

Initial-Boundary Value Problems of Fokas' Transform Method

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Abstract: A new, unified transform method for boundary value problems on linear and integrable nonlinear partial differential equations was recently introduced by Fokas. We consider initial boundary value problems for linear, constant-coefficient evolution equations of arbitrary order on a finite domain. We use Fokas' method to fully characterize well-posed problems. We derive an implicit solution to the IBVP. We also discuss the global relation and classification of boundary condition.

Key words: Unified Transform Method, Nonlinear Partial Differential Equations, IBVP.

1. Introduction

In this paper we develop the major steps of Fokas' unified transform method for solving initial-boundary value problems on linear evolution partial differential equations. In section 2 we define IBVP. Implicit solution is given in section 3. And in section 4 we discuss about the global relation and classification of boundary condition. In this paper we assume throughout that the initial-boundary value problem being studied is well-posed in the sense that it admits a unique, smooth solution. Fokas' method does yield a way of checking well-posedness of a problem.

2. The IBVP:

We consider the partial differential equation

$$\partial_t q(x, t) + a(-\iota \partial_x^n) q(x, t) = 0 \quad \text{for } (x, t) \in \Omega = [0, 1] \times [0, T], \quad (1)$$

where $n \geq 2$ and $a = e^{i\theta}$ for some $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ if n is even or $a = \pm \iota$ if n is odd. This choice of a is motivated by our interest in well-posed problems; the reverse heat equation exhibits instantaneous blow-up, for example. We study the initial-boundary value problem on this partial differential equation with initial condition

$$q(x, 0) = q_0(x), \quad (2)$$

where $q_0 : [0, 1] \rightarrow \mathbb{R}$ is a given, sufficiently smooth initial datum, and n linearly independent boundary conditions,

$$\sum_{j=0}^{n-1} \alpha_{kj} \partial_x^j q(0, t) + \sum_{j=0}^{n-1} \beta_{kj} \partial_x^j q(1, t) = h_k(t), \quad (3)$$

indexed by $k \in 1, 2, \dots, n$ where the $h_k : [0, T] \rightarrow \mathbb{R}$ are given, sufficiently smooth boundary data and the boundary coefficients, α_{kj} and β_{kj} are given real constants. We also require that the boundary data are compatible with the initial datum in the sense that

$$\sum_{j=0}^{n-1} \alpha_{kj} \frac{d^j}{dx^j} q_0(0) + \sum_{j=0}^{n-1} \beta_{kj} \frac{d^j}{dx^j} q_0(1) = h_k(0). \quad (4)$$

This class of initial-boundary value problems includes many problems of physical importance. The partial differential equation (1) is the heat equation for a finite uniform rod when $n = 2, a = 1$. A linearization of the Schrodinger equation is given by $n = 2, a = \iota$ and a linearized Korteweg-de Vries equation appears for $n = 3, a = -\iota$.

It is possible to weaken the smoothness requirements on our initial and boundary data provided we weaken the smoothness requirements on the solution accordingly. Such weaker smoothness conditions are considered in [5] for problems whose spatial domain is the half-line.

3. An implicit solution to the IBVP

The initial-boundary value problem (1)-(4) may, under the assumption that it is well-posed, be solved using Fokas' unified transform method. Theorem 1, below, does not give the full solution but only the first steps. To obtain the full solution the transformed boundary functions $\tilde{f}_j$ and $\tilde{g}_j$ must be determined from the boundary conditions. The power of Theorem 1 is that it gives a representation of the solution q on the whole of Ω in terms of the value of q and its normal derivatives on three sides of the boundary of Ω .

Theorem 1. *Let the initial-boundary value problem (1) - (4) be well-posed in the sense that it admits a unique smooth solution. Then its solution q may be expressed as the sum of three integrals,*

$$\begin{aligned} q(x, t) = & \frac{1}{2\pi} \left(\int_{\mathbb{R}} e^{\iota \rho x - a \rho^n t} \hat{q}_0(\rho) d\rho - \int_{\partial D^+} e^{\iota \rho x - a \rho^n t} \sum_{j=0}^{n-1} c_j(\rho) \tilde{f}_j(\rho) d\rho \right) \\ & - \frac{1}{2\pi} \left(\int_{\partial D^-} e^{\iota \rho(x-1) - a \rho^n t} \sum_{j=0}^{n-1} c_j(\rho) \tilde{g}_j(\rho) d\rho, \right) \end{aligned} \quad (5)$$

where the integrands are given in terms of the initial datum and the boundary functions by

$$\begin{cases} \tilde{f}_j(\rho) = \int_0^T e^{a\rho^n s} f_j(s) ds, & \tilde{g}_j(\rho) = \int_0^T e^{a\rho^n s} g_j(s) ds, \\ f_j(t) = \partial_x^j q(0, t), & g_j(t) = \partial_x^j q(1, t), \\ \hat{q}_0(\rho) = \int_0^1 e^{-\iota\rho y} q_0(y) dy, & c_j(\rho) = -a\rho^n (\iota\rho)^{-(j+1)}, \\ D^\pm = D \cap \mathbb{C}^\pm, & D = \rho \in \mathbb{C} \operatorname{Re}(a\rho^n) < 0. \end{cases} \quad (6)$$

