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n-COLOR OVERPARTITIONS, TWISTED DIVISOR FUNCTIONS, AND ROGERS-RAMANUJAN IDENTITIES

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: In the early 90's Andrews discussed a certain q-series whose coefficients are determined by a twisted divisor function. We provide several other examples of this nature. All of these q-series can be interpreted combinatorially in terms of n-color overpartitions, as can some closely related series occurring in identities of the Rogers-Ramanujan type.

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1. Introduction

In 1988 Andrews, Dyson, and Hickerson [16] made an extensive study of a q-series that first came to light in Ramanujan's lost notebook [12,13],

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = 1+q-q^2+2q^3-2q^4+q^5+q^7-2q^8+2q^{10}+\cdots . \tag{1.1}$$

If $r(n)$ denotes the coefficient of q^n in this series, then $r(n)$ has a rather simple combinatorial interpretation - as the number of partitions of n into distinct parts with even rank minus the number with odd rank. Recall that the rank of a partition is the largest part minus the number of parts. On the other hand, Andrews, Dyson, and Hickerson showed that $r(n)$ is almost always 0 and assumes every integer infinitely often, facts which may be deduced from a multiplicative "exact" formula relating $r(24n+1)$ to the arithmetic of $\mathbb{Z}[\sqrt{6}]$.

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Subsequently, in a study again related to Ramanujan's lost notebook, Andrews [14,15] wrote of another series which was "quite reminiscent" of (1.1), having multiplicative coefficients which are almost always 0. This series is

$$\sum_{n\geq 1} \frac{q^{n^2}(1-q)(1-q^2)\cdots(1-q^{n-1})}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})} = q+q^3+q^4+2q^7+q^9+q^{12}+2q^{13}+\cdots. \tag{1.2}$$

The "mystery" of this series was immediately explained by Andrews himself, who showed that the multiplicativity is dictated by a twisted divisor function through the identity

$$\sum_{n\geq 1} \frac{(q)_{n-1}q^{n^2}}{(q;q^2)_n(-q)_n} = \sum_{n\geq 1}\Big(\sum_{d|n}\Big(\frac{d}{3}\Big)\Big)q^n. \tag{1.3}$$

Here we have employed the usual Legendre symbol and rewritten Andrews' series using the standard q-series notation [23].

The first goal of this paper is to exhibit several other q-series which resemble (1.2) both in their q-series definitions and in their having multiplicative coefficients arising from twisted divisor functions.

Theorem 1.1. We have

$$\sum_{n\geq 1} \frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(q;q^2)_n(-q)_n} = -\sum_{n\geq 1}\Big(\sum_{d|n}\Big(\frac{d}{5}\Big)\Big)q^n, \tag{1.4}$$

$$\sum_{n\geq 0} \frac{(q)_{n}(-1)^n q^{n(n+1)/2}}{(q;q^2)_{n+1}(-q)_n} = \sum_{n\geq 0}\Big(\sum_{d|5n+1}\Big(\frac{d}{5}\Big)\Big)q^n, \tag{1.5}$$

and

$$\sum_{n\geq 0} \Big(\frac{(q)_{n}q^{n^2+n}}{(q;q^2)_{n+1}(-q)_n} = \sum_{n\geq 1}\Big(\sum_{d|3n+1}\Big(\frac{d}{3}\Big)\Big)q^n. \tag{1.6}$$

The proofs of these identities are presented in the next section using Bailey pairs. Equation (1.5) is rather straightforward, while for the remaining two we shall use Bailey pairs related to seventh order mock theta functions and pass through real quadratic fields, as in [16,20,25].

Of course identities like these are most fully appreciated when their combinatorial implications are well understood. Thus the second goal of this paper is to develop the combinatorics necessary to interpret the q-series occurring in (1.3)-(1.5) in terms of n-color overpartitions. These extend the n-color partitions whose study was initiated in the 80's by Agarwal, Andrews, and Bressoud

[1,3,4,7,8,9], inspired by work of Baxter [7, Section 3]. The basic definitions and fundamental generating functions (Theorem 3.1 and 3.2) for n-color overpartitions are contained in Section 3.

