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ON CERTAIN PRODUCT FORMULAS OF BASIC HYPERGEOMETRIC SERIES

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Abstract: In this paper, proofs of certain product formulas of basic hypergeometric series deduced from the identities due to Srivastava and Jain [2] have been given.

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1. Introduction, Notations and Definitions

For real or complex q, $|q| < 1$, let

$$(\lambda; q)_\mu = \prod_{r=0}^{\infty} \frac{(1-\lambda q^r)}{(1-\lambda q^{\mu+r})} \quad \text{for arbitrary } \lambda \text{ and } \mu, \tag{1.1}$$

so that

$$(\lambda; q)_n = \begin{cases} 1, & n = 0 \\ (1-\lambda)(1-\lambda q)...(1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, ...\} \end{cases} \tag{1.2}$$

and

$$(\lambda; q)_\infty = \prod_{r=0}^{\infty} (1-\lambda q^r). \tag{1.3}$$

A generalized q-hypergeometric function is defined by (cf.[3; Chapter 3])

$$ {}_{m+1}\Phi_{m+r}\left[\begin{array}{c} a_1, a_2, ..., a_{m+1}; q; z \\ b_1, b_2, ..., b_{m+r} \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_{m+1}; q)_n z^n (-1)^{nr} q^{rn(n-1)/2}}{(b_1, b_2, ..., b_{m+r}; q)_n (q; q)_n}, \tag{1.4} $$

where $(a_1, a_2, ..., a_{m+1}; q)_n = (a_1; q)_n (a_2; q)_n ... (a_{m+1}; q)_n$ and series in (1.4) converges for $|z| < \infty$ if r is a positive integer and for $|z| < 1$ when $r = 0$, provided no zeros appear in the denominator.

In 1986 Srivastava and Jain established six identities [2; (2.1)-(2.5)] which yield following five product formulas as $A_n \to 1$.

$$ {}_2\Phi_1\left[\begin{array}{c} a, -\frac{b}{a}; q; x \\ b \end{array}\right] {}_2\Phi_1\left[\begin{array}{c} a, -\frac{b}{a}; q; -x \\ b \end{array}\right] = {}_4\Phi_3\left[\begin{array}{c} a^2, \frac{b^2}{a^2}, -b, -bq; q^2; x^2 \\ b^2, b, bq \end{array}\right]. \tag{1.5} $$

$$ {}_2\Phi_1\left[\begin{array}{c} a, -a; q; x \\ a^2 \end{array}\right] {}_2\Phi_1\left[\begin{array}{c} b, -b; q; -x \\ b^2 \end{array}\right] = {}_4\Phi_3\left[\begin{array}{c} ab, -ab, abq, -abq; q^2; x^2 \\ a^2q, b^2q, a^2b^2 \end{array}\right]. \tag{1.6} $$

$$ {}_2\Phi_1\left[\begin{array}{c} a^2, a^2q; q^2; x \\ a^4 \end{array}\right] {}_2\Phi_1\left[\begin{array}{c} b^2, b^2q; q^2; xq \\ b^4q^2 \end{array}\right] = {}_4\Phi_3\left[\begin{array}{c} ab, -ab, ab\sqrt{q}, -ab\sqrt{q}; q; x \\ a^2b^2, -a^2, -b^2q \end{array}\right]. \tag{1.7} $$

$$ {}_2\Phi_1\left[\begin{array}{c} a, \frac{b}{aq}; q^2; x \\ b \end{array}\right] {}_2\Phi_1\left[\begin{array}{c} a, \frac{b}{aq}; q^2; xq \\ b \end{array}\right] = {}_4\Phi_3\left[\begin{array}{c} a, \frac{b}{aq}, \sqrt{\frac{b}{q}}, -\sqrt{\frac{b}{q}}; q; xq \\ b, \sqrt{b}, -\sqrt{b} \end{array}\right]. \tag{1.8} $$

$$ {}_2\Phi_1\left[\begin{array}{c} a, b; q; \frac{cx}{ab} \\ c \end{array}\right] {}_1\Phi_0\left[\begin{array}{c} \frac{c}{ab}; q; x \\ - \end{array}\right] = {}_2\Phi_1\left[\begin{array}{c} \frac{c}{a}, \frac{c}{b}; q; x \\ c \end{array}\right]. \tag{1.9} $$

Main aim of this paper is to give proofs of these product formulas. Following results are needed in our analysis.

