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PARTITION GENERATING FUNCTIONS AND CONTINUED FRACTIONS

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Dedicated to Prof. A.K. Agarwal on his 70^{th} Birth Anniversary

Abstract: In this paper, making use of known result, certain continued fraction representations for partition generating functions have been established.

Keyword and Phrases: Partition, Partition generating function, Continued fraction, Rogers-Fine Identity, Lambert series, Generalized Lambert Series.

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1. Introduction, Notations and Definitions

As usual, for a and q complex numbers with $|q| < 1$, define

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{r=0}^{n-1}(1-aq^r), \qquad \text{for} \quad n \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \qquad |q|<1$$

and

$$(a_1;q)_n(a_2;q)_n...(a_r;q)_n = (a_1,a_2,...,a_r;q)_n$$

An ${}_r\Phi_s$ basic hypergeometric series is defined by ([3], (2.7.2) p. 347) and [1]: (1.2.22), p. 4)

$${}_r\Phi_s\left[\begin{array}{c} a_1,a_2,...,a_r;q;z \\ q,b_1,b_2,...,b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n z^n}{(b_1,b_2,...,b_s;q)_n}\{(-1)^n q^{n(n-1)/2}\}^{1+s-r}$$

Ramanujan's Notebooks, specially the second and 'Lost' Notebooks contain a large number of identities and results involving continued fractions. In this paper, we shall establish certain results involving partition generating functions and continued fractions.

2. Main Results

In this section, we shall establish the continued fraction representation of the ratio of two ${}_2\Phi_1$ series

$$\frac{{}_2\Phi_1\left[\begin{matrix}\alpha,\beta q;z\\ \gamma\end{matrix}\right]}{{}_2\Phi_1\left[\begin{matrix}\alpha,\beta;zq\\ \gamma\end{matrix}\right]}=\frac{\sum_{n=0}^{\infty}\frac{(\alpha,\beta q;q)_n z^n}{(q,\gamma;q)_n}}{\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n z^n q^n}{(q,\gamma;q)_n}}$$

$$=\frac{1}{1-}\,\frac{z(1-\alpha)/(1-\gamma)}{1-}\,\frac{z(\alpha-\gamma)(1-\beta q)/(1-\gamma)(1-\gamma q)}{1-}$$

$$\frac{zq(\beta-\gamma)(1-\alpha q)/(1-\gamma q)(1-\gamma q^2)}{1-}\,\frac{zq(\alpha-\gamma q)(1-\beta q^2)/(1-\gamma q^2)(1-\gamma q^3)}{1-\quad\ldots} \tag{2.1}$$

Proof. (2.1)

$$\frac{\sum_{n=0}^{\infty}\frac{(\alpha,\beta q;q)_n z^n}{(q,\gamma;q)_n}}{\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n z^n q^n}{(q,\gamma;q)_n}}=\frac{1}{1-\dfrac{\sum_{n=0}^{\infty}\frac{(\alpha,\beta q;q)_n z^n}{(q,\gamma;q)_n}-\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n z^n q^n}{(q,\gamma;q)_n}}{\sum_{n=0}^{\infty}\frac{(\alpha,\beta q;q)_n z^n}{(q,\gamma;q)_n}}}$$

$$=\frac{1}{1-}\,\frac{z(1-\alpha)/(1-\gamma)}{\left\{\dfrac{\sum_{n=0}^{\infty}\frac{(\alpha,\beta q;q)_n z^n}{(q,\gamma;q)_n}}{\sum_{n=0}^{\infty}\frac{(\alpha q,\beta q;q)_n z^n}{(q,\gamma q;q)_n}}\right\}}$$

$$= \frac{1}{1-} \frac{z(1-\alpha)/(1-\gamma)}{1 - \cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{(\beta q;q)_n z^n}{(q;q)_n} \left\{ \frac{(\alpha q;q)_n}{(\gamma q;q)_n} - \frac{(\alpha;q)_n}{(\gamma;q)_n} \right\}}{\displaystyle\sum_{n=0}^{\infty} \frac{(\alpha q, \beta q;q)_n z^n}{(q,\gamma q;q)_n}}}$$

$$= \frac{1}{1-} \frac{z(1-\alpha)/(1-\gamma)}{1-} \frac{z(\alpha-\gamma)(1-\beta q)/(1-\gamma)(1-\gamma q)}{\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{(\alpha q, \beta q;q)_n z^n}{(q,\gamma q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(\alpha q, \beta q^2;q)_n z^n}{(q,\gamma q^2;q)_n}}}$$

Iterating this process, we get the required result (2.1).

3. Special Cases

In this section, we shall deduce interesting special cases of (2.1).