In Section 4, we then present the combinatorial versions of identities (1.3)-(1.6) as Corollaries 4.5-4.8. We shall also be able to give nice combinatorial interpretations of some identities of the Rogers-Ramanujan type, to which the series in (1.3)-(1.6) are closely related. These are

$$\sum_{n\geq 0}\frac{(-1)_n q^{n^2}}{(q;q^2)_n(q)_n}=\frac{(-q)_\infty(q^3;q^3)_\infty}{(q)_\infty(-q^3;q^3)_\infty}, \tag{1.7}$$

$$\sum_{n\geq 0}\frac{(-1)_n q^{n(n+1)/2}}{(q;q^2)_n(q)_n}=\frac{(-q)_\infty(q^5;q^5)_\infty}{(q)_\infty(-q^5;q^5)_\infty}, \tag{1.8}$$

$$\sum_{n\geq 0}\frac{(-q)_n q^{n(n+1)/2}}{(q;q^2)_{n+1}(q)_n}=\frac{(-q)_\infty(q^3,q^7,q^{10};q^{10})_\infty}{(q)_\infty}, \tag{1.9}$$

and

$$\sum_{n\geq 0}\frac{(-q)_n q^{n^2+n}}{(q;q^2)_{n+1}(q)_n}=\frac{(-q)_\infty(q,q^5,q^6;q^6)_\infty}{(q)_\infty}, \tag{1.10}$$

which are [14,Eq.(4.6)], [29, Eq.(5)], [30, Eq.(45)] and [30, Eq.(22)], respectively. The combinatorial versions of these identities are Corollaries 4.1-4.4. We close the paper with some remarks and ideas for future projects.

2. Proof of Theorem 1.1.

Here we prove the three identities in Theorem 1.1 using the method of Bailey pairs. Recall that two sequences (α_n,β_n) are said to form a Bailey pair with respect to a if for all $n\geq 0$ we have

$$\beta_n \;=\; \sum_{r=0}^{n}\frac{\alpha_n}{(q)_{n-r}(aq)_{n+r}}. \tag{2.1}$$

The following allows us to prove identities using Bailey pairs:

Lemma 2.1([17], Cor. 2.1). If (α_n,β_n) is a Bailey pair with respect to a, then

$$\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n=\frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}, \tag{2.2}$$

provided both sides converge absolutely.

We begin by treating the last equation in Theorem 1.1, which is the easiest. By [15, Eq.(3.16),(3.17)], the sequences (α_n, β_n) form a Bailey pair with respect to q, where

$$\alpha_n = \frac{(-1)^{n-1}(2n+1)q^{n(n+1)/2}}{(1-q)}$$

and

$$\beta_n = -\frac{(q)_n}{(q;q^2)_{n+1}(-q)_n}.$$

Inserting this into Lemma 2.1 with $a = q$ and $\rho_1, \rho_2 \to \infty$ yields

$$\begin{aligned}\sum_{n\geq 0} \frac{(q)_n q^{n^2+n}}{(q;q^2)_{n+1}(-q)_n} &= \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n(2n+1)q^{3n(n+1)/2} \\ &= \frac{(q^3;q^3)_\infty^3}{(q)_\infty},\end{aligned}$$

the last equality following from a well-known identity of Jacobi. Now, the modular form $q(q^9;q^9)_\infty^3/(q^3;q^3)_\infty$ is the twisted Eisenstein series whose coefficient of q^{3n+1} is precisely the coefficient of q^n on the right hand side of (1.6). So this identity is established.

