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REVIEW AND GENERALIZATION OF THE CONCEPTS OF INFINITE SERIES WITH RELATED TERMINOLOGIES FROM T.M. APOSTOL'S BOOK MATHEMATICAL ANALYSIS

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Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: The objective of this paper is to explore the general concept of infinite series and definition of few terminologies on the basis of the book Mathematical Analysis written by T. M. Apostol in 1974. Generally, the ordered set of numbers followed by any fixed rule is called a sequence and the combination of terms of sequences by using addition or subtraction sign with a fixed rule is called a series. In precise sense, a sequence defined by the function $f : \mathbb{N} \to \mathbb{R}$ and is denoted by $\{f(x)\}$ or $\{a_n\}$. The values of $a_1, a_2...$ are called the terms of the sequence $\{a_n\}$.

Keyword and Phrases: Series, Limit superior, Limit inferior,Convergent, Divergent.

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1. Introduction

The book entitled Mathematical Analysis (2nd Ed) was written by T. M. Apostol in 1974 is a very popular book. It is very lucid book in the field of mathematical analysis with regarding language and arrangement of matter. This book is used as the text book in different universities for bachelors degree and masters degree. Here, we are trying to expressed our experiences from students life to as a teacher because of it continuous handling for teaching learning activities in Tribhuvan University. Here, we are tried to simplify the proof and clarify the terminologies used in eighth chapter series and product.

Now, we start from a very basic term sequence. Generally, the ordered set of numbers followed any fixed rule is called a sequence and the combination of terms of sequences by using addition or subtraction sign with a fixed rule is called a series. In precise sense, a sequence defined by the function F from the set of natural numbers N to the set of real numbers R. It is denoted by $f(x)$ or $\{a_n\}$ and the values of $\{a_1, a_2, ...\}$ are called the terms of the sequence $\{a_n\}$. A sequence which contains finite number of terms is called finite sequence and a sequence having infinite number of terms is called an infinite sequence. Correspondingly we can define for the series [1].

Limit Superior

Let $\{a_n\}$ be a sequence of real numbers. Suppose there is a real number A satisfying the following conditions:

1. $\forall \epsilon > 0 \exists N : n > N \Rightarrow a_n < A + \epsilon$ and
2. given $\epsilon > 0$ and given $m > o$,$\exists$ an integer $M : n > m \Rightarrow a_n > A - \epsilon$

Then, the real number$A$ is said to be the limit superior (upper limit) of $\{a_n\}$ and is denoted by A= $\limsup_{n\to\infty} a_n$. It is note that (1) $\Rightarrow\{a_1, a_2, ...\}$ is bounded above. If the set $\{a_1, a_2, ...\}$ is not bounded above then we define

$$\limsup_{n\to\infty} a_n = \infty.$$

In other words, (1) means that ultimately all terms of the sequence lie to the left of$A + \epsilon$. If the set $\{a_1, a_2, ...\}$ bounded above but not below and of $\{a_n\}$ has no finite limit superior then we have

$$\limsup_{n\to\infty} a_n = \infty.$$

(2) Means that infinitely many terms lie to the right of $A - \epsilon$. Thus it is clear that there cannot be more than an A which satisfies the conditions (1) and (2).

Limit Inferior

A real number A is said to be the limit inferior of $\{a_n\}$ if A satisfies the above two conditions:

$$\lim_{n\to\infty} inf a_n = -limsup b_n,$$

where $a_n = b_n$ for $n = 1, 2, \ldots$ [2].

Examples

1. $(-1)^n \left(1 + \frac{1}{n}\right)$ Therefore,

$$\lim_{n\to\infty} lim \sup a_n = -1,$$

and

$$\lim_{n\to\infty} \sup a_n = 1$$

2. $a_n = (-1)^n$
   Therefore,

$$\lim_{n\to\infty} inf a_n = -1,$$

   and

$$\lim_{n\to\infty} \sup a_n = 1$$

3. $(-1)^n_n$ Therefore,

$$\lim_{n\to\infty} inf a_n = -\infty,$$

   and

$$\lim_{n\to\infty} \sup a_n = \infty$$

4. $a_n = n^2 \left(\frac{n\lambda}{2}\right)$ Therefore, [3]

$$\lim_{n\to\infty} inf a_n = 0,$$

   and

$$\lim_{n\to\infty} \sup a_n = \infty$$

Infinite Series

The ordered pair $(\{a_n\}, \{s_n\})$is said to be an infinite series where, $s_n$equal to the nth partial sum of the series.The series is said to be the sequence$\{s_n\}$is convergent or divergent.
symbolically,

$$\begin{aligned} s_n &= a_1 + a_2 + ... + a_n = \sum_{k=1}^{\infty} a_k \\ &= a_1 + a_2 + ... + a_n + ..., \\ &= a_1 + a_2 + ... \end{aligned}$$

Note that here k is a dummy variable [4].

