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# A THEOREM CHARACTERIZING KEY HYPERGEOMETRIC TRANSFORMATIONS IN RAMANUJAN'S DEVELOPMENT OF CLASSICAL AND ALTERNATIVE THEORIES OF ELLIPTIC FUNCTIONS[1]

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*Dedicated to Professor G.E. Andrews on his seventieth birthday*

## 1. Introduction

In his famous paper on modular equations and approximations to $\pi$, Ramanujan [2] remarks that "There are corresponding theories in which $q$ is replaced by one or the other of the functions

$$q_3 := \exp\Big(-\frac{2\pi}{\sqrt{3}}\cdot\frac{K_3'}{K_3}\Big), \quad q_4 := \exp\Big(-\frac{\pi\sqrt{2}K_4'}{K_4}\Big)$$

and

$$q_6 := \exp\Big(-\frac{2\pi K_6'}{K_6}\Big),$$

where

$$K_3 := K_3(k) := {}_2F_1 Big(\frac{1}{3},\frac{2}{3};1;k\Big) \quad K_4 := K_4(k) := {}_2F_1\Big(\frac{1}{4},\frac{3}{4};1;k\Big)$$

and

$$K_6 := K_6(k) := {}_2F_1\Big(\frac{1}{6},\frac{5}{6};1;k\Big)."$$

Here $K_j'$ stands for $K_j(k')$ with $k' = \sqrt{1-k^2}$ and $q$ stands for the classical base given by $q = q_2 := \exp\Big(-\frac{\pi K'}{K}\Big)$ with $K := K(k) := {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;k\Big)$. In fact on six pages, pp. 257-262, of his second notebook [3], Ramanujan gives approximately 50 results without proofs in these theories. All these results have been proved by Berndt, Bhargava and Garvan in [1], where the theory based on $q_r$ is called theory to base $q_r$ or theory to signature $r$. Among these results are the following hypergeometric transformations (1.1)-(1.8), which play a key role

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in the development of the alternative theories especially in establishing numerous modular equations, and have been given differing treatments in [1], including use of computer package MAPLE, parameterization and evaluation theorems belonging to the above theories and other hypergeometric transformations. Our purpose here is, to obtain a common theorem which characterizes not only all the key transformations (1.1) - (1.8) but all such transformations. However, we demonstrate the working of our theorem by applying it to (1.1) - (1.8) only, and without recourse to computers.

$$\text{(1)}\quad \left(1+p+p^2\right)\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)=\sqrt{1+2p}\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right) \tag{1.1}$$

where

$$\alpha = \frac{p^3(2+p)}{1+2p} \quad\text{and}\quad \beta=\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\ ;$$

$$\text{(2)}\quad {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)=(1+p)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right) \tag{1.2}$$

where

$$\alpha = \frac{p(3+p)^2}{2(1+p)^3} \quad\text{and}\quad \beta=\frac{p^2(3+p)}{4}\ ;$$

$$\text{(3)}\quad {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-p}{1+2p}\right)^3\right)=(1+2p)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;p^3\right); \tag{1.3}$$

$$\text{(4)}\quad (2+2p-p^2)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)=2(1+4p+p^2)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right) \tag{1.4}$$

where

$$\alpha=\frac{27p(1+p)^4}{2(1+4p+p^2)^3} \quad\text{and}\quad \beta=\frac{27p^4(1+p)}{2(2+2p-p^2)^3}\ ;$$

$$\text{(5)}\quad {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-\left(\frac{1-p}{1+3p}\right)^2\right)=\sqrt{1+3p}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;p^2\right); \tag{1.5}$$

$$\text{(6)}\quad \sqrt{27-18p-p^2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\alpha\right)=3\sqrt{3+6p-p^2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right) \tag{1.6}$$

where

$$\alpha=\frac{64p}{(3+6p-p^2)^2} \quad\text{and}\quad \beta=\frac{64p^3}{(27-18p-p^2)^2}\ ;$$

$$\text{(7)}\quad {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2p}{1+p}\right)=\sqrt{1+p}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;p^2\right); \tag{1.7}$$

and

$$(8) \quad \sqrt{1+2p}\ {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\beta\right) = \sqrt{1+p+p^2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right); \tag{1.8}$$

where

$$\alpha = \frac{p(2+p)}{1+2p} \quad \text{and} \quad \beta = \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \ .$$

In all the above, it is assumed that $0 \leq p < 1$.

Transformation (1.1), along with its companion found in [1,3], is helpful in going over from some of the classical results (signature 2) to the results in the cubic theory (theory to base $q_3$) and vice-versa. Transformations (1.1) and (1.2) have been proved in [1] by employing various parameterization and evaluation theorems in theories to signature 2 and 3. Transformation (1.2) along with its companion, also found in [1, 3], gives us some modular equations in the cubic theory [1,3]. Transformation (1.3) is crucial in establishing the "cubic" analogue of the well known quartic theta function identity of Jacobi which plays a fundamental role in the classical theories of theta and elliptic functions. (1.3) has been proved in [1] as a special case of a more general transformation found by the use of MAPLE. This cubic modular equation in turn is important in establishing certain inversion, evaluation and parameterization theorems in the cubic theory. Transformation (1.4) also belongs to the cubic theory and that along with its companion found in [1, 3] is helpful in establishing further modular equations in the cubic theory [1,3]. (1.4) has been proved in [1] via (1.2) and by employing further parameterizations. Transformations (1.5), (1.6) and (1.7) belong to the quartic theory (theory to base $q_4$) and (1.7) is useful in obtaining some of the theorems in the theory to base $q_4$ from those in classical theory and vice-versa. (1.7) is the same as Entry (33)(i) of chapter 11 of [3] and its proof is included here for completeness. In [1], (1.5) has been proved via (1.7) and a companion formula based on Entry (33)(iv) of chapter 11 of [3]. Further, the proof of (1.6) given in [1] is based on parameterization results in quartic theory. Similarly, transformation (1.8) belongs to sextic theory (theory to base $q_6$) and it is helpful in casting theorems in classical theory into those in theory of signature 6 and vice-versa [1,3]. Proof of (1.8) furnished in [3] is based on a transformation taking ${}_2F_1\left(\frac{1}{4},\frac{1}{4};1;z\right)$ to ${}_2F_1\left(\frac{1}{12},\frac{5}{12};1;-27z^2(z-4)^{-3}\right)$.

