan's equivalent of Jacobi's theta-function, defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}$$
$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab|<1,$$

with $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$. The identity (1.7) is a particular case of W. N. Bailey's ${}_6\psi_6$ very-well poised summation formula [1]. For details see [14].

In his Lost Notebook [11, pp. 51−53], Ramanujan recorded many identities involving integrals of theta-functions and incomplete elliptic integral of the first kind. For example:

$$If\ v := q\frac{f_1^4 f_{14}^4}{f_2^4 f_7^4}\ with\ c = \frac{\sqrt{13+16\sqrt{2}}}{7},\ then$$

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt = \frac{1}{\sqrt{8\sqrt{2}}}\int_{2\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} \frac{d\varphi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}. \quad (1.8)$$

S. Raghavan and S. S. Rangachari [9], have proved all these identities of Ramanujan. Their proofs depend on the theory of modular forms. Berndt, H. H. Chan and S.-S. Huang [5] have proved all the identities of Ramanujan using the tools known to Ramanujan. In Section 3 of this paper, we establish certain new identities which are analogous to (1.8). We close this section by recalling certain identities which we need in sequel.

Certain special cases of $f(a,b)$ are given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Following Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty.$$

One can easily show that

$$\varphi(q) = \frac{{f_2}^5}{{f_1}^2 {f_4}^2}, \qquad \varphi(-q) = \frac{{f_1}^2}{f_2}, \qquad \psi(q) = \frac{{f_2}^2}{f_1}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$\chi(q)=\frac{f_2^2}{f_1f_4},\qquad \chi(-q)=\frac{f_1}{f_2}\qquad\text{and}\qquad f(q)=\frac{{f_2}^3}{f_1f_4}. \tag{1.9}$$

The Gauss hypergeometric series is denoted by ${}_2F_1(a,b;c;z)$ and is defined by

$${}_2F_1(a,b;c;z):=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots.$$

Suppose that

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)} \tag{1.10}$$

holds for some positive integer n. The relation between α and β induced by the equation (1.10) is called a modular equation of degree n and in such equation we say that β is of nth degree over α. The multiplier m of the modular equation is defined by

$$m:=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}.$$

If β has degree 3 over α then from [3, pp. 232, Eq. (5.1)], we have

$$\left(\frac{\beta^3}{\alpha}\right)^{1/8}=\frac{m-1}{2},\quad \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}=\frac{m+1}{2},$$

$$\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}=\frac{3-m}{2m}\quad and\quad \left(\frac{\alpha^3}{\beta}\right)^{1/8}=\frac{3+m}{2m}. \tag{1.11}$$

Transforming the above modular equations into theta function identities by using [3, pp. 122−123, Entry 10(i)(iii), 11(i)], we find that

$$\frac{\varphi^2(q)}{\varphi^2(q^3)}-1=4q\frac{\varphi(q)\psi^3(q^3)}{\psi(q)\varphi^3(q^3)},\qquad \frac{\varphi^2(q)}{\varphi^2(q^3)}+1=2\frac{\varphi(q)\varphi^3(-q^6)}{\varphi^3(q^3)\varphi(-q^2)},$$

$$3-\frac{\varphi^2(q)}{\varphi^2(q^3)}=\frac{2\varphi^3(-q^2)}{\varphi(q)\varphi(q^3)\varphi(-q^6)}\quad and\quad 3+\frac{\varphi^2(q)}{\varphi^2(q^3)}=\frac{4\psi(q)}{\varphi(q)\varphi(q^3)\psi(q^3)}. \tag{1.12}$$

2. Ramanujan's Eisenstein series of level 3 and 6

Theorem 2.1. *We have*

$$-P_1+P_2+3P_3-3P_6=-24q\psi^2(q)\psi^2(q^3). \tag{2.1}$$

Proof. Replacing q by q^6 in (1.7), we obtain

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^{6n}}{(1-aq^{6n})^2}-\frac{bq^{6n}}{(1-bq^{6n})^2}\right]=af_6^6\frac{f\left(-ab,\frac{-q^6}{ab}\right)f\left(\frac{-b}{a},\frac{-aq^6}{b}\right)}{f^2\left(-a,\frac{-q^6}{a}\right)f^2\left(-b,\frac{-q^6}{b}\right)}. \tag{2.2}$$