A full proof of Theorem 1 involves some technicality and considerable care with the smoothness of the data. We present below an outline of the proof to give the essential argument of Fokas' unified transform method, as applied to linear two-point initial-boundary value problems such as these. The full proof may be found in [4] with the general argument appearing in [2]. There are also treatments in [1] and [3].

The first step of the proof of Theorem 1 is the expression of the partial differential equation (1) in the form of a Lax pair. This is accomplished in Lemma 1. After this, particular solutions of the Lax pair may be used to specify a Riemann-Hilbert problem. The solution of that Riemann-Hilbert problem yields the result.

LEMMA 1. *The partial differential equation (1) is equivalent to the compatibility condition,*

$$\partial_t \partial_x \mu = \partial_x \partial_t \mu \quad (7)$$

of the Lax pair

$$\partial_t \mu + a\rho^n \mu = \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q, \quad (8)$$

$$\partial_x \mu - \iota\rho \mu = q, \quad (9)$$

for the function $\mu(x, t, \rho)$, where $c_j(\rho)$ are the functions defined in equations (6).

Proof: We take the x partial derivative of equation (8),

$$\begin{aligned} \partial_x \partial_t \mu &= -a\rho^n [\partial_x \mu + \sum_{j=0}^n (\iota\rho)^{-j} \partial_x^j q] \\ &= -a\rho^n [q + \iota\rho \mu + \sum_{j=0}^n (\iota\rho)^{-j} \partial_x^j q] \end{aligned} \quad (10)$$

the latter equality being justified by equation (9). Similarly, we take the t partial derivative of equation (9),

$$\partial_t \partial_x \mu = \partial_t q + \iota\rho \partial_t \mu \quad (11)$$

$$= \partial_t q - a\rho^n [\iota\rho\mu + \sum_{j=0}^{n-1} (\iota\rho)^{-j} \partial_x^j q], \quad (12)$$

by equation (8). The compatibility condition (7) is equivalent to the right hand sides of equations (10) and (12) being equal, which is equivalent to the partial differential equation (1).

Outline proof of Theorem 1. We break the derivation into three steps.

Solutions of the Lax Pair: The first step is to find particular integrals of the Lax pair (8)-(9). We rewrite equation (9) as

$$\partial_x(e^{-\iota\rho x}\mu) = e^{-\iota\rho x}q$$

which has particular integral

$$\mu(x, t, \rho) = \int_{x*}^x e^{\iota\rho(x-y)} q(y, t) dy + e^{\iota\rho(x-x*)} \mu(x*, t, \rho), \quad (13)$$

where $\mu(x*, t, \rho)$ is a solution of equation (8) at $x = x*$. Taking a particular integral of equation (8) in the same way we obtain

$$\mu(x*, t, \rho) = \int_{t*}^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(x*, s) ds. \quad (14)$$

Substituting equation (14) into equation (13) yields

$$\mu*(x, t, \rho; x*, t*) = \int_{x*}^x e^{\iota\rho(x-y)} q(y, t) dy + e^{\iota\rho(x-x*)} \int_{t*}^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(x*, s) ds.$$

The function $\mu*$ is a solution of the Lax pair (8)-(9) for any particular choice of the pair $(x*, t*) \in \Omega$. The open sets of complex numbers $D^\pm$ are defined in (6). We also define the open sets

$$E^\pm = \mathbb{C}^\pm \cap E, \quad E = \rho \in \mathbb{C} : \operatorname{Re}(a\rho^n) > 0.$$