In what follows, Section 2 is devoted to our main theorem, Sections 3, 4 and 5 respectively to transformations in cubic theory, quartic and sextic theories. Incidental to our work and for completeness, we record in Section 6 the differential equations governing (1.1)-(1.8), without going over the details of their derivations.

## 2. Main Theorem

We straight away go to our main theorem

**Theorem 2.1.** Let $a$ and $b$ constants with $0 < a,\ b < 1$ and let $\alpha = \alpha(p), \beta = \beta(p),\ k = k(p)$ and $l = l(p)$ be rational functions of $p$ with $\alpha(0) = \beta(0) = 0$ and $k(0) = l(0)$ such that $p = 0$ is a simple pole with residue 1 of each of the functions $C_{f'}$ and $C_{g'}$ defined in (2.5) below and such that $p = 0$ is either a simple pole or point of analyticity of each of $C_f$ and $C_g$ defined in (2.6) below. Then the transformation

$$f(p) := \frac{{}_2F_1\left(a, 1-a; 1; \alpha\right)}{\sqrt{k}} = \frac{{}_2F_1\left(b, 1-b; 1; \beta\right)}{\sqrt{l}} =: g(p) \tag{2.1}$$

holds if and only if the following conditions are satisfied,

$$B = \frac{CA}{t} \tag{2.2}$$

and

$$\frac{b(1-b)\beta'}{B} = \frac{a(1-a)\alpha\,'}{A} + T\left[\log\left(\frac{AT}{\sqrt{t}}\right)\right]', \tag{2.3}$$

where $C$ is some constant and

$$t = \frac{l}{k}\ ,\ \ A = \frac{\alpha(1-\alpha)}{\alpha\,'}\ ,\ \ B = \frac{\beta(1-\beta)}{\beta'}\ ,\ \ T = \frac{1}{2}(\log t)'$$

and

$$(\cdot)' = \frac{d}{dp}(\cdot).$$

Equivalently, $f$ and $g$ both satisfy the same differential equation

$$u'' + C_u' u' + C_u u \ = \ 0 \tag{2.4}$$

with

$$C_{f'} := \frac{(Ak)'}{(Ak)} = \frac{(Bl)'}{(Bl)} =: C_{g'} \tag{2.5}$$

and

$$C_f := \frac{k'}{k}\left\{\log\left(\frac{Ak'}{\sqrt{k}}\right)\right\}' - \frac{a(1-a)\alpha}{A} = \frac{l'}{l}\left\{\log\left(\frac{Bl'}{\sqrt{l}}\right)\right\}' - \frac{b(1-b)\beta'}{B} =: C_g \tag{2.6}$$

**Proof.** Firstly, we have the well known Gaussian differential equation

$$\frac{d}{d\alpha}\left\{\alpha(1-\alpha)\frac{d}{d\alpha}{}_2F_1\left(a, 1-a; 1; \alpha\right)\right\} = a(1-a)\ \ {}_2F_1\left(a, 1-a; 1; \alpha\right).$$

This becomes, on using the notations described in (2.1) and (2.3),

$$\left(A\sqrt{k}f' + \frac{Ak'}{2\sqrt{k}}\,f\right)' = \alpha' a(1-a)\sqrt{k}f,$$

or,

$$A\sqrt{k}f'' + \frac{Ak'}{\sqrt{k}}f' + \left\{\left(\frac{Ak'}{2\sqrt{k}}\right)' - \alpha a(1-a)\sqrt{k}\right\} f = 0,$$

or, on slight manipulation,

$$f'' + \frac{(Ak)'}{(Ak)}f + \left\{\frac{k'}{k}\left(\log\left(\frac{Ak'}{\sqrt{k}}\right)\right)' - \frac{a(1-a)\alpha'}{A}\right\} f = 0. \tag{2.7}$$

Thus $f$ satisfies (2.4) with $u = f$ and $C_{f'}$ and $C_f$ as defined in (2.5) and (2.6). Similarly, $g$ also satisfies (2.4) with $u = g$ and $C_{g'}$ and $C_g$ as defined in (2.5) and (2.6).

Now, it is easy to verify that , under the conditions given on $a, b, \alpha, \beta, k$ and $l$, $p = 0$ is a regular singular point of (2.4) in each of the cases $u = f,\ u = g$ and that the associated indicial equation is $r^2 = 0$. (For completeness we give the details of this at the end of the proof of this theorem.) Thus one member of a fundamental set of solutions has a logarithmic singularity at $p = 0$. Since $f(0) = g(0)(= 1)$ by the definitions of $f$ and $g$ in (2.1), it follows that for (2.1) to hold, it is necessary and sufficient that the two differential equations obtained by putting $u = f$ and $u = g$ in (2.4) are identical. But this is the same as saying that $f$ and $g$ satisfy (2.4)-(2.6).

Now, (2.5) is equivalent to (2.2), as seen by integrating (2.5). That (2.6) is equivalent to (2.3) is seen on slight manipulations, using the definition of $A, B$ and $T$.

