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A FRAGMENT ON EULER'S CONSTANT IN RAMANUJAN'S LOST NOTEBOOK

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Dedicated to our close friend George Andrews on his 70th birthday

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Abstract: A formula for Euler's constant found in Ramanujan's lost notebook and also in a problem he submitted to the *Journal of the Indian Mathematical Society* is proved and discussed.

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1. Introduction

Like many mathematicians, Ramanujan was evidently fascinated with Euler's constant γ. He wrote only one paper on Euler's constant [10], [11], but published with his lost notebook [13, pp. 274–277] are two partial manuscripts devoted to γ. The first author and D.C. Bowman [4] previously examined one of these partial manuscripts [13, pp.274–275], and in this paper the remaining fragment is examined. Both partial manuscripts are related to early interests of Ramanujan. The first partial manuscript is related to Frullani integrals, which were featured in Ramanujan's *Quarterly Reports* [1,2], [3, pp.295–336], written in the year prior to Ramanujan's departure for Cambridge. The second partial manuscript is related to the first problem that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [8], [11, p. 322] and to the first six entries of Chapter 2 in his second notebook [11], [3, pp.25–35]. Moreover the second partial manuscript gives Ramanujan's solution to another problem [9,] [11, p. 325] that he submitted to the *Journal of the Indian Mathematical Society*. No solution to this problem

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was ever published. The formula for γ in this problem was also recorded in Ramanujan's second notebook as Entry 16 of Chapter 8 [12], [3, p.196]. In [3], we gave a solution based on material in Chapter 2 of Ramanujan's second notebook [12], [3, pp.25–35], where he considers a more general series and derives several elegant theorems and examples. The solution that Ramanujan gives in his lost notebook is not fundamentally different from that given by the first author in [3], but since it is more self contained and independent of our considerations in [3, pp.25–35], for those readers not desiring to read the aforementioned material in Chapter 2 and only interested in a direct route to Ramanujan's formula for Euler's constant, we provide Ramanujan's solution in this short note. Thus, in the next section, we mildly correct Ramanujan's claim and give his proof while providing a few additional details. We conclude our short paper by using Ramanujan's formula to numerically calculate γ. Generalizations of Ramanujan's formula for γ are briefly discussed in [4, p.23].

2. Ramanujan's Formula for γ

At the top of page 276 in [13], Ramanujan writes

$$\begin{aligned}\gamma &= \log 2 - \tfrac{2}{3^3-3} - 4\left(\frac{1}{6^3-6} + \frac{1}{9^3-9} + \frac{1}{12^3-12}\right)\\ &\quad -6\left(\frac{1}{15^3-15} + \frac{1}{18^3-18} + \cdots + \frac{1}{39^3-39}\right) - \cdots ,\end{aligned}$$

$$\text{the last term of the } n\text{th group being } \frac{1}{\left(\frac{3^n+3}{2}\right)^3} - \frac{1}{\frac{3^n+3}{2}}. \tag{2.1}$$

Ramanujan's assertion (2.1) needs to be slightly corrected. The *first*, not the last, term of the nth group is $\dfrac{1}{\left(\frac{3^n+3}{2}\right)^3} - \dfrac{1}{\frac{3^n+3}{2}}$. We give a more precise statement of Ramanujan's claim.

Theorem 2.1.

$$\gamma = \log 2 - \sum_{n=1}^{\infty} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{2.2}$$

Proof. It is easily checked that, for each positive integer k,

$$\frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1} = \frac{1}{k} + \frac{2}{(3k)^3 - 3k}. \tag{2.3}$$

Set $k = 1, 2, \ldots, n$ in (2.3) and add the n equalities to find that

$$\sum_{k=2}^{3n+1} \frac{1}{k} \;=\; \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{2}{(3k)^3 - 3k},$$

i.e.,

$$\sum_{k=1}^{2m+1} \frac{1}{m+k} \;=\; 1 + \sum_{k=1}^{m} \frac{2}{(3k)^3 - 3k}. \tag{2.4}$$