We turn to (1.5), which is somewhat more involved. We begin with a Bailey pair with respect to q related to the seventh order mock theta functions [11, Lemma 12, $i = 2$],

$$\alpha_{2n} = \frac{1}{1-q}\Big(q^{3n^2+n}\sum_{|j|\leq n} q^{-j^2} + 2q^{3n^2+2n}\sum_{j=0}^{n-1} q^{-j^2-j}\Big),$$

$$\alpha_{2n+1} = -\frac{1}{1-q}\Big(2q^{3n^2+4n+1}\sum_{j=0}^{n} q^{-j^2-j} + q^{3n^2+5n+2}\sum_{|j|\leq n} q^{-j^2}\Big),$$

and

$$\beta_n = \frac{1}{(q;q^2)_{n+1}(-q)_n}.$$

Inserting this Bailey pair into Lemma **??** with $\rho_1 = q$ and $\rho_2 \to \infty$ and simplifying gives

$$\begin{aligned}\sum_{n\geq 0} \tfrac{(q)_n(-1)^n q^{n(n+1)/2}}{(q;q^2)_{n+1}(-q)_n} &= \sum_{\substack{n\geq 0 \\ |j|\leq n}} q^{5n^2+2n-j^2} + \sum_{\substack{n\geq 0 \\ |j|\leq n}} q^{5n^2+8n+3-j^2} \\ &\quad + \sum_{n\geq 0}\sum_{j=-n}^{n-1} q^{5n^2+3n-j^2-j} \\ &\quad + \sum_{n\geq 0}\sum_{j=-n-1}^{n} q^{5n^2+7n+2-j^2-j}.\end{aligned} \tag{2.3}$$

We shall now proceed to demonstrate that the right hand side above is precisely the generating function for the number of inequivalent elements of (positive) norm $5k+1$ in the ring of integers of $\mathbb{Q}(\sqrt{5})$. Recall that this ring of integers is $\mathbb{Z}[(1+\sqrt{5})/2]$, and the norm function is $N(x+y\sqrt{5}) = x^2 - 5y^2$. The solutions to $x^2 - 5y^2 = 5k+1$ come in two flavors: x and y integers with $x \equiv \pm 1 \pmod 5$ or x and y half integers with $2y$ odd and $2x \equiv \pm 3 \pmod{10}$. It turns out that the first two sums above correspond to solutions of the first type and the second two sums to solutions of the second type.

To see this, first note that the element $x+y\sqrt{5}$ of $\mathbb{Z}[(1+\sqrt{5})/2]$ whose norm is 1 with x and y minimal positive is $3/2+\sqrt{5}/2$. Hence, by [16, Lemma 3], for any $m>0$ each equivalence class of solutions of $u^2-5v^2=m$ contains a unique $u+v\sqrt{5}$ with

$$u > 0 \ \text{ and } \ -u < v \leq u. \tag{2.4}$$

Next, in the first two sums on the right hand side of (2.3), we let $q = q^5$ and multiply by q. The exponent of q in the first sum is $(5n+1)^2 - 5j^2$ and in the second sum it is $(5n+4)^2-5j^2$. As mentioned above, these are the two possible ways to represent a number of the form $5k+1$ as x^2-5y^2 with x and y integers. Moreover, replacing u by $5n+1$ or $5n+4$ and v by j in (2.4), we see that the inequalities $n \geq 0$ and $|j| \leq n$ in the two sums guarantee that we have at most one element of norm $5k+1$.

The treatment of the second two sums is similar, so we omit the details. Here one regards the solution to $x^2-5y^2=5k+1$ with x and y half-integers as a solution to $x^2-5y^2=20k+4$ with y odd and $x \equiv \pm 3 \pmod{10}$. Then q is replaced by q^{20}, a factor of q^4 is multiplied, and one proceeds as above.

To finish the proof of identity (1.5), we may quote from [18] that if $a(m)$ denotes the number of elements of norm m in $\mathbb{Z}[(1+\sqrt{5})/2]$, then (i) $a(mn) = a(m)a(n)$ if m and n are relatively prime, (ii) $a(5^r)=1$, (iii) $a(p^r) = r+1$ for primes $p \equiv \pm 1 \pmod 5$ and (iv) $a(p^r) = (1+(-1)^r)/2$ for primes $p \equiv \pm 2 \pmod 5$. It is easy to see that this is equivalent to counting divisors d of m weighted by the Legendre symbol $\frac{d}{5}\Big)$.

