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# HARMONIC MORPHISMS ON ALMOST *GF-r* CONTACT METRIC STRUCTURE MANIFOLDS

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**Abstract:** In this paper we have studied almost morphisms on an almost contact metric structure manifolds in almost $GF$-$r$ contact metric manifolds. Some other interesting result on cartesian product manifold $M \times R^r$ have also been established where $M$ is an almost $GF$-$r$ contact metric manifold of dimension $(2m + r)$.

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## 1. Preliminaries

Let $(M, g)$ and $(N, h)$ be two $C^\infty$ manifolds and $F : M \to N$ be a $C^\infty$ map. Let $\nabla$ and $\bar{\nabla}$ be Levi-Civita connections on $M$ and $N$ respectively. The second fundamental form $\alpha_F$ of $F$ given by

$$\alpha_F(X, Y) \;=\; \bar{\nabla}_x dF(Y) - dF(\nabla_x Y) \tag{1.1}$$

for all $X, Y$ tangents to $M$. Here $\bar{\nabla}$ is the connection induced by $F$ on the bundle $F^{-1}(TN)$, $TN$ denotes the tangents bundle of $N$.

The tension field $\tau(F)$ of $F$ is defined as trace of second fundamental form $\alpha_F$, i.e.,

$$\tau(F) \;=\; \sum_{i=1}^{m} \alpha_F(e_i, e_i)_p \tag{1.2}$$

where $\{e_1, e_2, \cdots, e_m\}$ is an orthonormal basis for the tangent space $T_p(M)$ of $M$ at $p \in M$. We say that a map $F : (M, g) \to (N, h)$ between two manifolds $M$ and $N$ is harmonic if and only if $\tau(F) = 0$ [2,3,6].

A map $F : (M, g) \to (N, h)$ is called harmonic morphism if it maps germs of the harmonic function to germs of the harmonic function, i.e, $\Delta f = 0$ implies that $\Delta(f \circ F) = 0$ for all functions $f$ defined on an open set $V \subset N$ [1,4].

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Let $V_p = \ker dF_p$ and $H_p = V_p^{\perp}$, the horizontal space at $p \in M$.

If $F$ has no critical points, then $F$ is called horizontal conformal at $p \in M$ if there exists a smooth function $\lambda : M \to (0, \infty)$ such that

$$\lambda^2(P)g(X,Y) \;=\; h[dF(X), dF(Y)], \quad \text{for all } p \in M \text{ and } X, Y \in H_p. \tag{1.3}$$

Let $M^{2m+r}$ be a $(2m+r)$-dimensional differentiable manifold of differentiability class $C^\infty$. It is said to posses an almost $GF$-$r$ contact metric structure if there exists a tensor field of type $(1,1)$, $r(C^\infty)$ contravariant vector fields $\xi^\alpha$ and $r(C^\infty)$ 1-forms $\eta_\alpha$ and a positive definite Riemannian metric $g$ satisfying [5].

$$\begin{array}{lrcl} \text{(i)} & \phi^2 & = & a^2 I + \eta_\alpha \otimes \xi^\alpha, \\ \text{(ii)} & \phi\xi^\alpha & = & 0, \\ \text{(iii)} & \eta_\alpha \circ \phi & = & 0, \quad \text{and} \\ \text{(iv)} & \eta_\alpha(\xi^\beta) + a^2\delta_\alpha^\beta & = & 0, \end{array} \tag{1.4}$$

where $\alpha, \beta = 1, 2, \cdots, r$ and $\delta_\alpha^\beta$ denotes the Kronecker delta.

The positive definite Riemannian metric $g$ satisfies

$$\begin{array}{lrcl} \text{(i)} & g(\phi X, \phi Y) & = & a^2 g(X,X) + \eta_\alpha(X)\eta_\alpha(Y) \;=\; 0, \quad \text{and} \\ \text{(ii)} & g(\xi^\alpha, Y) & = & \eta_\alpha(Y). \end{array}$$

## 2. Harmonic Morphisms on Almost $GF$-$r$ Contact Metric Manifold

Let $M(\phi, \xi^\alpha, \eta_\alpha, a, g)$ be an almost $GF$-$r$ contact metric manifold of dimension $(2m+r)$ and $N(J,h)$ be a $2n$-dimensional almost Hermitian manifold. We call a mapping $F : M \to N$ to be a $(\phi, J)$-holomorphic if

$$dF \circ \phi \;=\; J \circ dF. \tag{2.1}$$

Now we have the following theorem.