$$ {}_4\Phi_3\left[\begin{array}{c} b^2x^4q^{2+2n}, x^2, x^2q, q^{-2n}; q^2; q^2 \\ bx^2q, bx^2q^2, x^4q^2 \end{array}\right] = \frac{x^{2n}(-q; q)_n (bq; q)_n}{(-x^2q; q)_n (bx^2q; q)_n}. \tag{1.10} $$

[4; (2.3.2), p.1029]

$$ {}_4\Phi_3\left[\begin{array}{c} q^{-n}, -\frac{q^{1-n}}{xy}, x, y; q; q \\ -xy, \frac{q^{1-n}}{x}, \frac{q^{1-n}}{y} \end{array}\right] = \begin{cases} 0, & \text{if n=2m+1,} \\ \dfrac{(x^2, y^2, xy, xyq, q; q^2)_m}{(x^2y^2, x, xq, y, yq; q^2)_m}. \end{cases} \tag{1.11} $$

[4; (2.3), p.1023]

$$ {}_4\Phi_3\left[\begin{array}{c} a, b, c, q^{-n}; q; q \\ e, g, h \end{array}\right] = \frac{\left(\frac{g}{c}, \frac{eg}{ab}; q\right)_n}{\left(g, \frac{eg}{abc}; q\right)_n} {}_4\Phi_3\left[\begin{array}{c} \frac{e}{a}, \frac{e}{b}, c, q^{-n}; q; q \\ e, \frac{cq^{1-n}}{g}, \frac{cq^{1-n}}{h} \end{array}\right] \tag{1.12} $$

[4; (2.4), p.1023]

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, \frac{q^{-n-1}}{x^2}, y, -y; q; q \\ \frac{q^{-n}}{x}, -\frac{q^{-n}}{x}, y^2 \end{matrix}\right] = \frac{(q;q)_n (x^2y^2q^2;q^2)_n (x^2q^2;q^2)_m (y^2q^2;q^2)_m}{(x^2q^2;q)_n (y^2q;q)_n (x^2y^2q^2;q^2)_m (q^2;q^2)_m}, \quad (1.13) $$

[4; (2.12), p.1025]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4\Phi_3\left[\begin{matrix} a^2, b^2, \frac{q^{1-2n}}{a^2b^2}, q^{-2n}; q^2; q^2 \\ a^2b^2q, \frac{q^{2-2n}}{a^2}, \frac{q^{2-2n}}{b^2} \end{matrix}\right] = \frac{(a^2, b^2, -q; q)_n \left(\frac{a^2b^2}{q}; q\right)_{2n}}{\left(\frac{a^2b^2}{q}, a^2, b^2; q^2\right)_n (a^2b^2q; q)_n}. \quad (1.14) $$

[5; (4.3), p.653]

2. Proofs of Product Formulas (1.5)-(1.9)

In this section we shall prove the product formulas mentioned in previous section.

(a) Proof of (1.5)

Left hand side of (1.5) is

$$ = \sum_{r=0}^{\infty} \frac{\left(a, -\frac{b}{a}; q\right)_r}{(q, b; q)_r} \sum_{s=0}^{\infty} \frac{\left(a, -\frac{b}{a}; q\right)_s x^{r+s} (-1)^s}{(q, b; q)_s} $$

Putting $s - r$ for s we have

$$ = \sum_{s=0}^{\infty} \frac{\left(a, -\frac{b}{a}; q\right)_s (-x)^s}{(q, b; q)_s} \sum_{r=0}^{s} \frac{\left(a, -\frac{b}{a}, q^{-s}, \frac{q^{1-s}}{b}; q\right)_r q^r}{\left(q, b, \frac{q^{1-s}}{a}, -\frac{a}{b}q^{1-s}; q\right)_r}. \quad (2.1) $$

Replacing x by a, y by $-\frac{b}{a}$ in (1.11) and then using it to sum the inner series in (2.1) we get the right hand side of (1.5).