Setting $a=q^3$ and $b=-1$ in (2.2), we find that

$$2\sum_{n=0}^{\infty}\frac{q^{6n+3}}{(1-q^{6n+3})^2}+\sum_{n=-\infty}^{\infty}\frac{q^{6n}}{(1+q^{6n})^2}=\frac{\varphi^4(q^3)}{4}. \tag{2.3}$$

Setting $a=q$ and $b=-1$ in (2.2), we find that

$$\sum_{n=0}^{\infty}\frac{q^{6n+1}}{(1-q^{6n+1})^2}+\sum_{n=0}^{\infty}\frac{q^{6n+5}}{(1-q^{6n+5})^2}+\sum_{n=-\infty}^{\infty}\frac{q^{6n}}{(1+q^{6n})^2}=\frac{\varphi(q)\varphi(q^3)\varphi^3(-q^3)}{4\varphi(-q)}.$$

Subtracting the above from (2.3), we obtain

$$\sum_{n=0}^{\infty}\left[\frac{q^{2n+1}}{(1-q^{2n+1})^2}-3\frac{q^{6n+3}}{(1-q^{6n+3})^2}\right]=-\frac{\varphi^4(q^3)}{4}\left[\frac{\varphi(q)\varphi^3(-q^3)}{\varphi(-q)\varphi^3(q^3)}-1\right]. \tag{2.4}$$

If β is of third degree over α and m is the multiplier for degree 3, then from Entry 10(i) and (ii) of Chapter 17 [3, p. 122], we have

$$\frac{\varphi(q)\varphi^3(-q^3)}{\varphi(-q)\varphi^3(q^3)}-1=\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}-1.$$

Employing (1.10) in the above, we find that

$$\begin{aligned}\frac{\varphi(q)\varphi^3(-q^3)}{\varphi(-q)\varphi^3(q^3)}-1&=\left(\frac{m+1}{2}\right)^2-1\\&=\frac{(m+3)(m-1)}{4}\\&=m(\alpha\beta)^{1/4}.\end{aligned}$$

Employing [3, p. 123, Entry 11(iii)] in the above and then using $\psi^2(q)=\varphi(q)\psi(q^2)$, we obtain

$$\frac{\varphi^4(q^3)}{4}\left[\frac{\varphi(q)\varphi^3(-q^3)}{\varphi(-q)\varphi^3(q^3)}-1\right]=q\psi^2(q)\psi^2(q^3).$$

Using this in the right-hand side of (2.5), expanding each summands of the resulting identity into geometric series, interchanging the order of summation and summing the inner geometric series and then using the definition of P_n, we obtain (2.1).

Theorem 2.2. *We have*

$$P_1 - 4P_2 - 3P_3 + 12P_6 = 6\varphi^2(-q)\varphi^2(-q^3). \tag{2.5}$$

Proof. Setting $a = -q^2$ and $b = -q^6$ in (2.2) and then after some simplification, we find that

$$\sum_{n=0}^{\infty} \frac{q^{2n+2}}{(1+q^{2n+2})^2} - 3\sum_{n=0}^{\infty} \frac{q^{6n+6}}{(1+q^{6n+6})^2} = \frac{1}{4} - \frac{\varphi^2(-q^2)\varphi^2(-q^6)}{4}.$$

Replacing q by $q^{1/2}$ in the above, expanding each summands of the resulting identity into geometric series, interchanging the order of summation and summing the inner geometric series and then using the definition of P_n, we obtain (2.6).

Proof of (1.2). The following theta function identity is equivalent to Ramanujan's modular equation of degree 3 [3, p. 230, Entry 5(ii)]:

$$4q\psi(q^2)\psi(q^6) = \varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3). \tag{2.6}$$

Multiplying the above by $4\varphi(q)\varphi(q^3)$ and then employing $\psi^2(q) = \varphi(q)\psi(q^2)$, we obtain

$$16q\psi^2(q)\psi^2(q^3) = 4\varphi^2(q)\varphi^2(q^3) - 4\varphi(q)\varphi(-q)\varphi(q^3)\varphi(-q^3).$$

Adding $\varphi^2(-q)\varphi^2(-q^3)$ to both sides of the above, we find that

$$\varphi^2(-q)\varphi^2(-q^3) + 16q\psi^2(q)\psi^2(q^3) = \left(2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3)\right)^2. \tag{2.7}$$

Using [3, p. 223, Entry 3(ii)] in the above, we have

$$2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3) = 1 + 6\left(\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \cdots\right). \tag{2.8}$$

Multiplying (2.1) by 4 and then adding the resulting identity with (2.6), we obtain

$$-P_1 + 3P_3 = 2\left(\varphi^2(-q)\varphi^2(-q^3) + 16q\psi^2(q)\psi^2(q^3)\right).$$

Using (2.7) in the above, we find that

$$-P_1 + 3P_3 = 2\left(2\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3)\right)^2. \tag{2.9}$$

Employing (2.8) in the above and then using the definition of P_n, we obtain (1.2).