For completeness of the proof, we verify that $p = 0$ is indeed a regular singular point of (2.4) in the case $u = f$ and that the associated indicial equation is $r^2 = 0$. The other case $u = g$ is similar. Now, (2.4) with $u = f$ is

$$f'' + C_{f'}f' + C_f f \ = \ 0 \tag{2.8}$$

where, under the given conditions on $a, b, \alpha, \beta, k$ and $l$, the coefficients $C_{f'}$ and $C_f$ have the power series expansions

$$C_{f'} \ = \ \frac{b_1(p)}{p} \ = \ \frac{1}{p}\Big(b_{10} + b_{11}p + b_{12}p^2 + \cdots\Big), \ \text{ with } \ b_{10} = 1, \tag{2.9}$$

$$C_f = \frac{b_2(p)}{p^2} = \frac{1}{p^2}\left(b_{20} + b_{21}p + b_{22}p^2 + \cdots\right), \ \text{with } \ b_{20} = 0, \tag{2.10}$$

where $b_1(p)$ and $b_2(p)$ are analytic at $p=0$. Thus, in the neighbourhood of $p=0$, (2.8) becomes

$$p^2 f'' + p b_1(p)\, f' + b_2(p)\, f = 0. \tag{2.11}$$

Functions $b_1(p)$ and $b_2(p)$ being analytic at $p=0$, it is clear that $p=0$ is a regular singular point of (2.11) and the associated indicial equation is

$$r(r-1) + b_{10} r + b_{20} \;=\; r(r-1) + r + 0 = r^2 = 0.$$

This completes the proof of the theorem.

## 3. Transformations in Cubic Theory

In this section we demonstrate the working of our results of Section 2 by obtaining some of Ramanujan's hypergeometric transformations in cubic theory.

**Theorem 3.1(p.258, [3]).** We have

$$\left(1+p+p^2\right)_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right) = \sqrt{1+2p}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)$$

where

$$\alpha \;=\; \frac{p^3(2+p)}{1+2p} \quad \text{and} \quad \beta \;=\; \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}.$$

given $0 \le p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = \frac{1}{2}\,,\; l = (1+p+p^2)^2,\; b = \frac{1}{3}\,,\; k = 1+2p$$

so that

$$t = \frac{l}{k} = \frac{(1+p+p^2)^2}{(1+2p)}\,,\; 1-\alpha = \frac{(1-p)(1+p)^3}{(1+2p)}\,,\; \alpha\,' = \frac{6p^2(1+p)^2}{(1+2p)^2}\,,$$

$$T = \frac{3p(1+p)}{(1+2p)(1+p+p^2)}\,,$$

$$1-\beta = \frac{(2+p)^2(1+2p)^2(1-p)^2}{4(1+p+p^2)^3}\,,\; \beta' = \frac{27p(1-p^2)(1+2p)(2+p)}{4(1+p+p^2)^4},$$

$$A = \frac{1}{6}\,p\,(1-p^2)(2+p),\; B = \frac{p(1-p^2)(2+p)(1+2p)}{4(1+p+p^2)^2}.$$

Putting $y = 1+p+p^2$, we can rewrite

$$l = y^2,\; \beta = \frac{27(y-1)^2}{4y^3}\,,\; t = \frac{y^2}{1+2p}\,,\; \alpha\,' = \frac{6(y-1)^2}{(4y-3)}\,,$$

$$T = \frac{3(y-1)}{y(1+2p)}\ ,\ 1-\beta = \frac{(4y-3)(3-y)^2}{4y^3}\ ,\ \beta' = \frac{27(y-1)(1+2p)(3-y)}{4y^4},$$

$$A = \frac{1}{6}\,(y-1)(3-y),\ B = \frac{(y-1)(1+2p)(3-y)}{4y^2}.$$

Thus,

$$tB = \frac{(y-1)(3-y)}{4} \ \text{ and } \ CA = C\cdot\frac{(y-1)(3-y)}{6}$$

and they are equal with $C = \frac{3}{2}$ . Thus (2.2) holds. Now,

$$\frac{b(1-b)\beta'}{B} - \frac{a(1-a)\alpha\,'}{A} = \left[\frac{6}{y^2(4y-3)} - \frac{9(y-1)}{(3-y)(4y-3)^2}\right]$$

$$= \frac{3(y-1)}{y(4y-3)}\left[-\frac{2}{y} + \frac{2}{y-1} + \frac{1}{y-3} - \frac{1}{4y-3}\right],$$

where as $T\left(\log\left(\frac{AT}{\sqrt{t}}\right)\right)'$ equals,

$$\frac{d}{dy}\left(\log\frac{y}{\sqrt{1+2p}}\right)\left[\frac{d}{dy}\log\left(-\frac{(y-1)^2(y-3)}{2y^2\sqrt{1+2p}}\right)\right]$$

$$= \left[\frac{1}{y} - \frac{1}{(1+2p)^2}\right]\left[-\frac{2}{y} + \frac{2}{y-1} + \frac{1}{y-3} - \frac{(dp/dy)}{(1+2p)}\right]$$

$$= \left[\frac{1}{y} - \frac{1}{4y-3}\right]\left[-\frac{2}{y} + \frac{2}{y-1} + \frac{1}{y-3} - \frac{1}{4y-3}\right]$$

$$= \frac{3(y-1)}{y(4y-3)}\left[-\frac{2}{y} + \frac{2}{y-1} + \frac{1}{y-3} - \frac{1}{4y-3}\right],\ \text{also.}$$

Thus (2.3) holds and the proof of the theorem is complete, provided we show that the singularities at $p = 0$ of $C_f$ and $C_{f'}$ (or of $C_g$ and $C_{g'}$) are as in the statement of the Theorem 2.1. Indeed, on substituting for $a, k, A$ and $\alpha'$, we have

$$C_{f'} = [log(Ak)]' = \left[log\left\{\frac{1}{6}p\,(1-p^2)(2+p)(1+2p)\right\}\right]'$$

$$= \frac{1}{p} + \text{ function analytic at } 0.$$

Thus $p = 0$ is a simple pole of $C_{f'}$ with residue 1, as required.

Also,

$$C_f = \frac{k'}{k}\left\{(logA)' + \frac{k''}{k'} - \frac{1}{2}\,\frac{k'}{k}\right\} - \frac{a(1-a)\alpha'}{A}$$

$$= \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A}$$

$$= \frac{2}{1+2p} \cdot \frac{1}{p} + \text{ function analytic at } 0.$$

$$= \frac{2}{p} + \text{ function analytic at } 0.$$

Thus, $p = 0$ is a simple pole of $C_f$, as required. Similarly, $C_{g'}$ and $C_g$ each have simple pole at $p = 0$ with residue 1 for $C_{g'}$.