**Theorem 2.1.** For a manifold an almost $GF$-$r$ contact metric structure $M(\phi, \xi^\alpha, \eta_\alpha, a, g)$ and almost Hermitian manifold $N(J,h)$, the following results are equivalent

$$\text{(i)} \;\; \operatorname{div} J \;=\; 0 \qquad \text{(ii)} \;\; Tr_g\beta \;=\; 0 \tag{2.2}$$

where

$$\beta(X,Y) \;=\; (\bar{\nabla}_x J)dF(Y) \;\; \text{for all } X, Y \text{ tangents to } M. \tag{2.3}$$

**Proof.** Let $M$ be $(2m+r)$ dimensional and $N$ of dimension $2n$ $(m > n)$. Let $\{Y_1, Y_2, \cdots, Y_n, JY_1, JY_2, \cdots, JY_n\}$ be the local Hermitian frame for $TN$ and

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$\{Y_1^*, Y_2^*, \cdots, Y_n^*\}$ be unique horizontal lifts of $\{Y_1, Y_2, \cdots, Y_n\}$. We set $X_i = \lambda Y_i^*$ where $\lambda$ is dilation of $F$. We extend $\{X_1, X_2, \cdots, X_J\}$ to a local $\phi$-adapted frame on $TM$ as $\{X_1, X_2, \cdots, X_n, X_{n+1}, \cdots X_m, \phi X_1, \phi X_2, \cdots, \phi X_n, \phi X_{n+1}, \cdots, \phi X_m, \xi_1, \xi_2, \cdots, \xi_r\}$. On the other hand as $F$ is $(\phi, J)$ holomorphic, we have $dF(\xi_1) = dF(\xi_2) = \cdots = dF(\xi_r) = 0$. Again since $(X_1, X_2, \cdots, X_n, \phi X_{n+1}, \cdots, \phi X_m)$ are all vertical vector fields, using the definition of $\bar{\nabla}$, we have

$$\begin{aligned} Tr_g\beta = & \sum_{i=1}^{n}(\bar{\nabla}_{X_i}J)dF(X_i) + \sum_{i=1}^{n}(\bar{\nabla}_{\phi X_i}J)dF(\phi X_i) \\ = & \lambda^2 \sum_{i=1}^{n}\{(\bar{\nabla}_{Y_i}J)Y_i + (\bar{\nabla}_{JY_i}J)JY_i\} \\ = & \lambda^2 \mathrm{div}\, J. \end{aligned}$$

Hence the proposition.

Let us call $GF$-$r$ contact metric structure $M$ as nearly $GF$-$r$ Trans-Sasaki if

$$\begin{aligned} r\{(\nabla_X\phi)Y + (\nabla_Y\phi)X\} = & \alpha \sum_{i=1}^{r}\{2a^2 g(X,Y)\xi_i - \eta_i(Y)X - \eta_i(X)Y\} \\ & -\beta \sum_{i=1}^{r}\{\eta_i(Y)\phi(X) + \eta_i(X)\phi(Y)\}. \end{aligned} \tag{2.5}$$

Now we prove the following theorem.

**Theorem 2.2.** Let $M(\phi, \xi^\alpha, \eta_\alpha, a, g)$ be nearly $GF$-$r$ Trans-Sasaki manifold, $N(J, h)$ be a Quasi-Kaëhler manifold and $F : M \to N$ a $(\phi, J)$ holomorphic map. Then the tension field $\tau(F)$ vanishes and thus $F$ is harmonic map.

**Proof.** Let $\{e_1, e_2, \cdots, e_m, \phi e_1, \phi e_2, \cdots, \phi e_m, \xi_1, \xi_2, \cdots, \xi_r\}$ be an orthonormal $\phi$-adapted local frame on $TM$. Then

$$\mathrm{div}\,\phi = \sum_{i=1}^{n}\{(\nabla_{e_i}\phi)e_i + (\nabla_{\phi e_i}\phi)\phi e_i\} + \sum_{i=1}^{r}(\nabla_{\xi_i}\phi)\xi_i \tag{2.6}$$

From equation (2.5) we have

$$\begin{aligned} (\nabla_{e_i}\phi)e_i = & \frac{a^2\alpha m}{r}\sum_{i=1}^{r}\xi_i \\ (\nabla_{\phi e_i}\phi)\phi e_i = & \frac{a^2\alpha m}{r}\sum_{i=1}^{r}\xi_i \quad \text{and} \quad (\nabla_{\xi_i}\phi)\xi_i = 0 \end{aligned}$$