Proof of (1.3). From [3, p. 462, Eq. 3.5], we have

$$\frac{\psi^4(q)+3q\psi^4(q^3)}{\psi(q)\psi(q^3)}=\varphi(q)\varphi(q^3)+4q\psi(q^2)\psi(q^6). \tag{2.10}$$

From (2.6), we have

$$2\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)=\varphi(q)\varphi(q^3)+4q\psi(q^2)\psi(q^6).$$

Using (2.7) in the above, we find that

$$2\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)=\frac{\psi^4(q)+3q\psi^4(q^3)}{\psi(q)\psi(q^3)}.$$

Using this in (2.9) and then using the definition of P_n, we obtain (1.3).

Proof of (1.4). From [3, p. 463], we have

$$\frac{\psi^4(q)+3q\psi^4(q^3)}{\psi(q)\psi(q^3)}=\left\{\frac{f_1^{12}+27qf_3^{12}}{f_1^3f_3^3}\right\}^{2/3}.$$

Using this in (1.3), we find (1.4).

Proof of (1.6). Adding (2.1) and (2.5), and using the definition of P_n, we find that

$$1+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-36\sum_{k=1}^{\infty}\frac{kq^{6k}}{1-q^{6k}}=4q\psi^2(q)\psi^2(q^3)+\varphi^2(-q)\varphi^2(-q^3). \tag{2.11}$$

Multiplying (2.6) by $\varphi(q)\varphi(q^3)$, and $\varphi(-q)\varphi(-q^3)$, and then adding the resulting identities, we find that

$$4q\psi^2(q)\psi^2(q^3)+\varphi^2(-q)\varphi^2(-q^3)=\varphi^2(q)\varphi^2(q^3)-4q\psi^2(-q)\psi^2(-q^3).$$

Using this in (2.11), we obtain (1.6).

Proof of (1.5). Let β has degree 3 over α and m is multiplier of degree 3, then from (1.10), we deduce that

$$m^2-1=4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}$$

and

$$\frac{9}{m^2}-1=6\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}.$$

Multiplying the above two, we find that

$$m^2+\frac{9}{m^2}=10-16\left(\alpha\beta(1-\alpha)(1-\beta)\right)^{1/4}.$$

Converting this formula into theta function identity and with the aid of [3, Ch. 17, Entry 10, 11, 12], we obtain

$$\varphi^8(q)+9\varphi^8(q^3)=10\varphi^4(q)\varphi^4(q^3)-64q\varphi^2(q)\psi^2(-q)\varphi^2(q^3)\psi^2(-q^3).$$

This implies

$$\left[\frac{\varphi^4(q)+3\varphi^4(q^3)}{4\varphi(q)\varphi(q^3)}\right]^2=\varphi^2(q)\varphi^2(q^3)-4q\psi^2(-q)\psi^2(-q^3).$$

Replacing q by $-q$ in (2.11) and then using the above in the resulting identity, we obtain (1.5).

3. Incomplete elliptic integrals of the first kind

Theorem 3.1. *Let* $v:=\dfrac{\varphi(-q)}{\varphi(-q^3)}$. *Then*

$$\frac{3}{2}\int_0^q\frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)\ \ dt\ \ =\ \ \int_0^{\sin^{-1}(v^2)}\frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\ d\theta.$$

Proof. By logarithmically differentiating v with respect to q, we find that

$$\frac{q}{v}\frac{dv}{dq}=\frac{1}{12}[P_1-P_2-3P_3+3P_6],$$

where P_n is as in (1.1). Using (2.1) in the above, we deduce that

$$\frac{1}{v}\frac{dv}{dq}=2q^{1/2}\psi^4(q^3)\left\{\frac{\psi^2(q)}{q^{1/2}\ \psi^2(q^3)}\right\}. \tag{3.1}$$