Finally, we turn to the first identity in Theorem 1.1. We begin again with a Bailey pair related to the seventh order mock theta functions, this time a Bailey pair with respect to 1 [11, Lemma 12, $i=0$],

$$\alpha_{2n} = q^{3n^2+n} \sum_{|j|\leq n} q^{-j^2} - q^{3n^2-n} \sum_{|j|<n} q^{-j^2},$$

$$\alpha_{2n+1} = -2q^{3n^2+4n+1} \sum_{j=0}^{n} q^{-j^2-j} + q^{3n^2+2n} \sum_{j=0}^{n-1} q^{-j^2-j},$$

and

$$\beta_n = \frac{1}{(q;q^2)_n(-q)_n}.$$

Instead of using this pair directly in Lemma 2.1, we first transform it using the following lemma:

Lemma 2.2(Lemma 2.2, [25]). If (α_n, β_n) is a Bailey pair with respect to 1, then (α_n^*, β_n^*) is a Bailey pair with respect to q, where $\alpha_0^* = \beta_0^* = 0$, and for $n \geq 1$,

$$\beta_n^* \ = \ \frac{-1}{(1-q^n)}\beta_n \tag{2.5}$$

and

$$\alpha_n^* = \frac{(-1)^n q^{n(n-1)/2}(1-q^{2n+1})}{(1-q)}\Big(n + \sum_{r=1}^{n} \frac{(-1)^{r+1}q^{-r(r-1)/2}\alpha_r}{(1-q^r)} + 2\sum_{r=1}^{n}\frac{q^r}{(1-q^r)}\Big). \tag{2.6}$$

Now if we insert our Bailey pair into Lemma 2.2 and then put this new pair into Lemma 2.1 with $\rho_1 = q$ and $\rho_2 \to \infty$, then on the "β side", we have the negative of the left hand side of (1.4). For the "α side", leaving the α_n^* in the form (2.6) for now, notice that we obtain an expression of the form

$$\sum_{n\geq 1} q^{n^2}(1-q^{2n+1})\sum_{r=1}^{n} f_r,$$

where the f_r comes from the parentheses in (2.6). Thus this collapses to $\sum_{n\geq 1} q^{n^2} f_n$, with

$$f_n = 1 + \frac{(-1)^{n+1}q^{-n(n-1)/2}\alpha_n}{(1-q^n)} + \frac{2q^n}{(1-q^n)}.$$

Now we insert the α_n we started with and after some simplification we may obtain the identity

$$-\sum_{n\geq 1}\frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(q;q^2)_n(-q)_n} = \sum_{n\geq 1}\sum_{j=-n+1}^{n} q^{5n^2-j^2} + \sum_{n\geq 0}\sum_{|j|\leq n} q^{5n^2+5n+1-j^2-j}. \tag{2.7}$$

Now we proceed as in the proof of (1.5) above. We shall discover that the right hand side of (2.7) is the generating function for the number of (inequivalent) elements of negative norm $-m$ in $\mathbb{Z}[(1+\sqrt{5})/2]$. Since this ring contains the element $1/2 + \sqrt{5}/2$ of norm -1, this is the same as counting elements of positive norm m. Then citing the formula of [18] as in the last paragraph of the proof of (1.5), we will have established (1.4).

So, we again quote Lemma 3 of [16], which tells us that for each equivalence class of elements of negative norm $-m$, there is a unique representative $j + n\sqrt{5}$ with

$$n > 0 \ \text{ and} - n < j \leq n. \tag{2.8}$$

But these are precisely the inequalities in the first sum of (2.7). This then takes care of the elements of norm $-m$ where this unique representative has j and n integral. The second sum takes care of the case when j and n are half-integers. To see this, we rewrite the exponent of q as $5((2n+1)/2)^2 - ((2j+1)/2)^2$. Then replacing n by $(2n+1)/2$ and j by $(2j+1)/2$ in (2.8) gives $n \geq 0$ and $|j| \leq n$, which are precisely the inequalities in our second sum. This completes the proof of (1.4) and of Theorem 1.1.