**Theorem 3.2.(p.258, [3]).**

$$_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) = (1+p)\ _2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)$$

where

$$\alpha = \frac{p(3+p)^2}{2(1+p)^3} \quad \text{and} \quad \beta = \frac{p^2(3+p)}{4},$$

given $0 \leq p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = b = \frac{1}{3},\ k = (1+p)^2,\ l = 1$$

so that,

$$1 - \alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3},\ \alpha' = \frac{3(3+p)(1-p)}{2(1+p)^4}$$

$$1 - \beta = \frac{(1-p)(2+p)^2}{4},\ \beta' = \frac{3p(2+p)}{4}$$

$$t = \frac{l}{k} = (1+p)^{-2},\ T = -\frac{1}{1+p},\ \frac{T}{\sqrt{t}} = -1$$

$$A = \frac{p(3+p)(1-p)(2+p)}{6(1+p)^2},\ B = \frac{p(3+p)(1-p)(2+p)}{12}.$$

Clearly, $tB = CA$ holds with $C = \frac{1}{2}$ and hence (2.2) holds. Further

$$\frac{b(1-b)\beta'}{B} - \frac{a(1-a)\alpha'}{A} = 2\left[\frac{1}{(3+p)(1-p)} - \frac{1}{p(1+p)^2(2+p)}\right]$$

$$= \frac{2}{(1+p)}\left[\frac{(1+p)}{(3+p)(1-p)} - \frac{1}{p(1+p)(2+p)}\right]$$

and $T\left(\log\left(\frac{AT}{\sqrt{t}}\right)\right)'$ also equals this, for it equals

$$-\frac{1}{(1+p)}\left[\frac{1}{p}+\frac{1}{3+p}+\frac{1}{p-1}+\frac{1}{2+p}-\frac{2}{1+p}\right]$$
$$=\frac{2}{(1+p)}\left[\frac{(1+p)}{(3+p)(1-p)}-\frac{1}{p(1+p)(2+p)}\right].$$

Thus (2.3) also holds and the proof of the theorem is complete, provided we show that the singularities at $p=0$ of $C_f$ and $C_{f'}$ (or of $C_g$ and $C_{g'}$) are as in the statement of the Theorem 2.1. Indeed, on substituting for $a, k, A$ and $\alpha'$, we have

$$C_{f'}=[log(Ak)]'=\frac{1}{p}+\text{ function analytic at 0}$$

and

$$C_f=\frac{k'}{k}(logA)'+\frac{k''}{k}-\frac{1}{2}\left(\frac{k'}{k}\right)^2-\frac{a(1-a)\alpha'}{A}$$
$$=\left[\frac{2}{p}+\text{ function analytic at 0}\right]-\left[\frac{2}{p}+\text{ function analytic at 0}\right].$$
$$=\text{ function analytic at 0.}$$

Thus $p=0$ is a simple pole of $C_f$, with residue 1, and a point of analyticity of $C_f$. Similarly $C_{g'}$ and $C_g$ have the required singularity at $p=0$.

**Theorem 3.3(p.258, [3]).** We have

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-p}{1+2p}\right)^3\right)=(1+2p)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;p^3\right),$$

given $0\le p<1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a=b=\frac{1}{3}\ ,\ k=1,\ l=(1+2p)^2,\ \alpha=p^3,\ \beta=1-\frac{(1-p)^3}{(1+2p)^3}$$

so that,

$$1-\alpha=1-p^3,\ \alpha\,'=3p^2,\ A=\frac{1}{3}\,p\,(1-p^3),$$

$$\beta=\frac{9p(1+p+p^2)}{(1+2p)^3}\ ,\ 1-\beta=\frac{(1-p)^3}{(1+2p)^3}\ ,\ \beta'=\frac{9(1-p)^2}{(1+2p)^4}\ ,$$

$$B=\frac{p(1-p^3)}{(1+2p)^2}\,,\ t=\frac{l}{k}=l=(1+2p)^2,\quad T=\frac{2}{1+2p}.$$

Now, clearly, (2.2) of Theorem 2.1 holds because

$$tB=p(1-p^3)$$

and

$$CA=C\cdot\frac{1}{3}\cdot p(1-p^3)$$

and so, $tB=CA$ holds for $C=3$.

It remains to show that (2.3) holds. We have,

$$\frac{b(1-b)\beta'}{B}-\frac{a(1-a)\alpha\,'}{A}=\frac{2(1+2p^2)(1-2p-2p^2)}{p(1-p^3)(1+2p)^2}. \tag{3.1}$$

Similarly, $T\left(\log\left(\frac{AT}{\sqrt{t}}\right)\right)'$ equals

$$\frac{2}{1+2p}\left[\left\{\log\left(\frac{1}{3}p(1-p^3)\right)\right\}'-\frac{4}{1+2p}\right]=\frac{2}{1+2p}\left[\left(\frac{1}{p}-\frac{3p^2}{1-p^3}\right)-\frac{4}{1+2p}\right]$$

$$=\frac{2(1+2p^2)(1-2p-2p^2)}{p(1+2p)^2(1-p^3)} \tag{3.2}$$

From (3.1) and (3.2) we have that (2.3) also holds.

Lastly, we have

$$C_{f'}=[log(Ak)]'=\frac{1}{p}+\text{ function analytic at 0}$$

and

$$C_f=\frac{k'}{k}(logA)'+\frac{k''}{k}-\frac{1}{2}\left(\frac{k'}{k}\right)^2-\frac{a(1-a)\alpha'}{A}$$

$$=-\frac{2p}{(1-p^3)}+\text{ function analytic at 0.}$$

Thus $p=0$ is a simple pole of $C_{f'}$ with residue 1 and a point of analyticity of $C_{f'}$. Similarly, $C_{g'}$ and $C_g$ have the required singularity at $p=0$. Thus the theorem is fully proved.