Replacing q by $-q$ in (1.12) and then employing (1.9) in the resulting identity, we find that

$$1-\frac{\varphi^2(-q)}{\varphi^2(-q^3)}=4q\frac{f_1f_{12}^3}{f_4f_3^3},\qquad \frac{\varphi^2(-q)}{\varphi^2(-q^3)}+1=2\frac{f_1^2f_4f_6^9}{f_2^3f_3^3f_{12}^3},$$

$$3-\frac{\varphi^2(-q)}{\varphi^2(-q^3)}=2\frac{f_2^7f_{12}}{f_1^2f_4^3f_3^2f_6}\quad and\quad 3+\frac{\varphi^2(-q)}{\varphi^2(-q^3)}=4\frac{f_1f_4^3f_6^2}{f_2^2f_3^3f_{12}}. \tag{3.2}$$

From the above identities, we deduce that

$$\frac{\psi^4(q)}{q\psi^4(q^3)}=\frac{9-v^4}{1-v^4},$$

which is equivalent to

$$\begin{aligned}\frac{\psi^2(q)}{q^{1/2}\psi^2(q^3)}&=\frac{\sqrt{(9-v^4)(1-v^4)}}{1-v^4}\\&=\frac{f_2^3f_3^9}{8qf_1^3f_6^9}\sqrt{(9-v^4)(1-v^4)}.\end{aligned}$$

Using this in (3.1) and then employing (1.9), we obtain

$$2v\frac{dv}{dq}=\frac{f_1f_2f_3f_6}{2q^{1/2}}\sqrt{(9-v^4)(1-v^4)}.$$

This implies

$$\frac{3}{2}\int_0^q\frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)dt=\int_0^v\frac{2v}{\sqrt{(9-v^4)(1-v^4)}}dv.$$

Setting $\sin\theta=v^2$ in right-hand side of the above, we get

$$\frac{3}{2}\int_0^q\frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)dt=\int_0^{sin^{-1}(v^2)}\frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\,d\theta.$$

This completes the proof.

Theorem 3.2. *Let* $v:=\frac{\psi(q)}{\psi(q^3)}$. *Then*

$$\frac{3}{2}\int_0^q\frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)\;\;dt=\int_0^{\sin^{-1}(v^2)}\frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\,d\theta.$$

Proof. Using (1.9) in the definition of v and then taking logarithm on both sides, we find that

$$\log v=-\frac{1}{4}\log q-\sum_{n=1}^{\infty}\log(1-q^n)+2\sum_{n=1}^{\infty}\log(1-q^{2n})+\sum_{n=1}^{\infty}\log(1-q^{3n})-2\sum_{n=1}^{\infty}\log(1-q^{6n}).$$

Differentiating the above with respect to q and then using the definition of P_n, we obtain

$$\frac{q}{v}\frac{dq}{dv} = -\frac{1}{24}\left[-P_1 + 4P_2 + 3P_3 - 12P_6\right].$$

Using (2.5) above, we get

$$\frac{q}{v}\frac{dq}{dv} = \frac{1}{4}\varphi^4(-q^3)\left\{\frac{\varphi^2(-q)}{\varphi^2(-q^3)}\right\}. \tag{3.3}$$

If $k := k(q) = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}$, then from [3, pp. 345−347], we have

$$1 + \frac{1}{k^3} = \frac{\psi^4(q)}{q\psi^4(q^3)} \tag{3.4}$$

and

$$1 - 8k^3 = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\frac{\varphi^2(-q)}{\varphi^2(-q^3)} = \frac{\sqrt{(1-v^4)(9-v^4)}}{1-v^4}.$$

Using (3.4) in the above, we see that

$$\frac{\varphi^2(-q)}{\varphi^2(-q^3)} = -\frac{\chi^9(-q^3)}{q\chi^3(-q)}\sqrt{(1-v^4)(9-v^4)}.$$

Using the above in (3.3) and then employing (1.9), we deduce that

$$2v\frac{dv}{dq} = \frac{1}{2q^{1/2}}f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{(1-v^4)(9-v^4)}.$$

This yields

$$\int_0^q \frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)dt = \frac{3}{2}\int_0^{v(q)} \frac{2v\ dv}{\sqrt{(1-v^4)(9-v^4)}}.$$

Setting $v^2 = \sin\theta$ in right-hand side of the above, we obtain the required result.