3. n-Color Overpartitions

In this section we define the basic notions associated with n-color overpartitions and determine some basic generating functions. An n-color partition is a partition in which each number n may appear in n colors, with parts ordered first according to size and then according to color. For example, there are 13 n-color partitions of 4,

$$(4_4), (4_3), (4_2), (4_1), (3_3, 1_1), (3_2, 1_1), (3_1, 1_1), (2_2, 2_2), (2_2, 2_1), (2_1, 2_1),$$
$$(2_2, 1_1, 1_1), (2_1, 1_1, 1_1), (1_1, 1_1, 1_1, 1_1).$$

Motivated by Baxter's solution of the hard hexagon model (see [7, Section 3]), the study of n-colored partitions was initiated in the 1980's by Agarwal, Andrews, and Bressoud. These partitions arose in identities of the Rogers-Ramanujan type (e.g. [1,3,4,7]) and they played an influential role in the development of the Bailey lattice [8, Section 5]. They have been related to a number of other combinatorial objects, such as lattice paths, generalized Frobenius partitions, and plane partitions (e.g. [2,5,6,9]). Indeed, the latter have the same generating function as unrestricted n-color partitions,

$$\prod_{n\geq 1} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \cdots \ .$$

An exposition of much of the above, as well as further results on n-color partitions, may be found in the book Partition Theory [10].

We define an n-color overpartition to be an n-color partition in which the final occurrence of a part n_j may be overlined. For example, there are 16 n-color overpartitions of 3,

$$\begin{aligned}&(3_3), (3_2), (3_1), (\overline{3}_3), (\overline{3}_2), (\overline{3}_1), (2_2, 1_1), (2_1, 1_1), (\overline{2}_2, 1_1), (2_2, \overline{1}_1),\\ &(\overline{2}_2, \overline{1}_1), (\overline{2}_1, 1_1), (2_1, \overline{1}_1), (\overline{2}_1, \overline{1}_1), (1_1, 1_1, 1_1), (1_1, 1_1, \overline{1}_1).\end{aligned} \tag{3.1}$$

The generating function for n-color overpartitions is clearly

$$\prod_{n\geq 1}\frac{(1+q^n)^n}{(1-q^n)^n}=1+2q+6q^2+16q^3+38q^4+88q^5+\cdots \tag{3.2}$$

We shall be interested in n-color overpartitions whose weighted difference is bounded below. The weighted difference of two adjacent parts m_i and n_j, denoted $((m_i,n_j))$, is defined to be $m-i-n-j$. For an n-colored overpartition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_s)$, we write $c(\lambda_j)$ for the color of λ_j. The next theorem provides the n-color overpartition-theoretic basis for interpreting q-series which contain the term $1/(q;q^2)_n$.

Theorem 3.1. If we define $f_m(r,s,k)$ by

$$\sum_{r,s,k\geq 0} f_m(r,s,k)a^rb^sq^k := \frac{(-a)_m q^{m(m+1)/2}}{(q;q^2)_m(bq)_m}, \tag{3.3}$$

then $f_m(r,s,k)$ is the number of n-color overpartitions λ of k into m parts such that

(i) the weighted differences satisfy $((m_i,n_j))\geq -1+\chi(n_j\text{overlined})$,

(ii) r is the number of overlined parts, and

(iii) $s = \lambda_1 - c(\lambda_1)+\sum_{j\geq 2}\Big(\chi(\lambda_j \text{ non-overlined})-2c(\lambda_j)\Big)$.

Proof. The proof is by classical methods of constructive partition theory. On the right side of (3.3), the term $q^{m(m+1)/2}/(bq)_m$ is the generating function for partitions μ_1 into m distinct parts, where the exponent of b tracks the largest part minus m. Notice that if we assign the color 1 to each part, then μ_1 is an n-color (over)partition whose weighted differences are at least -1, with equality if an only if two adjacent parts are consecutive integers.