The first three cases, $m = 1, 2, 3$, of (2.4) are, respectively,

$$\begin{aligned}
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= 1 + \frac{2}{3^3 - 3}, \\
\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{13} &= 1 + \frac{2}{3^3 - 3} + \frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}, \\
\frac{1}{4} + \frac{1}{15} + \cdots + \frac{1}{40} &= 1 + \frac{2}{3^3 - 3} + \cdots + \frac{2}{39^3 - 39}.
\end{aligned}$$

More generally, taking $m = 1, 2, \ldots, n$ in (**??**) and adding the n equalities, we find that

$$\begin{aligned}
\sum_{k=1}^{\frac{3^n - 1}{2}} \frac{1}{k} \;=\; & n + (n-1)\frac{2}{3^3 - 3} + (n-2)\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right) \\
& + (n-3)\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \cdots + \frac{2}{39^3 - 39}\right),
\end{aligned} \tag{2.5}$$

where there are n expressions on the right-hand side of (2.5). Now, from the standard definition of Euler's constant, as $n \to$ ı,

$$\begin{aligned}
\sum_{k=1}^{\frac{3^n - 1}{2}} \frac{1}{k} \;&=\; \log\left(\frac{3^n - 1}{2}\right) + \gamma + o(1) \\
&=\; n \log 3 - \log 2 + \gamma + o(1).
\end{aligned} \tag{2.6}$$

If we use (2.6) in (2.5), divide both sides of the resulting equality by n, and then let $n \to$ ı, we deduce that

$$\log 3 \;=\; 1 + \sum_{k=1}^{\infty} \frac{2}{(3k)^3 - 3k}. \tag{2.7}$$

(The identity (2.7) is also found in Section 2 of Chapter 2 in Ramanujan's second notebook [12]; see also [3, p.27]I.) Lastly, using (2.6) in (2.5), letting $n \to \infty$ while invoking (2.7), and rearranging, we readily arrive at (2.2) to complete the proof.

3. Numerical Calculations

Define

$$S_j := \sum_{n=1}^{j} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{3.1}$$

The first fourteen values of $-\gamma + \log 2 - S_j$ are given in the following table.

j	S_j	j	S_j
1	3.25982×10^{-2}	8	3.14043×10^{-8}
2	5.66401×10^{-3}	9	3.87176×10^{-9}
3	8.37419×10^{-4}	10	4.72684×10^{-10}
4	1.15710×10^{-4}	11	5.72414×10^{-11}
5	1.53668×10^{-5}	12	6.88472×10^{-12}
6	1.98621×10^{-6}	13	8.23230×10^{-13}
7	2.51665×10^{-7}	14	6.05812×10^{-14}

These calculations were carried out using Mathematica 5.2. The partial sums in (3.1) are taken with respect to the index n of the outer sum. Thus, (2.2) converges quite rapidly, with only 14 terms needed to determine γ up to an error of order 10^{-14}. If we regard (3.1), or (2.2), as a single sum, i.e., each partial sum contains only one additional term from the inner sum, then the computations take much longer.

Ramanujan's series for γ converges much more rapidly than the standard series definition for γ, namely,

$$\gamma = \lim_{n\to\infty} C_n, \qquad C_n := \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right). \tag{3.2}$$

To compare the use of (3.2) with that of (3.1), which we used in computing the previous table, we list the first fourteen values of $C_n - \gamma$ in the following table.

n	$C_n - \gamma$	n	$C_n - \gamma$
1	0.42278	7	0.069731
2	0.22964	8	0.061200
3	0.15751	9	0.054528
4	0.11982	10	0.049167
5	0.09668	11	0.044766
6	0.081025	12	0.041088
13	0.037969	14	0.035289

For several years, the most effective algorithm for computing γ has been that of Brent and McMillan [5]. The current world record (as of July, 2007) for calculating the digits of

$$\gamma = 0.5772156649015328606065120900824024310421593359399 2\ldots$$

is held by Kondo [7], who calculated 5 billion digits. A clear and informative expository paper on γ has been written by Gourdon and Sebah [6], who provide a table of computational records for calculating the digits of γ.

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