Theorem 3.3. *Let* $v{:=}q\dfrac{\chi^3(-q)}{\chi^9(-q^3)}$. *Then*

$$\frac{3}{2}\int_0^q \frac{1}{t^{1/2}}f(-t)f(-t^2)f(-t^3)f(-t^6)\ \ dt \ \ = \ \ \int_{\pi/2}^{\sin^{-1}\sqrt{1-8v}} \frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\ d\theta.$$

Proof. By logarithmically differentiating v with respect to q, we see that

$$\frac{q}{v}\frac{dv}{dq} = 1 - 3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 6\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} + 27\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} - 54\sum_{n=1}^{\infty}\frac{nq^{6n}}{1-q^{6n}}.$$

Using the definition of P_n above, we obtain

$$\frac{q}{v}\frac{dv}{dq} = \frac{1}{8}\left[P_1 - 2P_2 - 9P_3 + 18P_6\right]. \tag{3.6}$$

From [14], we have

$$P_1 - 2P_2 = \varphi^4(-q) - 2\varphi^4(q). \tag{3.7}$$

This implies

$$\begin{aligned} P_1 - 2P_2 - 9P_3 + 18P_6 &= \varphi^4(-q) - 2\varphi^4(q) - 9\varphi^4(-q^3) - 18\varphi^4(q^3) \\ &= \varphi^4(-q^3)\left\{\frac{\varphi^4(-q)}{\varphi^4(-q^3)} - 9\right\} - 2\varphi^4(q^3)\left\{\frac{\varphi^4(q)}{\varphi^4(q^3)} - 9\right\}. \end{aligned}$$

From (3.2), we have

$$9 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)} = 8\frac{f_2^5 f_6}{f_1 f_3^5}.$$

Using this in first curly bracket and using the same in second curly bracket with q replaced by $-q$, and then employing (1.9), we deduce that

$$P_1 - 2P_2 - 9P_3 + 18P_6 = -8q^{1/3}f_2^4\left[\frac{\chi^3(-q^3)}{q^{1/3}\chi(-q)} - 2\frac{\chi^3(q^3)}{q^{1/3}\chi(q)}\right]. \tag{3.8}$$

From [8], we have

$$u + w = -2u^2w^2, \tag{3.9}$$

where $u = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}$ and $w = u(-q)$. This implies

$$2u + w = u - 2u^2w^2.$$

From (3.9), we have

$$u + 2u^2w^2 = -w.$$

This implies

$$(u - 2u^2w^2)^2 = w^2 - 8u^3w^2 = w^2\frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \tag{3.10}$$

upon using (3.5) in the last equality. Thus we have

$$\begin{aligned} P_1 - 2P_2 - 9P_3 + 18P_6 &= -8q^{1/3}f_2^4\left[\frac{1}{u} + \frac{2}{w}\right] \\ &= -\frac{8q^{1/3}f_2^4}{uw}\left[w + 2u\right] \\ &= -\frac{8q^{1/3}f_2^4}{uw}\left[u - 2u^2w^2\right] \\ &= -\frac{8q^{1/3}f_2^4}{u}\frac{\varphi^2(-q)}{\varphi^2(-q^3)}, \end{aligned}$$

where we have used (3.10). Upon using (1.9) in the above, we get

$$P_1 - 2P_2 - 9P_3 + 18P_6 = -8\frac{f_1^3f_2^3}{f_3f_6}.$$

From (3.6) and the above, we have

$$\frac{q}{v}\frac{dv}{dq} = -\frac{f_1^3f_2^3}{f_3f_6}.$$

From (3.4), (3.5) and the above, we deduce that

$$\frac{dv}{dq} = -\frac{1}{q^{1/2}}\frac{f_1f_2f_3f_6}{\sqrt{v}}\sqrt{(v+1)(1-8v)}.$$

This yields

$$\int_0^q \frac{f(-t)f(-t^2)f(-t^3)f(-t^6))}{t^{1/2}}\,dt = -\int_0^v \frac{1}{v^{1/2}\sqrt{(v+1)(1-8v)}}\,dv \tag{3.11}$$

$$= -\int_0^v \frac{1}{v^{3/2}\sqrt{(\frac{1}{v}+1)(\frac{1}{v}-8)}}\,dv \tag{3.12}$$

Employing the method outlined in [7, pp. 311−312], we make the substitution $\frac{1}{v} = \frac{8}{(1-y^2)}$ in the above and obtain.

$$-\int_0^v \frac{1}{v^{3/2}\sqrt{(\frac{1}{v}+1)(\frac{1}{v}-8)}}\,dv = \int_1^{\sqrt{1-8v}} \frac{2}{3\sqrt{(1-y^2)(1-\frac{1}{9}y^2)}}\,dz.$$

Setting $y = \sin\theta$, we find that

$$\int_0^q \frac{f(-t)f(-t^2)f(-t^3)f(-t^6)}{q^{1/2}}\,dt = \frac{2}{3}\int_{\pi/2}^{\sin^{-1}\sqrt{1-8v}} \frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\,d\theta.$$

Which is the required result.