(i) Applying the property ([2], (2.3.14), p. 33), (2.1) takes the following form,

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(\alpha, \beta q;q)_n z^n}{(q,\gamma;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(\alpha, \beta;q)_n z^n q^n}{(q,\gamma;q)_n}} = \frac{1}{1-} \frac{z(1-\alpha)}{(1-\gamma)-} \frac{z(\alpha-\gamma)(1-\beta q)}{(1-\gamma q)-} \frac{zq(\beta-\gamma)(1-\alpha q)}{(1-\gamma q^2)-}$$

$$\frac{zq(\alpha-\gamma q)(1-\beta q^2)}{(1-\gamma q^3)- \quad ...} \tag{3.1}$$

(ii) Taking $\beta = 1$ and $\gamma = q$ in (3.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n z^n}{(q;q)_n} = \frac{(\alpha z;q)_\infty}{(z;q)_\infty} = \frac{1}{1-} \frac{z(1-\alpha)}{(1-q)-} \frac{z(\alpha-q)(1-q)}{(1-q^2)-} \frac{zq(1-q)(1-\alpha q)}{(1-q^3)-}$$

$$\frac{zq(\alpha-q^2)(1-q^2)}{(1-q^4)-} \frac{zq^2(1-q^2)(1-\alpha q^2)}{(1-q^5)-} \frac{zq^2(\alpha-q^3)(1-q^3)}{(1-q^6)-...} \tag{3.2}$$

(iii) Taking $\alpha = 0$ and $z = q$ in (3.2), we find

$$\frac{1}{(q;q)_\infty} = \frac{1}{1-} \frac{q}{(1-q)+} \frac{q^2(1-q)}{(1-q^2)-} \frac{q^2(1-q)}{(1-q^3)+} \frac{q^4(1-q^2)}{(1-q^4)-}$$

$$\frac{q^3(1-q^2)}{(1-q^5)+}\frac{q^6(1-q^3)}{(1-q^6)-}\frac{q^4(1-q^3)}{(1-q^7)+...} \tag{3.3}$$

Since $\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n$, where $p(n)$ is the number of unrestricted partition of n. (3.3) is the continued fraction representation of unrestricted partition generating functions $p(n)$.

(iv) Again, replacing q by q^2 and then taking $\alpha = 0$ and $z = q$ in (3.2), we get

$$\frac{1}{(q;q^2)_\infty} = \frac{1}{1-}\frac{q}{(1-q^2)+}\frac{q^3(1-q^2)}{(1-q^4)-}\frac{q^3(1-q^2)}{(1-q^6)+}\frac{q^7(1-q^4)}{(1-q^8)-}$$

$$\frac{q^5(1-q^4)}{(1-q^{10})+}\frac{q^{11}(1-q^6)}{(1-q^{12})-...} \tag{3.4}$$

where $\frac{1}{(q;q^2)}$ in the generating function of the partitions into odd parts.

(v) Putting $\frac{z}{\alpha}$ for z and then taking $\alpha \to \infty$ in (3.2), we get

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n-1)/2} z^n}{(q;q)_n} = (z;q)_\infty = \frac{1}{1+}\frac{z}{(1-q)-}\frac{z(1-q)}{(1-q^2)+}\frac{zq^2(1-q)}{(1-q^3)-}$$

$$\frac{zq(1-q^2)}{(1-q^4)+}\frac{zq^4(1-q^2)}{(1-q^5)-}\frac{zq^2(1-q^3)}{(1-q^6)+}\frac{zq^6(1-q^3)}{(1-q^7)-...} \tag{3.5}$$

(vi) For $z = -q$, (3.5) yields

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n} = (-q;q)_\infty = \frac{1}{1-}\frac{q}{(1-q)+}\frac{q(1-q)}{(1-q^2)-}\frac{q^3(1-q)}{(1-q^3)+}$$

$$\frac{q^2(1-q^2)}{(1-q^4)-}\frac{q^5(1-q^2)}{(1-q^5)+}\frac{q^3(1-q^3)}{(1-q^6)-}\frac{q^7(1-q^3)}{(1-q^7)+...} \tag{3.6}$$

where $(-q;q)_\infty$ generates the partitions into distinct parts. Since $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$, continued fractions in (3.4) and (3.6) are equivalent.