**Theorem 3.4(p.258, [3]).** We have

$$(2+2p-p^2)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)=2(1+4p+p^2)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)$$

where

$$\alpha = \frac{27p(1+p)^4}{2(1+4p+p^2)^3} \quad \text{and} \quad \beta = \frac{27p^4(1+p)}{2(2+2p-p^2)^3}\ ,$$

given $0 \leq p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = b = \frac{1}{3}\,, k = 4(1+4p+p^2)^2, l = (2+2p-p^2)^2,$$

so that,

$$t = \frac{l}{k} = \frac{(2+2p-p^2)^2}{4(1+4p+p^2)^2}\ ,$$

$$T = \frac{1}{2}\,(\log t)' = \frac{-6(1+p+p^2)}{(2+2p-p^2)(1+4p+p^2)}\ ,$$

$$1-\alpha = \frac{(1-p)^4(2+p)(1+2p)}{2(1+4p+p^2)^3}\ ,\ \alpha\,' = \frac{27(1-p^2)^3}{2(1+4p+p^2)^4}\ ,$$

$$1-\beta = \frac{(1-p)(2+p)^4(1+2p)}{2(2+2p-p^2)^3}\ ,\ \beta' = \frac{27p^3(p+2)^3}{2(2+2p-p^2)^4}\ ,$$

$$A = \frac{\alpha(1-\alpha)}{\alpha\,'} = \frac{p(1-p^2)(2+p)(1+2p)}{2(1+4p+p^2)^2},$$

$$\frac{AT}{\sqrt{t}} = \frac{-6p(1-p^2)(2+p)(1+2p)(1+p+p^2)}{(1+4p+p^2)^2\ (2+2p-p^2)^2},$$

$$\frac{d}{dp}\left(\log\frac{AT}{\sqrt{t}}\right) = \frac{1}{p} + \frac{2p}{p^2-1} + \frac{1}{p+2} + \frac{2}{2p+1} + \frac{2p+1}{1+p+p^2} - \frac{4(p+2)}{(p^2+4p+1)} - \frac{4(1-p)}{(2+2p-p^2)}\ .$$

Further,

$$B = \frac{\beta(1-\beta)}{\beta'} = \frac{p(1-p^2)(2+p)(1+2p)}{2(2+2p-p^2)^2}.$$

Now, (2.2) holds since, it is easily verified that

$$tB = CA \quad \text{for} \quad C = \frac{1}{4}.$$

Also,

$$\frac{b(1-b)\beta'}{B} - \frac{a(1-a)\alpha'}{A} = 6\left[\frac{p^2(p+2)^2}{(2+2p-p^2)^2(2p+1)(1-p^2)} - \frac{(1-p^2)^2}{p(1+4p+p^2)^2(2p+1)(p+2)}\right]$$

and

$$T\left(\log\frac{AT}{\sqrt{t}}\right)' = \frac{-6(1+p+p^2)}{(2+2p-p^2)(1+4p+p^2)}$$

$$\times\left[\frac{1}{p}+\frac{2p}{p^2-1}+\frac{1}{p+2}+\frac{2}{2p+1}+\frac{2p+1}{p^2+p+1}-\frac{4(p+2)}{(p^2+4p+1)}-\frac{4(1-p)}{(2+2p-p^2)}\right].$$

Thus, to complete the proof of the theorem, it is enough to prove that the above two expressions on the right sides are equal, or, equivalently,

$$\frac{(1-p^2)^2(2+2p-p^2)}{p(p+2)(2p+1)(p^2+p+1)(p^2+4p+1)}+\frac{p^2(p+2)^2(p^2+4p+1)}{(2p+1)(p^2-1)(p^2+p+1)(2+2p-p^2)}$$

$$=\frac{1}{p}+\frac{2p}{p^2-1}+\frac{1}{p+2}+\frac{2}{2p+1}+\frac{2p+1}{p^2+p+1}-\frac{4(p+2)}{(p^2+4p+1)}-\frac{4(p-1)}{(p^2-2p-2)}. \quad (3.3)$$

For this, we employ the following partial fraction expansions, the details of proofs of which, being routine, we omit:

$$\frac{(1-p^2)^2(2+2p-p^2)}{p(p+2)(2p+1)(p^2+p+1)(p^2+4p+1)}$$

$$= \frac{1}{p}+\frac{1}{p+2}+\frac{1}{2p+1}+\frac{p}{p^2+p+1}-\frac{4(p+2)}{(p^2+4p+1)}. \quad (3.4)$$

and

$$\frac{p^2(p+2)^2(p^2+4p+1)}{(2p+1)(p^2-1)(p^2+p+1)(2+2p-p^2)}$$

$$= \frac{1}{2p+1}+\frac{2p}{p^2-1}+\frac{p+1}{(p^2+p+1)}+\frac{4(p-1)}{(2+2p-p^2)}. \quad (3.5)$$

Adding (3.4) and (3.5) we have (3.3).

Now, for the nature of singularity of $C_{f'}$ and $C_f$ at $p=0$, we easily have

$$C_{f'} = [log(Ak)]' = \frac{1}{p} + \text{ function analytic at } 0$$

and

$$C_f = \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A}$$

$$= \frac{5}{p} + \text{ function analytic at } 0.$$

Thus $C_f$ and $C_{f'}$, and similarly $C_g$ and $C_{g'}$, have the required type of singularity at $p=0$. Thus the theorem is fully proved.