(b) Proof of (1.6)

Left hand side of (1.6) is

$$ = \sum_{r=0}^{\infty} \frac{(a, -a; q)_r}{(q, a^2; q)_r} \sum_{s=0}^{\infty} \frac{(b, -b; q)_s x^{r+s} (-1)^s}{(q, b^2; q)_s} $$

Putting $s - r$ for r we have

$$ = \sum_{s=0}^{\infty} \frac{(b, -b; q)_s (-x)^s}{(q, b^2; q)_s} \sum_{r=0}^{s} \frac{\left(a, -a, q^{-s}, \frac{q^{1-s}}{b^2}, q^{-s}; q\right)_r q^r}{\left(q, a^2, \frac{q^{1-s}}{b}, -\frac{q^{1-s}}{b}; q\right)_r}. $$

$$= \sum_{s=0}^{\infty} \frac{(b,-b;q)_s(-x)^s}{(q,b^2;q)_s} \; {}_4\Phi_3 \left[\begin{matrix} a,-a,\frac{q^{1-s}}{b^2},q^{-s};q;q \\ a^2,\frac{q^{1-s}}{b},-\frac{q^{1-s}}{b} \end{matrix} \right] \tag{2.2}$$

Taking $b=-a$, $c=\frac{q^{1-n}}{b^2}$, $e=a^2$, $g=\frac{q^{1-n}}{b}$ and $h=\frac{q^{1-n}}{-b}$ in (1.12) we have

$${}_4\Phi_3 \left[\begin{matrix} a,-a,\frac{q^{1-n}}{b^2},q^{-n};q;q \\ a^2,\frac{q^{1-n}}{b},-\frac{q^{1-n}}{b} \end{matrix} \right] = (-1)^n \; {}_4\Phi_3 \left[\begin{matrix} a,-a,\frac{q^{1-n}}{b^2},q^{-n};q;q \\ a^2,\frac{q^{1-n}}{b},-\frac{q^{1-n}}{b} \end{matrix} \right],$$

which shows that ${}_4\Phi_3 \left[\begin{matrix} a,-a,\frac{q^{1-n}}{b^2},q^{-n};q;q \\ a^2,\frac{q^{1-n}}{b},-\frac{q^{1-n}}{b} \end{matrix} \right]$ vanishes when n is odd. Its value exists when n is even.

Now, replacing x by $\frac{b}{q}$ and y by a in (1.13) and then using it to sum the inner ${}_4\Phi_3$ series in (2.2) we get the right hand side of (1.6) after some simplifications.

(c) Proof of (1.7)

Left hand side of (1.7) is

$$\sum_{r=0}^{\infty} \frac{(b^2,b^2q;q^2)_r q^r}{(q^2,b^4q^2;q^2)_r} \sum_{s=0}^{\infty} \frac{(a^2,a^2q;q^2)_s}{(q^2,a^4;q^2)_s} x^{s+r}$$

Putting $s-r$ for s we have

$$= \sum_{s=0}^{\infty} \frac{(a^2,a^2q;q^2)_s x^s}{(q^2,a^4;q^2)_s} \sum_{r=0}^{s} \frac{(b^2,b^2q;q^2)_r \left(q^{-2s},\frac{q^{2-2s}}{a^4};q^2\right)_r q^{2r}}{(q^2,b^4q^2;q^2)_r \left(\frac{q^{2-2s}}{a^2},\frac{q^{1-2s}}{a^2};q^2\right)_r}$$

$$= \sum_{s=0}^{\infty} \frac{(a^2,a^2q;q^2)_s x^s}{(q^2,a^4;q^2)_s} \; {}_4\Phi_3 \left[\begin{matrix} b^2,b^2q,q^{-2s},\frac{q^{2-2s}}{a^4};q^2;q^2 \\ b^4q^2,\frac{q^{1-2s}}{a^2},\frac{q^{2-2s}}{a^2} \end{matrix} \right]. \tag{2.3}$$

Replacing x by b and b by $\dfrac{1}{a^2b^2q^{2s}}$ in (1.10) and then using it to sum the inner ${}_4\Phi_3$ series in (2.3) we get the right hand side of (1.7) after some simplification.

(d) Proof of (1.8)

Left hand side of (1.8) is

$$= \sum_{r=0}^{\infty} \frac{\left(a,\frac{b}{aq};q^2\right)_r}{(q^2,b;q^2)_r} \sum_{s=0}^{\infty} \frac{\left(a,\frac{b}{aq};q^2\right)_s}{(q^2,b;q^2)_s} x^{s+r} q^s$$