(vii) Taking $\beta = 1$ in (3.1), we have

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n z^n}{(\gamma;q)_n} = \frac{1}{1-}\frac{z(1-\alpha)}{(1-\gamma)-}\frac{z(\alpha-\gamma)(1-q)}{(1-\gamma q)-}\frac{zq(1-\gamma)(1-\alpha q)}{(1-\gamma q^2)-}$$

$$\frac{zq(\alpha-\gamma q)(1-q^2)}{(1-\gamma q^3)-}\frac{zq^2(1-\gamma q)(1-\alpha q^2)}{(1-\gamma q^4)-}\frac{zq^2(\alpha-\gamma q^2)(1-q^3)}{(1-\gamma q^5)-...} \tag{3.7}$$

(viii) Taking $\gamma=\alpha q$ in (3.7), we have

$$(1-\alpha)\sum_{n=0}^{\infty}\frac{z^n}{(1-\alpha q^n)}=\frac{1}{1-}\frac{z(1-\alpha)}{(1-\alpha q)-}\frac{z\alpha(1-q)^2}{(1-\alpha q^2)-}\frac{zq(1-\alpha q)^2}{(1-\alpha q^3)-}$$
$$\frac{z\alpha q(1-q^2)^2}{(1-\alpha q^4)-}\frac{zq^2(1-\alpha q^2)^2}{(1-\alpha q^5)-}\frac{zq^2\alpha(1-q^3)^2}{(1-\alpha q^6)-...} \tag{3.8}$$

(ix) Replacing q by q^5 and then taking $\alpha=q^j$ and $z=q^i$ in (3.8), we have

$$(1-q^{\alpha})\sum_{n=0}^{\infty}\frac{q^{ni}}{(1-q^{5n+j})}=\frac{1}{1-}\frac{q^i(1-q^j)}{(1-q^{5+j})-}\frac{q^{i+j}(1-q^5)^2}{(1-q^{10+j})-}\frac{q^{i+5}(1-q^{5+i})^2}{(1-q^{15+j})-}$$
$$\frac{q^{i+j+5}(1-q^{10})^2}{(1-q^{2i+j})-...}, \tag{3.9}$$

which is continued fraction representation of Lambert series.

(x) Taking $\frac{z}{\alpha}$ for z in (3.7) and then taking $\alpha\to\infty$ and $\gamma=0$, we find

$$\sum_{n=0}^{\infty}(-)^n q^{n(n-1)/2}z^n=\frac{1}{1+}\frac{z}{1-}\frac{z(1-q)}{1+}\frac{zq^2}{1-}\frac{zq(1-q^2)}{1+}\frac{zq^4}{1-}\frac{zq^2(1-q^3)}{1+...} \tag{3.10}$$

(xi) For $z=-q$, (3.10) yields

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}=\Phi(q)=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{1}{1-}\frac{q}{1+}\frac{q(1-q)}{1-}\frac{q^3}{1+}\frac{q^2(1-q^2)}{1-}\frac{q^5}{1+...} \tag{3.11}$$

where $\Phi(q)$ is Ramanujan's theta function.

(xii) For $z=q$, (3.10) yields

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}=\frac{1}{1+}\frac{q}{1-}\frac{q(1-q)}{1+}\frac{q^3}{1-}\frac{q^2(1-q^2)}{1+}\frac{q^5}{1-...} \tag{3.12}$$

where $\sum_{n=0}^{\infty}(-)^n q^{n(n-1)/2}$ is false theta function.

(xiii) Replacing q by q^2, z by $\frac{z}{\alpha}$ and then taking $\alpha\to\infty$ in (3.2), we get

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n-1)}z^n}{(q^2;q^2)_n}=(z;q^2)_\infty=\frac{1}{1+}\frac{z}{(1-q^2)-}\frac{z(1-q^2)}{(1-q^4)+}\frac{zq^4(1-q^2)}{(1-q^6)-}$$

$$\frac{zq^2(1-q^4)}{(1-q^8)+}\frac{zq^8(1-q^4)}{(1-q^{10})-}\frac{zq^4(1-q^6)}{(1-q^{12})+...} \tag{3.13}$$

(xiv) For $z=-q$, (3.13) yields

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n}=(-q;q^2)_\infty=\frac{1}{1-}\frac{q}{(1-q^2)+}\frac{q(1-q^2)}{(1-q^4)-}\frac{q^5(1-q^2)}{(1-q^6)+}$$

$$\frac{q^3(1-q^4)}{(1-q^8)-}\frac{q^9(1-q^4)}{(1-q^{10})+}\frac{q^5(1-q^6)}{(1-q^{12})-...} \tag{3.14}$$

where $(-q;q^2)_\infty$ generates the partition into distinct odd parts.
(xv) Again taking $z=-q^2$ in (3.13) we get

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}=(-q^2;q^2)_\infty=\frac{1}{1-}\frac{q^2}{(1-q^2)+}\frac{q^2(1-q^2)}{(1-q^4)-}\frac{q^6(1-q^2)}{(1-q^6)+}$$

$$\frac{q^4(1-q^4)}{(1-q^8)-}\frac{q^{10}(1-q^4)}{(1-q^{10})+}\frac{q^6(1-q^6)}{(1-q^{12})-...} \tag{3.15}$$

where $(-q^2;q^2)_\infty$ generates the partitions into distinct even parts.

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