## 4. Transformations in Quartic Theory

**Theorem 4.1(p.260).** We have

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\alpha\right) = \sqrt{1+3p}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right) $$

where

$$ \alpha = 1-\left(\frac{1-p}{1+3p}\right)^2 \quad \text{and} \quad \beta = p^2\ , $$

given $0 \le p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$ a=b=\frac{1}{4}\ ,\ k=1+3p, l=1,\ \alpha = \frac{8p(1+p)}{(1+3p)^2}\ , $$

so that,

$$ t = \frac{l}{k} = \frac{1}{1+3p}\ , $$

$$ T = \frac{1}{2}\ (\log t)' = -\frac{3}{2(1+3p)}\ ,\ 1-\alpha = \left(\frac{1-p}{1+3p}\right)^2,\ \alpha\,' = \frac{8(1-p)}{(3p+1)^3}\ , $$

$$ 1-\beta = 1-p^2\ ,\ \beta' = 2p\ , $$

$$ A = \frac{\alpha(1-\alpha)}{\alpha\,'} = \frac{p(1-p^2)}{(1+3p)}\ , B = \frac{\beta(1-\beta)}{\beta'} = \frac{1}{2}\ p\ (1-p^2), $$

$$ \frac{AT}{\sqrt{t}} = \frac{-3p(1-p^2)}{2(1+3p)^{3/2}}\ , $$

$$ \frac{d}{dp}\left(\log\frac{AT}{\sqrt{t}}\right) = \frac{1}{p} + \frac{2p}{p^2-1} - \frac{9}{2(3p+1)}. $$

Thus

$$ T\left(\log\frac{AT}{\sqrt{t}}\right)' = -\frac{3(9p^3+6p^2+3p-2)}{4p(p^2-1)(3p+1)^2} \tag{4.1} $$

and

$$ \frac{a(1-a)\beta'}{B} - \frac{b(1-b)\alpha\,'}{A} = \frac{3}{16}\left[\frac{4p}{p(1-p^2)} - \frac{8(1-p)}{(3p+1)^2p(1-p^2)}\right] $$

$$ = -\frac{3}{4}\frac{(9p^3+6p^2+3p-2)}{p(p^2-1)(3p+1)^2} \tag{4.2} $$

and,

$$\frac{tB}{A} = \frac{1}{(1+3p)} \cdot \frac{p(1-p^2)}{2} \cdot \frac{(1+3p)}{p(1-p^2)} = \frac{1}{2} = \text{constant C.} \tag{4.3}$$

From (4.3) and by (4.1) and (4.2) we have that (2.2) and (2.3) hold.

Lastly, we easily have

$$C_{f'} = [log(Ak)]' = \frac{1}{p} + \text{ function analytic at 0}$$

and

$$\begin{aligned} C_f &= \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A} \\ &= \frac{3}{2p} + \text{ function analytic at 0.} \end{aligned}$$

Thus $C_f$ and $C_{f'}$, and similarly $C_g$ and $C_{g'}$ have the desired type of singularity at $p = 0$. Thus the theorem is fully proved.

**Theorem 4.2(p.260).** We have

$$\sqrt{27-18p-p^2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\alpha\right) = 3\sqrt{3+6p-p^2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right)$$

where

$$\alpha = \frac{64p}{(3+6p-p^2)^2} \quad \text{and} \quad \beta = \frac{64p^3}{(27-18p-p^2)^2},$$

given $0 \le p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = b = \frac{1}{4},\ k = 9(3+6p-p^2), l = 27-18p-p^2,$$

so that,

$$t = \frac{l}{k} = \frac{27-18p-p^2}{9(3+6p-p^2)}, T = \frac{1}{2}(\log t)' = \frac{-12(p^2-2p+9)}{(27-18p-p^2)(3+6p-p^2)},$$

$$1-\alpha = \frac{(1-p)^3(9-p)}{(3+6p-p^2)^2},\ 1-\beta = \frac{(1-p)(9-p)^3}{(27-18p-p^2)^2},$$

$$\alpha' = \frac{192(1-p)^2}{(3+6p-p^2)^3},\ \beta' = \frac{64p^2(9-p)^2}{(27-18p-p^2)^3},$$

$$A = \frac{\alpha(1-\alpha)}{\alpha'} = \frac{p(1-p)(9-p)}{3(3+6p-p^2)},\ B = \frac{\beta(1-\beta)}{\beta'} = \frac{p(1-p)(9-p)}{(27-18p-p^2)},$$

$$\frac{AT}{\sqrt{t}} = \frac{-12p(1-p)(9-p)(p^2-2p+9)}{(3+6p-p^2)^{3/2}\,(27-18p-p^2)^{3/2}}\,,$$

$$\left(\log\frac{AT}{\sqrt{t}}\right)' = \frac{1}{p} + \frac{1}{p-1} + \frac{1}{p-9} + \frac{2(p-1)}{p^2-2p+9} + \frac{3(p-3)}{3+3p-p^2} + \frac{3(p+9)}{27-18p-p^2}, \tag{4.4}$$

Now,

$$\frac{1}{T}\left(\frac{b(1-b)\beta'}{B} - \frac{a(1-a)\alpha\,'}{A}\right) = -\frac{(p^2+18p-27)(p^2-6p-3)}{12(p^2-2p+9)}\cdot\frac{3}{16}\left[\frac{\beta'}{B} - \frac{\alpha\,'}{A}\right]$$

$$= \left[\frac{-p(p-9)(p^2-6p-3)}{(p-1)(p^2-2p+9)(p^2+18p-27)} + \frac{9(p-1)(p^2+18p-27)}{p(p-9)(p^2-6p-3)(p^2-2p+9)}\right]. \tag{4.5}$$

After some routine calculations one can obtain the following partial fraction expansions

$$\frac{-p(p-9)(p^2-6p-3)}{(p-1)(p^2-2p+9)(p^2+18p-27)} = \frac{1}{p-1} + \frac{p}{p^2-2p+9} + \frac{-3p-27}{p^2+18p-27}$$

and

$$\frac{9(p-1)(p^2+18p-27)}{p(p-9)(p^2-6p-3)(p^2-2p+9)} = \frac{1}{p} + \frac{1}{p-9} + \frac{-3p+9}{p^2-6p-3} + \frac{p-2}{p^2-2p+9}.$$

Substituting these in (4.5), we have that

$$\frac{1}{T}\left(\frac{b(1-b)\beta'}{B} - \frac{a(1-a)\alpha\,'}{A}\right)$$

$$= \frac{1}{p} + \frac{1}{p-1} + \frac{1}{p-9} + \frac{2(p-1)}{p^2-2p+9} - \frac{3(p-3)}{p^2-6p-3} - \frac{3(p+9)}{p^2+18p-27}.$$

This and (4.4) prove that (2.3) holds.