Following Proposition 3.1 of [2] we choose the the points $x*, t*$ to be the four corners of Ω , defining the functions $\mu Y(x, t, \rho)$ for $Y \in D^\pm, E^\pm$:

$$\begin{aligned} \mu_E + (x, t, \rho) &= \int_0^x e^{\iota\rho(x-y)} q(y, t) dy + e^{\iota\rho x} \int_0^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(0, s) ds, \\ \mu_D + (x, t, \rho) &= \int_0^x e^{\iota\rho(x-y)} q(y, t) dy - e^{\iota\rho x} \int_t^T e^{a\rho^n(s-t)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(0, s) ds, \\ \mu_D - (x, t, \rho) &= - \int_x^1 e^{-\iota\rho(y-x)} q(y, t) dy - e^{\iota\rho(x-1)} \int_t^T e^{a\rho^n(s-t)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(1, s) ds, \\ \mu_E - (x, t, \rho) &= - \int_x^1 e^{-\iota\rho(y-x)} q(y, t) dy + e^{\iota\rho(x-1)} \int_0^t e^{-a\rho^n(t-s)} \sum_{j=0}^{n-1} c_j(\rho) \partial_x^j q(1, s) ds, \end{aligned}$$

These functions are all entire in ρ and are all particular solutions of the Lax pair. The μY are indexed with the sets $D^\pm, E^\pm$ because property $\mu Y(x, y, \rho) \longrightarrow 0$ as $\rho \longrightarrow \infty$ from within Y holds.

Riemann-Hilbert Problem: For any $Y, Z \in D^\pm, E^\pm$ let the function

$$\mu Y Z = \mu_j - \mu_k.$$

Then μ_{jk} is a solution of the homogeneous Lax pair

$$\begin{aligned} \partial_t \mu + a \rho^n \mu &= 0, \\ \partial_x \mu - \iota \rho \mu &= 0. \end{aligned}$$

Taking particular integrals of the homogeneous Lax pair, in the same way as is done above for the inhomogeneous Lax pair, we deduce that

$$\mu_{jk}(x, t, \rho) = e^{\iota \rho x - a \rho^n t} X_{jk}(\rho), \quad (15)$$

where the function X_{jk} may be easily obtained by evaluating equation (15) at $x = t = 0$. This yields, in particular,

$$\begin{aligned} \mu_{D+E} + (x, t, \rho) &= -e^{\iota \rho x - a \rho^n t} \sum_{j=0}^{n-1} c_j(\rho) \tilde{f}_j(\rho), \\ \mu_{D-E} - (x, t, \rho) &= -e^{\iota \rho(x-1) - a \rho^n t} \sum_{j=0}^{n-1} c_j(\rho) \tilde{g}_j(\rho), \\ \mu_{E+E} - (x, t, \rho) &= -e^{\iota \rho x - a \rho^n t} \hat{q}_0(\rho). \end{aligned}$$

As the sets $D^\pm, E^\pm$ are each comprised of finitely many simply connected components, are disjoint and have union $\mathbb{C} \setminus (\partial D^+ \cup \partial D^- \cup \mathbb{R})$, it makes sense to define a 'jump function' $M : \Omega \times (\partial D^+ \cup \partial D^- \cup \mathbb{R}) \longrightarrow \mathbb{C}$ by

$$M(x, t, \rho) = \mu_{D+E} + (x, t, \rho) \chi_{\partial D^+} + \mu_{D-E} - (x, t, \rho) \chi_{\partial D^-} + \mu_{E+E} - (x, t, \rho) \chi_{\mathbb{R}},$$

where each χ_j is the indicator function for the set j . Then M represents the jumps on the boundaries of the domains of definition of the μY .

4. The Global Relation:

The global relation is derived from Green's Theorem and represents an equation relating the t-transforms of the boundary functions, defined in (6) to the Fourier transforms of the initial datum, $q_0 x$, and final function, $q_T(x) = q(x, T)$. We define these Fourier transforms as

$$\begin{aligned} \hat{q}_0(\rho) &= \int_0^1 e^{-\iota \rho x} q_0(x) dx = \int_{\mathbb{R}} e^{-\iota \rho x} q_0(x) \chi_{[0,1]} dx, \\ \hat{q}_T(\rho) &= \int_0^1 e^{-\iota \rho x} q(x, T) dx, \end{aligned}$$

where $\rho \in \mathbb{C}$.

Global relation : *Let $q : \Omega \rightarrow \mathbb{R}$ be a formal solution to an initial-boundary value problem specified by the partial differential equation (1) and initial condition (2). Then the functions $\widehat{q}_0, \widehat{q}_T$ defined above and the functions $\widetilde{f}_j$ and $\widetilde{g}_j$, given by (6) satisfy*

$$\sum_{j=0}^{n-1} c_j(\rho) \left(\widetilde{f}_j(\rho) - e^{-\iota\rho} \widetilde{g}_j(\rho) \right) = \widehat{q}_0(\rho) - e^{a\rho^n T} \widehat{q}_T(\rho), \quad \rho \in \mathbb{C}. \quad (16)$$

The global relation is useful because of the particular form of the spectral transforms of the boundary functions.

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