Putting $s-r$ for s we have

$$= \sum_{s=0}^{\infty} \frac{\left(a, \frac{b}{aq}; q^2\right)_s (xq)^s}{(q^2, b; q^2)_s} \sum_{r=0}^{s} \frac{\left(a, \frac{b}{aq}, q^{-2s}, \frac{q^{2-2s}}{b}; q^2\right)_r q^{2r}}{\left(q^2, b, \frac{q^{2-2s}}{a}, \frac{aq^{3-2s}}{b}; q^2\right)_r}$$

$$= \sum_{s=0}^{\infty} \frac{\left(a, \frac{b}{aq}; q^2\right)_s (xq)^s}{(q^2, b; q^2)_s} \, {}_4\Phi_3 \left[\begin{matrix} a, \frac{b}{aq}, q^{-2s}, \frac{q^{2-2s}}{b}; q^2; q^2 \\ b, \frac{q^{2-2s}}{a}, \frac{aq^{3-2s}}{b} \end{matrix} \right]. \tag{2.4}$$

Now, replacing a^2 by a, b^2 by b/aq in (1.14) and then using it to sum the inner ${}_4\Phi_3$ series in (2.4) we get the right hand side of (1.8) after some simplifications.

(e) Proof of (1.9)

Left hand side of (1.9) is

$$\sum_{r=0}^{\infty} \frac{(a, b; q)_r}{(q, c; q)_r} \left(\frac{c}{ab}\right)^r \sum_{s=0}^{\infty} \frac{\left(\frac{c}{ab}; q\right)_s x^{s+r}}{(q; q)_s}$$

Putting $s-r$ for s we have

$$= \sum_{s=0}^{\infty} \frac{\left(\frac{c}{ab}; q\right)_s x^s}{(q; q)_s} \, {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-s}; q; q \\ c, \frac{ab}{c} q^{1-s} \end{matrix} \right]. \tag{2.5}$$

Now, summing the inner ${}_3\Phi_2$-series by making use of the summation formulae [Slater 3; App.] we get the right hand side of (1.9).

3. Special Cases

We shall deduce certain interesting results from the product formulas (1.5)-(1.9).

(i) Taking $a = 1$ in (1.6) we have

$${}_2\Phi_1 \left[\begin{matrix} b, -b; q; -x \\ b^2 \end{matrix} \right] = {}_4\Phi_3 \left[\begin{matrix} b, -b, bq, -bq; q^2; x^2 \\ q, b^2 q, b^2 \end{matrix} \right]. \tag{3.1}$$

(ii) Taking $a = 1$ in (1.7) we get

$${}_2\Phi_1 \left[\begin{matrix} b^2, b^2 q; q^2; xq \\ b^4 q^2 \end{matrix} \right] = {}_4\Phi_3 \left[\begin{matrix} b, -b, b\sqrt{q}, -b\sqrt{q}; q; x \\ b^2, -1, -b^2 q \end{matrix} \right]. \tag{3.2}$$

(iii) For $a = q$, (1.8) yields

$$\sum_{n=0}^{\infty} \frac{\left(q, \frac{b}{q^2}; q^2\right)_n x^n}{(q^2, b; q^2)_n} \sum_{n=0}^{\infty} \frac{\left(q, \frac{b}{q^2}; q^2\right)_n x^n q^n}{(q^2, b; q^2)_n} = \sum_{n=0}^{\infty} \frac{\left(\frac{b}{q^2}; q\right)_n \left(\sqrt{\frac{b}{q}}, -\sqrt{\frac{b}{q}}; q\right)_n x^n q^n}{(b, \sqrt{b}, -\sqrt{b}; q)_n} \tag{3.3}$$

(iv) Taking $a, b \to \infty$ in (1.9) we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)} c^n x^n}{(c;q)_n (q;q)_n} = (x;q)_\infty \sum_{n=0}^{\infty} \frac{x^n}{(c,q;q)_n} \tag{3.4}$$

For $c = q$, (3.4) yields

$$\sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q;q)_n^2} = (x;q)_\infty \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n^2}, \tag{3.5}$$

which is known result [1; (6.7.1) p. 170]
Taking $x = q$ in (3.5) we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n^2} = (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n^2}. \tag{3.6}$$

For $c = -q$, (3.4) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^n}{(q^2;q^2)_n} = (x;q)_\infty \sum_{n=0}^{\infty} \frac{x^n}{(q^2;q^2)_n}. \tag{3.7}$$

Putting $c = q$ and $x = zq$ in (3.4) we have

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q;q)_n^2} = (zq;q)_\infty \sum_{n=0}^{\infty} \frac{z^n q^n}{(q;q)_n^2}, \tag{3.8}$$

which is a known result [1; (6.7.2) p. 169]

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