Now, (2.2) also holds with $C = \frac{1}{3}$, as is easily seen by substituting the expressions for $A, B$ and $t$ obtained in the beginning of the proof.

Lastly, we easily have

$$C_{f'} = [log(Ak)]' = \frac{1}{p} + \text{ function analytic at } 0$$

and

$$C_f = \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A}$$

$$= \frac{2}{3p} + \text{ function analytic at 0.}$$

Hence, $C_{f'}$ and $C_f$, and similarly $C_{g'}$ and $C_g$ have the desired singularities at $p = 0$. Thus the proof of the theorem is complete.

**Theorem 4.3(p.260).** We have

$$_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) = \sqrt{1+p}\ \ _2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \beta\right)$$

where

$$\alpha = \frac{2p}{1+p} \quad \text{and} \quad \beta = p^2,$$

given $0 \le p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = \frac{1}{2}\ ,\ b = \frac{1}{4}\ ,\ k = 1 + p,\ l = 1,$$

so that,

$$t = \frac{l}{k} = \frac{1}{1+p}\ ,\ T = \frac{1}{2}\,(\log t)' = \frac{-1}{2(1+p)}\ ,$$

$$1 - \alpha = \frac{1-p}{1+p}\ , \alpha\,' = \frac{2}{(1+p)^2}\ ,\ 1 - \beta = 1 - p^2\ ,\ \beta' = 2p,$$

$$A = \frac{\alpha(1-\alpha)}{\alpha\,'} = p(1-p),\ B = \frac{\beta(1-\beta)}{\beta'} = \frac{1}{2}\,p\,(1-p^2)$$

$$\frac{AT}{\sqrt{t}} = -\frac{p(1-p)}{2\sqrt{1+p}}\ ,$$

$$\frac{d}{dp}\left(\log\frac{AT}{\sqrt{t}}\right) = \frac{(3p^2+3p-2)}{2p(p^2-1)}\ .$$

Thus,

$$T\left(\log\frac{AT}{\sqrt{t}}\right)' = \frac{(3p^2+3p-2)}{4p(1+p)(1-p^2)}\ ,$$

which also equals $\dfrac{b(1-b)\beta'}{B} - \dfrac{a(1-a)\alpha\,'}{A}$ , because it equals

$$\frac{3}{4}\cdot\frac{p}{p(1-p^2)} - \frac{1}{2(1+p)^2p(1-p)} = \frac{(3p^2+3p-2)}{4p(1+p)(1-p^2)}.$$

Thus (2.3) holds.

That (2.2) holds is trivial. For,

$$\frac{tB}{A} = \frac{1}{1+p} \cdot \frac{1}{2} \frac{p(1-p^2)}{p(1-p)} = \frac{1}{2} = \text{constant C},$$

as required.

Lastly, we have

$$C_{f'} = [\log(Ak)]' = \frac{1}{p} + \text{ function analytic at 0}$$

and

$$\begin{aligned} C_f &= \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A} \\ &= \frac{1}{2p} + \text{ function analytic at 0.} \end{aligned}$$

Thus $C_{f'}$ and $C_f$, and similarly $C_{g'}$ and $C_g$, have the desired type of singularity at $p = 0$. Thus the proof of the theorem is complete.

## 5. Transformations in Sextic Theory

**Theorem 5.1(p.262).** We have

$$\sqrt{1+p+p^2}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) = \sqrt{1+2p}\ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \beta\right)$$

where

$$\alpha = \frac{p(2+p)}{1+2p} \quad \text{and} \quad \beta = \frac{27p^2(p+1)^2}{4(1+p+p^2)^3},$$

given $0 \leq p < 1$.

**Proof.** In the notations of Theorem 2.1 we take

$$a = \frac{1}{2}\ ,\ b = \frac{1}{6}\ ,\ k = 1+2p,\ l = 1+p+p^2,$$

so that,

$$t = \frac{l}{k} = \frac{1+p+p^2}{1+2p}\ ,$$

$$T = \frac{1}{2}\,(\log t)' = \frac{1}{2}\left[\frac{(2p^2+2p-1)}{(2p+1)(p^2+p+1)}\right],$$

$$1-\alpha = \frac{1-p^2}{1+2p}\ ,\ 1-\beta = \frac{(p-1)^2(p+2)^2(2p+1)^2}{4(1+p+p^2)^3}\ ,$$

$$\alpha\,' = \frac{2(1+p+p^2)}{(1+2p)^2}\ ,\ \beta' = -\frac{27p(p^2-1)(p+2)(2p+1)}{4(p^2+p+1)^4}\ ,$$

$$A = \frac{\alpha(1-\alpha)}{\alpha\,'} = \frac{p(2+p)(1-p^2)}{2(1+p+p^2)}\ ,\ B = \frac{\beta(1-\beta)}{\beta'} = -\frac{p(p^2-1)(p+2)(2p+1)}{4(1+p+p^2)^2}\ ,$$

$$\frac{AT}{\sqrt{t}} = \frac{p(2+p)(1-p^2)(2p^2+2p-1)}{4(p^2+p+1)^{5/2}\,(1+2p)^{1/2}}\ ,$$

$$\left(\log\frac{AT}{\sqrt{t}}\right)' = \frac{1}{p}+\frac{1}{p+2}+\frac{2p}{p^2-1}+\frac{2(2p+1)}{2p^2+2p-1}-\frac{5(2p+1)}{2(p^2+p+1)}-\frac{1}{(2p+1)},\ \ (5.1)$$

