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EXTENDED VALUES OF RAMANUJAN'S TAU FUNCTION

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Dedicated to Professor G.E. Andrews on his seventieth birthday

Abstract: Present paper concerns mainly with verification and extension of table for $\tau(1), \tau(2), \tau(3), \tau(4), \cdots, \tau(29), \tau(30)$ of Ramanujan. Our extended table for $\tau(31), \tau(32), \cdots, \tau(37)$ is obtained without using certain arithmetical functions defined by Ramanujan and also the theory of elliptic functions.

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1. Introduction

In this paper, we obtain the values of $\tau(31), \tau(32), \tau(33), \tau(34), \tau(35), \tau(36)$ and $\tau(37)$, where $\tau(n)$ is Tau function of Ramanujan, defined as follows:

$$\sum_{n=1}^{\infty} \tau(n)\ x^n \ = \ x\left\{\prod_{n=1}^{\infty}(1-x^n)\right\}^{24} \tag{1.1}$$

Ramanujan [3, p.196, Table(V); see also 1,2] calculated the values of $\tau(1), \tau(2), \tau(3), \tau(4), \cdots, \tau(29), \tau(30)$, by means of the theory of elliptic functions and certain arithmetical functions such as $F_{r,s}(x)$, $\Phi_{r,s}(x)$, $E_{r,s}(n), \sigma_s(n)$, Riemann's Zeta function $\zeta(n)$, greatest integer function $[x]$, theory of symbols o, O, continued fraction, asymptotic expansion, some trigonometrical identities, inequalities, Gamma function, theory of order of error terms, number theory, convergence and divergence of infinite series.

We have obtained the values of $\tau(1), \tau(2), \tau(3), \tau(4), \cdots, \tau(36), \tau(37)$ without using the theory of elliptic functions and certain arithmetical functions etcetera. In this sequence, we consider the whole square of power series $\sum_{n=0}^{\infty} b_n x^n$ and collect the terms upto x^{36}. Thus we have:

$$\Big[b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8+b_9x^9+b_{10}x^{10}+b_{11}x^{11}+b_{12}x^{12}$$
$$+b_{13}x^{13}+b_{14}x^{14}+b_{15}x^{15}+b_{16}x^{16}+b_{17}x^{17}+b_{18}x^{18}+b_{19}x^{19}+b_{20}x^{20}+\cdots+b_{31}x^{31}+b_{32}x^{32}$$
$$+b_{33}x^{33}+b_{34}x^{34}+b_{35}x^{35}+b_{36}x^{36}+\cdots\Big]^2$$
$$= \Big[b_0^2+b_1^2x^2+b_2^2x^4+b_3^2x^6+b_4^2x^8+b_5^2x^{10}+b_6^2x^{12}+\cdots+b_{17}^2x^{34}+b_{18}^2x^{36}+\cdots\Big]$$
$$+\Bigg[2(b_0)\Bigg\{\sum_{m=1}^{36} b_mx^m\Bigg\}+\cdots\Bigg]+\Bigg[2(b_1x)\Bigg\{\sum_{m=2}^{35} b_mx^m\Bigg\}+\cdots\Bigg]+\Bigg[2(b_2x^2)\Bigg\{\sum_{m=3}^{34} b_mx^m\Bigg\}+\cdots\Bigg]$$
$$+\cdots+\Big[2(b_{16}x^{16})\ \Big\{b_{17}x^{17}+b_{18}x^{18}+b_{19}x^{19}+b_{20}x^{20}\Big\}+\cdots\Big]$$
$$+\Big[2(b_{17}x^{17})\ \Big\{b_{18}x^{18}+b_{19}x^{19}\Big\}+\cdots\Big]+\cdots \tag{1.2}$$

2. Verification and Extension

Consider the expanded form of (1.1), we have

$$\sum_{n=1}^{\infty}\tau(n)\ x^n = x\{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^{36})\cdots\}^{24} \tag{2.1}$$

$$= x\{(1-x)^3(1-x^2)^3(1-x^3)^3(1-x^4)^3\cdots(1-x^{36})^3\cdots\}^8$$
$$= x\ T^8 = x\{(T^2)^2\}^2 \tag{2.2}$$

where

$$T = (1-x)^3(1-x^2)^3(1-x^3)^3(1-x^4)^3\cdots(1-x^{36})^3\cdots\cdots \tag{2.3}$$

Now considering the product of first thirty six polynomials in (2.3) and collecting the terms upto x^{36}, we get

$$T = +1-3x+5x^3-7x^6+9x^{10}-11x^{15}+13x^{21}-15x^{28}+17x^{36}-\cdots \tag{2.4}$$

It is to be noted that the coefficients in (2.4) are alternatively positive and negative such that the sequence $1,3,5,7,9,\cdots$ form arithmetic progression. Suppose the powers of x (i.e. the sequence $0,1,3,6,10,15,\cdots$) are generated by $F(k)$, Therefore

$$T = \sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)x^{F(k)} \tag{2.5}$$