Now the factor $1/(q;q^2)_m$ contributes a partition μ_2 into odd parts at most $2m-1$. These are added to μ_1 in the following way: For a part $2t+1$ of μ_2, we add 2 to each of the first t largest parts of μ_1 and then to the $t+1$st largest part we add 1 to both the part size and the color. It is clear that this operation leaves all weighted differences fixed. It is also easy to see that this procedure is invertible, since the colors indicate exactly where an odd part has been added.

Finally, the factor $(-a)_m$ gives a partition μ_3 into distinct non-negative parts less than m, where the exponent of a counts the number of parts. We add this bijectively to our object in the usual way, for each part t of μ_3 adding 1 to the t largest parts and then overlining the next part. This doesn't change the weighted differences except when the smaller part picks up an overline. This is accounted for by the characteristic function in condition (i).

To finish the proof, it is clear that r is the number of overlined parts and the number of parts is m. In order to see why condition (iii) holds, we define

$$\sigma \ = \ \lambda_1 - c(\lambda_1) + \sum_{j\geq 2}\Big(\chi(\lambda_j \text{ non-overlined}) - 2c(\lambda_j)\Big).$$

In μ_1, we have $s = \lambda_1 - m = \sigma$ since there are no overlined parts and all the colors are equal to one. When we apply μ_2, every part $2t+1$ adds 2 to λ_1 and 1 to both λ_{t+1} and its color. If $t = 0$, both λ_1 and $c(\lambda_1)$ increase by 1, so σ does not change; if $t > 0$, λ_1 increases by 2 and $c(\lambda_{t+1})$ increases by 1, so σ does not change either. When we apply μ_3, every part $t > 0$ adds 1 to λ_1 and overlines λ_{t+1}, so σ is not modified. If $t = 0$, λ_1 is not modified; it becomes overlined, but since the summation in the definition of σ begins at $j = 2$, this does not change the value of σ. Hence, we have $s = \sigma$ in the final n-color overpartition, so condition (iii) is verified.

For series with the term $1/(q;q^2)_{n+1}$, we shall consider $(n+1)$-color overpartitions. Here the number n may occur in any of the $n+1$ colors $\{1, 2, \ldots, n+1\}$.

Theorem 3.2. If we define $g_m(r, s, k)$ by

$$\sum_{r,s,k\geq 0} g_m(r,s,k)a^r b^s q^k := \frac{(-aq)_m q^{m(m+1)/2}}{(q;q^2)_{m+1}(bq)_m}, \tag{3.4}$$

then $g_m(r, s, k)$ is the number of $(n+1)$-color overpartitions λ of k into $m+1$ non-negative parts such that (i) the smallest part has the form i_{i+1} (overlined or not), (ii) the largest part is non-overlined, (iii) the weighted differences satisfy $((m_i, n_j)) \geq -1 + \chi(n_j \text{ overlined})$, (iv) r is the number of overlined parts, and (v) $s \ = \ 1 + \lambda_1 - c(\lambda_1) + \sum_{j\geq 2}\Big(\chi(\lambda_j \text{ non-overlined}) - 2c(\lambda_j)\Big)$.

Proof. We proceed more or less as in Theorem 3.1. This time, however, the term $q^{m(m+1)/2}/(bq)_m$ will contribute a partition μ_1 into $m+1$ distinct non-negative parts with color 1 (inserting a 0_1). The exponent of b again starts off as the largest part minus m. The factor $1/(q;q^2)_{m+1}$ contributes a partition μ_2 into odd parts at most $2m+1$, and this is applied to μ_1 just as before. Note that the smallest part will always have the form i_{i+1}. Finally, the factor $(-aq)_m$ contributes a partition μ_3 into distinct parts at most m, which is also applied as before. Since μ_3 cannot contain 0 as a part, we never have the largest part overlined. The verification of condition (v) is very similar to the verification of condition (iii) in Theorem 3.1.