Theorem 3.4. *Let* $v:=\dfrac{\chi^9(-q)}{\chi^3(-q^3)}$. *Then*

$$\frac{3}{2}\int_0^q \frac{1}{t^{1/2}} f(-t)f(-t^2)f(-t^3)f(-t^6)\,dt \;=\; \int_{\pi/2}^{\sin^{-1}\sqrt{v-1}} \frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\,d\theta.$$

Proof. From (1.9), we have

$$v = \frac{f_1^9 f_6^3}{f_2^9 f_3^3}.$$

By logarithmically differentiating v with respect to q, we obtain

$$\frac{q}{v}\frac{dv}{dq} = -9\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} + 2\sum_{n=1}^{\infty}\frac{nq^{6n}}{1-q^{6n}}\right].$$

Using the definition of P_n in the above, we have

$$\frac{q}{v}\frac{dv}{dq} = \frac{3}{8}\left[P_1 - 2P_2 - P_3 + 2P_6\right].$$

Employing (3.7) in the above, we find that

$$\frac{q}{v}\frac{dv}{dq} = \frac{3}{8}\left[\varphi^4(-q) - 2\varphi^4(q) - \varphi^4(-q^3) + 2\varphi^4(q^3)\right]. \tag{3.13}$$

We have

$$\varphi^4(q)-2\varphi^4-(q)-\varphi^4(q^3)+2\varphi^4(-q^3) = \varphi^4(q^3)\left[\frac{\varphi^4(q)}{\varphi^4(q^3)} - 2\frac{\varphi^4(-q)}{\varphi^4(q^3)} - 1 + 2\frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right].$$

Now transforming the theta functions inside the bracket into modular equations, we have

$$\varphi^4(q) - 2\varphi^4(-q) - \varphi^4(q^3) + 2\varphi^4(-q^3) = \varphi^4(q^3)\left[-m^2 + 2m^2\alpha - 2\beta + 1\right].$$

Employing (1.10) in the above to eliminate α and β, we have

$$\varphi^4(q) - 2\varphi^4(-q) - \varphi^4(q^3) + 2\varphi^4(-q^3) = -3\varphi^4(q^3)\left[\frac{(m-1)(m+1)}{m}\right].$$

Employing (1.10) in right-hand side of the above, we have

$$\varphi^4(-q) - 2\varphi^4(q) - \varphi^4(-q^3) + 2\varphi^4(q^3) = -3\varphi^4(q^3)\left[\frac{4}{m}\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}\right].$$

Now transforming the right-hand side of the above into theta function and then replacing q by $-q$, we find that

$$\varphi^4(-q) - 2\varphi^4(q) - \varphi^4(-q^3) + 2\varphi^4(q^3) = -24q\frac{f_3^3 f_6^3}{f_1 f_2}.$$

Using this in (3.13), we find that

$$\frac{q}{v}\frac{dv}{dq} = -9q\frac{f_3^3 f_6^3}{f_1 f_2}. \tag{3.14}$$

From (3.2), we have

$$\frac{\varphi^4(-q^3)}{\varphi^4(-q)}\left[9 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}\right] = \frac{8}{v}.$$

This implies

$$9\frac{\varphi^4(-q^3)}{\varphi^4(-q)} = \frac{8+v}{v}. \tag{3.15}$$

From (3.4) and (3.5), we have

$$\left[1 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}\right]\left[\frac{\psi^4(q)}{q\psi^4(q^3)} - 1\right] = 8.$$

Using (3.15) in the above, we find that

$$\frac{\psi^4(q)}{q\psi^4(q^3)} = \frac{9}{1-v}. \tag{3.16}$$

From (1.9), (3.15) and (3.16), we have

$$\frac{f_6^3 f_3^3}{f_1 f_2} = \frac{f_1 f_2 f_3 f_6}{9q^{1/2}}\sqrt{\frac{(1-v)(8+v)}{v}}.$$