$$\begin{aligned}
&\frac{1}{T}\left[\frac{b(1-b)\beta'}{B}-\frac{a(1-a)\alpha\,'}{A}\right]\\
&\quad= \frac{2(2p+1)(p^2+p+1)}{(2p^2+2p-1)}\left[\frac{15}{4(p^2+p+1)^2}+\frac{(p^2+p+1)^2}{p(p+2)(p^2-1)(2p+1)^2}\right]\\
&\quad= \frac{15(2p+1)}{2(2p^2+2p-1)(p^2+p+1)}+\frac{2(p^2+p+1)^3}{p(p+2)(p^2-1)(2p+1)(2p^2+2p-1)}\\
&\quad= \frac{1}{p}+\frac{1}{p+2}+\frac{1}{p+1}+\frac{1}{p-1}-\frac{1}{(2p+1)}+\frac{2(2p+1)}{2p^2+2p-1}-\frac{5(2p+1)}{2(p^2+p+1)},\ \ (5.2)
\end{aligned}$$

on using the partial fraction expansions

$$\frac{15(2p+1)}{2(2p^2+2p-1)(p^2+p+1)} = -\frac{5(2p+1)}{2(p^2+p+1)}+\frac{5(2p+1)}{2p^2+2p-1}$$

and

$$\frac{2(p^2+p+1)^3}{p(p+2)(p^2-1)(2p+1)(2p^2+2p-1)} = \frac{1}{p}+\frac{1}{p+2}+\frac{1}{p+1}+\frac{1}{p-1}-\frac{1}{(2p+1)}-\frac{3(2p+1)}{2p^2+2p-1}$$

which are routinely established.

From (5.1) and (5.2) we have that (2.3) holds. Further, it is easily seen that (2.2) is also satisfied with $C = -\frac{1}{2}$.

Lastly, we easily have

$$C_{f'} = [log(Ak)]' = \frac{1}{p} + \text{ function analytic at } 0$$

and

$$C_f = \frac{k'}{k}(logA)' + \frac{k''}{k} - \frac{1}{2}\left(\frac{k'}{k}\right)^2 - \frac{a(1-a)\alpha'}{A}$$

$$= \frac{3}{2p} + \text{ function analytic at } 0.$$

Hence $C_{f'}$ and $C_f$, and similarly $C_{g'}$ and $C_g$, have the desired type of singularity at $p = 0$. Thus the proof of the theorem is complete.

## 6. Differential Equations Governing The Transformations (1.1)-(1.8)

The following table provides the differential equations governing the respective transformations established in Sections 3-5. We omit the routine details leading to the differential equations. In table Transformation denote Trans.

| | | | |
|---|---|---|---|
| Trans. 1.1 | $\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{\sqrt{1+2p}}$<br>$= \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)}{1+p+p^2}$ | $\frac{p^3(2+p)}{1+2p}$ and<br>$\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}$ | $p(1-p^2)(2+p)(1+2p)u''$<br>$+2(1+5p-10p^3-5p^4)u'$<br>$+2(1+2p)(1-2p-2p^2)u=0$ |
| Trans. 1.2 | ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)$<br>$=(1+p)\ \ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)$ | $\frac{p(3+p)^2}{2(1+p)^3}$ and<br>$\frac{p^2(3+p)}{4}$ | $p(1-p)(2+p)(3+p)(1+p)^2u''$<br>$+2(1+p)(3-4p-6p^2-4p^3-p^4)u'$<br>$-2(3+p)(1-p)u=0$ |
| Trans. 1.3 | $(1+2p)\ \ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)$<br>$=\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)$ | $p^3$ and<br>$1-\left(\frac{1-p}{1+2p}\right)^3$ | $p(1+2p)^2(1-p^3)u''$<br>$+(1+2p)(1-2p-4p^3-4p^4)u'$<br>$-2(1-p)^2u=0$ |
| Trans. 1.4 | $\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)}{2(1+4p+p^2)}$<br>$= \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)}{2+2p-p^2}$ | $\frac{27p(1+p)^4}{2(1+4p+p^2)^3}$ and<br>$\frac{27p^4(1+p)}{2(2+2p-p^2)^3}$ | $p(1-p^2)(2+p)(1+2p)(1+4p+p^2)$<br>$(2+2p-p^2)(1+p+p^2)u''$<br>$+2(1+5p-10p^3-5p^4)(1+4p+p^2)$<br>$(2+2p-p^2)(1+p+p^2)u'$<br>$+2(1+2p)[(1+p+p^2)(1-7p-6p^2$<br>$+2p^3+p^4)+9p(1-p^2)(2+p)]u=0$ |
| Trans. 1.5 | $\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)}{\sqrt{1+3p}}$<br>$=\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\alpha\right)$ | $p^2$ and<br>$1-\left(\frac{1-p}{1+3p}\right)^2$ | $4p(1-p^2)u''+4(1-3p^2)u'-3pu=0$ |
| Trans 1.6 | $\frac{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\alpha\right)}{3\sqrt{3+6p-p^2}}$<br>$= \frac{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right)}{\sqrt{27-18p-p^2}}$ | $\frac{64p}{(3+6p-p^2)^2}$ and<br>$\frac{64p^3}{(27-18p-p^2)^2}$ | $p(1-p)(p-9)u''+(3p^2-20p+9)u'$<br>$+(p-3)u=0$ |
| Trans. 1.7 | $\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{\sqrt{1+p}}$<br>$=\ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right)$ | $\frac{2p}{1+p}$ and<br>$p^2$ | $4p(1-p^2)u''+4(1-3p^2)u'-3pu=0$ |
| Trans. 1.8 | $\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{\sqrt{1+2p}}$<br>$= \frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\beta\right)}{\sqrt{1+p+p^2}}$ | $\frac{p(2+p)}{1+2p}$ and<br>$\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}$ | $p(2+p)(1-p^2)(1+2p)(1+p+p^2)u''$<br>$-6(p^6+3p^5-5p^4-15p^3-21p^2-13p+5)u'$<br>$-(p^4+2p^3+4p^2+3p-1)(1+2p)u=0$ |

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