4. The Combinatorial Versions

We now use the discussion of the previous section to give combinatorial versions of identities (1.7)-(1.10) and (1.3)-(1.6) (in this order).

Corollary 4.1. Let $A_1(k)$ denote the number of n-color overpartitions of k whose weighted differences satisfy $((m_i, n_j)) \geq \chi(n_j \text{ overlined})$. Let $B_1(k)$ denote the number of overpartitions of k into parts not divisible by 3. Then $A_1(k) = B_1(k)$.

Proof. This follows from (1.7) and Theorem 3.1. Specifically, the summand on the left hand side of (1.7) is the right hand side of (3.3) with $a = b = 1$ multiplied by $q^{m(m-1)/2}$. This multiplication corresponds to adding a staircase $(m-1, m-2, \ldots, 0)$ to the relevant n-color overpartitions, which simply augments the minimum weighted difference in condition (i) by 1. Hence the left hand side of (1.7) is the generating function for $A_1(k)$, while the right hand side is obviously the generating function for $B_1(k)$.

Corollary 4.2. Let $A_2(k)$ denote the number of n-color overpartitions of k whose weighted differences satisfy $((m_i, n_j)) \geq -1 + \chi(n_j \text{ overlined})$. Let $B_2(k)$ denote the number of overpartitions of k into parts not divisible by 5. Then $A_2(k) = B_2(k)$.

Proof. This follows immediately from the case $a = b = 1$ of Theorem 3.1 and equation (1.8).

Corollary 4.3. Let $A_3(k)$ denote the number of $(n + 1)$-color overpartitions of k such that (i) the largest part is non-overlined, (ii) the smallest part has the form i_{i+1} (overlined or not), and (iii) the weighted differences satisfy $((m_i, n_j)) \geq -1 + \chi(n_j \text{ overlined})$. Let $B_3(k)$ denote the number of overpartitions of k whose non-overlined parts are not congruent to 0 or ± 3 modulo 10. Then $A_3(k) = B_3(k)$.

Proof. This follows immediately from equation (1.9) and the case $a = b = 1$ of Theorem 3.2.

Corollary 4.4. Let $A_4(k)$ denote the number of $(n + 1)$-color overpartitions of k into non-negative parts such that (i) the largest part is non-overlined, (ii) the smallest part has the form i_{i+1} (overlined or not), and (iii) the weighted differences satisfy $((m_i, n_j)) \geq \chi(n_j \text{ overlined})$. Let $B_4(k)$ denote the number of overpartitions of k whose non-overlined parts are not congruent to 0 or ± 1 modulo 6. Then $A_4(k) = B_4(k)$.

Proof. This follows from equation (1.10) and the case $a = b = 1$ of Theorem 3.2 in the same way that Corollary 4.1 followed from equation (1.7) and the case $a = b = 1$ of Theorem 3.1.

Corollary 4.5. Let $A_5(k)$ denote the number of n-color overpartitions of k counted by $A_1(k)$ with $\lambda_1 - c(\lambda_1)$ even minus the number with $\lambda_1 - c(\lambda_1)$ odd. Then

$$A_5(k) = 2\sum_{d|k} \left(\frac{d}{3}\right).$$

Proof. Setting $b = -1$, replacing $(-a)_m$ by $2(q)_{m-1}$, and multiplying by $q^{m(m-1)/2}(-1)^m$ on the right hand side of (3.3), keeping track of the corresponding weight, and then appealing to (1.3) gives the corollary.

Corollary 4.6. Let $A_6(k)$ denote the number of n-color overpartitions of k counted by $A_2(k)$ with $\lambda_1 - c(\lambda_1)$ even minus the number with $\lambda_1 - c(\lambda_1)$ odd. Then

$$A_6(k) = 2\sum_{d|k} \left(\frac{d}{5}\right).$$

Proof. This follows from equation (1.4) in the same way that Corollary 4.5 follows from (1.3).