Using this in (3.14), we get

$$\frac{dv}{dq} = -\frac{f_1 f_2 f_3 f_6}{q^{1/2}}\sqrt{(1-v)(8+v)v}.$$

This implies

$$\int_0^q \frac{1}{t^{1/2}} f(-t)f(-t^2)f(-t^3)f(-t^6)dt = -\int_0^v \frac{1}{\sqrt{(1-v)(8+v)v}}\, dv.$$

Putting $v = \cos^2\theta$ in the above, we find that

$$\frac{3}{2}\int_0^q \frac{1}{t^{1/2}} f(-t)f(-t^2)f(-t^3)f(-t^6)dt = \int_{\pi/2}^{\cos^{-1}\sqrt{v}} \frac{1}{\sqrt{1-\frac{1}{9}\sin^2\theta}}\, d\theta.$$

This completes the proof.

Theorem 3.5. *[5] Let* $v{:=}q\dfrac{f_3^4 f_6^4}{f_1^4 f_2^4}$. *Then*

$$\frac{dv}{dq} \;=\; \frac{f_1 f_2 f_3 f_6}{q^{1/2}}\;\sqrt{81v^3+14v^2+v}.$$

Proof. By logarithmically differentiating v with respect to q, we find that

$$\frac{q}{v}\frac{dv}{dq} = \frac{1}{6}\left[-P_1 - 2P_2 + 3P_3 + 6P_6\right].$$

Employing (2.1), (2.5) and (1.9) above, we obtain

$$\frac{q}{v}\frac{dv}{dq} = q^{1/2} f_1 f_2 f_3 f_6 \left[\frac{f_1^3 f_3^3}{q^{1/2} f_3^3 f_6^3} + \frac{8q^{1/2} f_2^3 f_6^3}{f_1^3 f_3^3}\right]. \tag{3.17}$$

From [4, Entry 52, p. 205], we have

$$\frac{f_1^3 f_3^3}{q^{1/2} f_3^3 f_6^3} - \frac{8q^{1/2} f_2^3 f_6^3}{f_1^3 f_3^3} = \frac{1}{\sqrt{v}} - 9\sqrt{v}.$$

This yields

$$\frac{f_1^3 f_3^3}{q^{1/2} f_3^3 f_6^3} + \frac{8q^{1/2} f_2^3 f_6^3}{f_1^3 f_3^3} = \frac{\sqrt{(9v+2\sqrt{v}+1)(9v-2\sqrt{v}+1)}}{\sqrt{v}}.$$

Using this in (3.17), we obtain

$$\frac{q}{v}\frac{dv}{dq} = \frac{f_1 f_2 f_3 f_6}{q^{1/2}}\sqrt{81v^3+14v^2+v}.$$

This completes the proof.

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References

[1] W. N. Bailey, Series of hypergeometric type which are infinite in both directions, *Quart. J. Math. (Oxford)*, **7**(1) (1936), 105−115.

[2] W. N. Bailey, A further note on two of Ramanujan's formulae, *Quart. J. Math. (Oxford)*, **3**(1) (1952), 158−160.

[3] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.

[4] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer, New York, 1994.

[5] B. C. Berndt, H. H. Chan and S.-S. Huang, Incomplete elliptic integrals in Ramanujan's Lost Notebook, *Amer. Math. Soc.*, Providence, RI, 2000, 79−126.

[6] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, Wiley (1987).

[7] A. Cayley, *An Elementary Treatise on Elliptic Functions*, Dover, New York, 1961.

[8] H. H. Chan, On Ramanujan's cubic continued fraction, *Acta Arith.*, **73**(4) (1995), 343−355.

[9] S. Raghavan and S. S. Rangachari, *On Ramanujan's elliptic integrals and modular identities*, 119−149. Oxford University Press, Bombay, 1989.

[10] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

[11] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[12] S. Ramanujan, Modular equation and approximation to π, *Quart. J. Math. (Oxford)* **45** (1914), 350−372.

[13] K. R. Vasuki, On some Ramanujan's Schläfli-type modular equations, *Aust. J. Math. Anal. Appl.*, **3**(2) (2006), 1−8.

[14] K. R. Vasuki and R. G. Veeresha, Ramanujan's Eisenstein series of level 7 and 14, *J. Number Theory*, **159** (2016), 59−75.