Now we shall find the function $F(k)$ using the following ordinary finite difference table:

k	$F(k)$	First Order	Second Order	Third Order	Fourth Order	Fifth Order	Sixth Order	Seventh Order	Eighth Order	$\cdots$
1	0									
		1								
2	1		1							
		2		0						
3	3		1		0					
		3		0		0				
4	6		1		0		0			
		4		0		0		0		
5	10		1		0		0		0	
		5		0		0		0		
6	15		1		0		0			
		6		0		0				
7	21		1		0					
		7		0						
8	28		1							
		8								
9	36									
$\vdots$										

Ordinary Finite Difference Table

Since second order ordinary differences are equal, therefore third and higher order differences are zero, and so $F(k)$ will be a polynomial of second degree. Thus:

$$F(k) \;=\; A + Bk + Ck^2 \tag{2.6}$$

where the unknowns A, B and C are to be calculated.

Now selecting any three values of k and also their corresponding values of $F(k)$ from above table, and putting them in (2.6), we get a system of three linear equations which on simplification gives $A = 0$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

Therefore suitable $F(k)$ is given by

$$F(k) \;=\; -\frac{1}{2}k + \frac{1}{2}k^2 \;=\; \frac{k(k-1)}{2}$$

Consequently (2.5) reduces to:

$$T \;=\; \sum_{k=1}^{\infty} (-1)^{k-1}(2k-1)x^{\frac{k(k-1)}{2}} \tag{2.7}$$

Now squaring the expansion in (2.4), using (1.2) and collecting the terms upto x^{36}, we have

$$T^2 = +1-6x+9x^2+10x^3-30x^4+11x^6+42x^7-70x^9+18x^{10}-54x^{11}+49x^{12}+90x^{13}$$
$$-22x^{15}-60x^{16}-110x^{18}+81x^{20}+180x^{21}-78x^{22}+130x^{24}-198x^{25}-182x^{27}-30x^{28}$$
$$-90x^{29}+121x^{30}+84x^{31}+210x^{34}-252x^{36}+\cdots \tag{2.8}$$

Further repeating the same process for $(T^2)^2$, we get

$$T^4 = +1-12x+54x^2-88x^3-99x^4+540x^5-418x^6-648x^7+594x^8+836x^9+1056x^{10}$$
$$-4104x^{11}-209x^{12}+4104x^{13}-594x^{14}+4256x^{15}-6480x^{16}-4752x^{17}-298x^{18}$$
$$+5016x^{19}+17226x^{20}-12100x^{21}-5346x^{22}-1296x^{23}-9063x^{24}-7128x^{25}$$
$$+19494x^{26}+29160x^{27}-10032x^{28}-7668x^{29}-34738x^{30}+8712x^{31}-22572x^{32}$$
$$21812x^{33}49248x^{34}-46872x^{35}+67562x^{36}+\cdots \tag{2.9}$$

Finally adopting the same procedure for $\{(T^2)^2\}^2$, we obtain

$$T^8 = +1-24x+252x^2-1472x^3+4830x^4-6048x^5-16744x^6+84480x^7-113643x^8$$
$$-115920x^9+534612x^{10}-370944x^{11}-577738x^{12}+401856x^{13}+1217160x^{14}$$
$$+987136x^{15}-6905934x^{16}+2727432x^{17}+10661420x^{18}-7109760x^{19}-4219488x^{20}$$
$$+12830688x^{21}+18643272x^{22}+21288960x^{23}-25499225x^{24}+13865712x^{25}$$
$$-73279080x^{26}+24647168x^{27}+128406630x^{28}-29211840x^{29}-52843168x^{30}$$
$$-196706304x^{31}+134722224x^{32}+165742416x^{33}-80873520x^{34}+167282496x^{35}$$
$$-182213314x^{36}+\cdots \tag{2.10}$$

Now on multiplying (2.10) by x and comparing the coefficients of $x, x^2, x^3, x^4, \cdots, x^{36}, x^{37}$ with the coefficients of left hand side of (2.2), we get the values of $\tau(1), \tau(2), \tau(3), \tau(4), \cdots, \tau(36), \tau(37)$ and are given in tabular form as follows:

3. Extended Table for $\tau(n)$; $n \in \{1, 2, 3, 4, 5, \cdots, 37\}$

n	$\tau(n)$	n	$\tau(n)$
1	+1	20	−7109760
2	−24	21	−4219488
3	+252	22	−12830688
4	−1472	23	+18643272
5	+4830	24	+21288960
6	−6048	25	−25499225
7	−16744	26	+13865712
8	+84480	27	−73279080
9	−113643	28	+24647168
10	−115920	29	+128406630
11	+534612	30	−29211840
12	−370944	31	−52843168
13	−577738	32	−196706304
14	+401856	33	+134722224
15	+1217160	34	+165742416
16	+987136	35	−80873520
17	−6905934	36	+167282496
18	+2727432	37	−182213314
19	+10661420		

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