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Thus we have

$$\operatorname{div}\phi \;=\; \frac{2a^2\alpha m}{r}\sum_{i=1}^{r}\xi_i.$$

On the other hand as $F$ is $(\phi, J)$-holomorphic, we have

$$dF(\phi\xi_1) \;=\; dF(\phi\xi_2) \;=\; \cdots \;=\; dF(\phi\xi_r) \;=\; 0 \tag{2.7}$$

since $N$ is a Quasi-Kahler manifold we obtain

$$\begin{aligned} Tr_g\beta = &\sum_{i=1}^{m}\{(\bar{\nabla}_{e_i}J)dF(e_i) + (\bar{\nabla}_{\phi e_i}J)dF(\phi e_i)\} + \sum(\bar{\nabla}_{\xi_i}J)\xi_i \\ &= 0 \end{aligned}$$

Hence from the formula

$$\begin{aligned} J(\tau(F)) = &\; dF(\operatorname{div}\phi) - Tr_g\beta \\ = &\; dF\left(\frac{2\alpha a^2 m}{r}\sum_{i=1}^{r}\xi_i\right) - 0 \\ = &\; 0 \end{aligned}$$

Hence we obtain that $J(\tau(F)) = 0 \;\Rightarrow\; \tau(F) = 0$. Thus $F$ is harmonic map.

## 3. The Product Manifold $M \times R^r$

Let $M(\phi, \xi^\alpha, \eta_\alpha, a, g)$ be an almost $GF$-$r$ contact metric manifold. Consider the cartesian product manifold $M \times R^r$, $R$ is the field of real members. If $(x^1, x^2, \cdots, x^r)$ be the local coordinates system in $R^r$, the tangent vector of $M \times R^r$ is of the form

$$\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right), \quad a^\alpha \in R, \quad \alpha = 1, 2, \cdots, r.$$

Let $J$ be a (1,1) tensor field on $M \times R^r$ given by

$$J\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right) \;=\; \left(\phi X - \frac{1}{a}\,\eta_\alpha(X)\xi^\alpha, a a^\beta \frac{\partial}{\partial x^\beta}\right) \tag{3.1}$$

where $\alpha, \beta$ be a positive integer, $1 \le \alpha,\ \beta \le r$.

Operating $J$ on equation (3.1) we have

$$J^2\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right) = \left(\phi(\phi X - \frac{1}{a}\eta_\alpha(X)\xi^\alpha) - \frac{1}{a}\,\eta_\beta\left(X - \frac{1}{a}\,\eta_\alpha(X)\xi^\alpha\right)\xi^\beta, \quad a^2 a^\beta \frac{\partial}{\partial x^\beta}\right) \tag{3.2}$$

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making use of (1.4) and after some simplification, we have

$$J^2\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right) \;=\; a^2\left(X, a^\beta, \frac{\partial}{\partial x^\beta}\right).$$

Hence $J$ gives the almost $GF$ complex structure on $M \times R^r$. Thus we have

**Theorem 3.1.** For an almost $GF$-$r$ contact structure manifold $M(\phi, \xi^\alpha, \eta_\alpha, a, g)$ the (1,1) tensor field $J$ defined by (3.1) gives an almost $GF$-complex structure on the cartesian product manifold $M \times R^r$. For an almost $GF$-$r$ contact metric structure $M$, we define the metric $h$ on $M \times R^r$ as follows

$$h\left[\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right), \left(Y, b^\beta \frac{\partial}{\partial x^\beta}\right)\right] \;=\; -\,g(X,Y) + a^\alpha b^\alpha \tag{3.3}$$

where $a^\alpha b^\beta \in R$, $1 \le \alpha, \beta \le r$.

Then in view of equation (3.1) and definition of almost $GF$-$r$ contact metric structure it is easy to show that

$$h\left[J\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right), J\left(Y, b^\beta \frac{\partial}{\partial x^\beta}\right)\right] \;=\; a^2 h\left[\left(X, a^\alpha \frac{\partial}{\partial x^\alpha}\right), \left(Y, b^\beta \frac{\partial}{\partial x^\beta}\right)\right]. \tag{3.4}$$

Hence the metric $h$ is $GF$ structure metric on $M \times R^r$ and $M \times R^r$ admits $GF$- structure.

**Theorem 3.2.** In view of the (1,1) tensor field $J$ given by the equation (3.1) and the metric $h$ given by (3.3), the product manifold $M \times R^r$ admits a $GF$ structure.

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