Corollary 4.7. Let $A_7(k)$ denote the number of $(n+1)$-color overpartitions of k counted by $A_3(k)$ with $\lambda_1 - c(\lambda_1)$ odd minus the number with $\lambda_1 - c(\lambda_1)$ even. Then

$$A_7(k) = \sum_{d|5k+1} \left(\frac{d}{5}\right).$$

Proof. We replace a and b by -1 in Theorem 3.2, multiply the right hand side of (3.4) by $(-1)^m$ and keep track of the weight.

Corollary 4.8. Let $A_8(k)$ denote the number of $(n+1)$-color overpartitions of k counted by $A_4(k)$ with $\lambda_1 - c(\lambda_1)$ odd minus the number with $\lambda_1 - c(\lambda_1)$ even. Then

$$A_8(k) = \sum_{d|3k+1} \left(\frac{d}{3}\right).$$

Proof. By now, this should be routine.

For an example to illustrate some of these corollaries, observe that $A_1(4) = B_1(4) = 10$, the overpartitions counted by $B_1(4)$ being

$$(4), (\overline{4}), (2,2), (2,\overline{2}), (2,1,1), (\overline{2},1,1), (2,1,\overline{1}), (\overline{2},1,\overline{1}), (1,1,1,1), (1,1,1,\overline{1}),$$

while the n-color overpartitions counted by $A_1(4)$ are

$$(4_4)(4_3), (4_2), (4_1), (\overline{4}_4), (\overline{4}_3), (\overline{4}_2), (\overline{4}_1), (3_1, 1_1), (\overline{3}_1, 1_1).$$

Notice also that $A_5(4) = 2 = 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{3}\right)$, as predicted by Corollary 4.5.

For another example, we have $A_4(4) = B_4(4) = 6$, the overpartitions counted by $B_4(4)$ being

$$(4), (\overline{4}), (3, \overline{1}), (\overline{3}, \overline{1}), (2, 2), (2, \overline{2}),$$

while the $(n+1)$-color overpartitions counted by $A_4(4)$ are

$$(4_5), (4_1, 0_1), (4_1, \overline{0}_1), (4_2, 0_1), (4_2, \overline{0}_1), (4_3, 0_1).$$

Notice also that $A_8(4) = 2 = \left(\frac{1}{3}\right) + \left(\frac{13}{3}\right)$, as predicted by Corollary 4.8.

5. Concluding Remarks

We wish to conclude with several remarks. First Theorem 1.1 emphasizes the point, made in [25] and based on the work in [20] and [25], that one can "sniff out" examples of q-series with behavior resembling that of (1.1) by appealing to the combinatorics of overpartitions. The combinatorics of ordinary partitions more typically leads to mock theta functions. For example, the series

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q^2)_n(-q)_n}$$

can be interpreted in terms of ordinary n-color partitions, and it is one of Ramanujan's 7th order mock theta functions [11, Eq. (7.22)].

Second, some of the products in equations (1.7)-(1.10) have appeared in other studies of overpartitions. Namely, the product in (1.7) occurs in the case $k = 3$ of [24, Theorem 1.1], the case $k = 2$ of [26, Theorem 1.2], Corollary 1.4 of [27], and Corollary 1.5 of [19], while the product in (1.8) occurs in the case $k = 3$ of [26, Theorem 1.2] and the product in (1.10) occurs in the case $k = 3$ of [24, Theorem 1.2].

Finally, this is probably just the beginning for n-color overpartitions and related objects. The generating function for unrestricted n-color overpartitions (3.2) turns up in studies of plane partitions (e.g. [21,22,31]) and one suspects that there is a rich theory of plane overpartitions analogous to that of ordinary plane partitions. Additionally, the Rogers-Ramanujan-type identities in (1.7)-(1.10) can undoubtedly be embedded in infinite families of identities in the same way as Rogers-Ramanujan-type identities for n-color partitions [8,9]. In fact, this can probably be further generalized to n-color overpartition pairs (c.f. [28]).

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