Some Related Theorems

Theorem 1. *Let $\sum a_n = a, \sum b_n = b$ be convergent series. Then, for every pair of constants α and β, the series $\sum(\alpha a_n + \beta b_n)$ converges to the sum $\alpha a + \beta b$.*

i.e.$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$.

Proof. Let $\sum a_n = a$, $\sum b_n = b$ be the convergent series. Then $\forall$ α , β the series $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ converges to the sum of αa and βb.

Now, $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$. Which follows the proof of the theorem.

Theorem 2. *Let $a_n \geq 0$ for each $n = 1, 2, ...$ Then, $\sum a_n$ converges if and only if, the sequence of partial sum is bounded above.*

Proof. Let $S_n = a_1 + a_2 + ... + a_n$. Then clearly S_n is monotonically increased. Let S_n is monotone & increasing convergent then it is bounded above. $\Rightarrow a_n$ is bounded above. Next, let S_n is bounded above then it has attains its least upper bounds say s. Since s is an upper bounds for S_n, $S_n \geq s, \forall n \in z^+$. Since s is the least upper bound, for any $\epsilon > 0$, $\exists$ a positive integer$N : s - \epsilon < S_N$. If S_n is monotonically increasing, $S_n \geq S_N$ for $n \geq N$. Then we have, $s - \epsilon < S_n \leq s$ $\Rightarrow |S_n - s| < \epsilon, n \geq N$. Therefor, $\lim_{n\to\infty} S_n = s$. Hence the series an is convergent.

Theorem 3 (Telescoping Series). *Let$\{a_n\}$ and $\{b_n\}$ be two sequences defined as, $a_n = b_{n+1} - b_n$ for n=1,2,...,then $\sum a_n$ converges if, and only if, $lim b_n$ exists, in which case we have $n \to \infty$. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(b_{n+1} - b_n) = b_{n+1} - b_n$*
$= b_{n+1} - b_1$ which follows the proof of the theorem.

$a_1 = b_2 - b_1$
$a_2 = b_3 - b_2$
$. = ...$
$. = ...$
$. = ...$
$a_{n-1} = b_n - b_{n-1}$
$a_n = b_{n+1} - b_n$
$\Rightarrow a_1 + a_2 + a_3 + a_4 + ... + a_n = b_2 - b_1 + b_3 - b_2 + ... + b_{n+1} - b_n$
$= b_{n+1} - b_1$.

Inserting and Removing Parentheses

Let P be a function whose domain range is a subset of the positive integers such that

1. $P(n) < p(n), n < m$ Let $\sum a_n$& $\sum b_n$ be related as, $b_1 = a_1 + a_2 + a_3 + ... + a_{p(1)}$.

2. $b_{n+1} = a_{p(n)+1} + a_{p(n)+2} + ... + a_{p(n+1)}, ifn = 1, 2...$ Then we say that $\sum b_n$ is obtained from $\sum a_n$ by inserting the parentheses and $\sum a_n$ is obtained from $\sum b_n$ by removing parentheses [5].

Theorem 4 (Cauchy Condition for Series). *The series $\sum a_n$ converges, if and only if for every $\epsilon > o$, there exists an integers $N : n \geq N$ implies that, i.*

$|a_{n+1} + ... + a_{n+p}| < \epsilon$ *for each* $p = 1, 2, ...$
Proof. Let $S_n = \sum_{n=1}^{\infty} a_n$, where $|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| < \epsilon$.
In other words, by Cauchys General principle of convergence (for sequences), the sequences if to each $\epsilon > 0, \exists n \in Z^+$. $|S_{n+p} - S_n| < \epsilon, \forall n \geq N \& p \geq 1$.
Or $|a_{n+1} + a_{n+2} + ... + a_{n+p}| < \epsilon$.
Now put p = 1 in (i) we find that $lim a_n = o$ is a necessary condition for $n = \infty$ Convergence of $\sum a_n$. Since this condition is not sufficient for this that we consider an example.

$$A_n = \frac{1}{n}$$

When $n = 2^m$ and $p = 2^m$ is (i) we have,

$$a_{n+1} + a_{n+2} + ... + a_{n+p} = \frac{1}{2^m + 1} + ... \frac{1}{2^m + 2^m} \geq \frac{2^m}{2^m + 2^m} \geq \frac{2^m}{2.2^m} = \frac{1}{2}$$

Therefore, the Cauchy condition can not satisfied for $\epsilon \leq \frac{1}{2}$
Thus, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called harmonic series.

Theorem 5. *If* $\sum a_n \to S$, *every* $\sum b_n$ *obtained from* $\sum a_n$ *by inserting parentheses also converges to s.*
Proof. Let $\sum a_n$ and $\sum b_n$ be two series which are related A.P., and $b_{n+1} = a_{p(n)+1} + a_{p(n)+2} + ... + a_{p(n+1)}$. If$n = 1, 2, ...$ Write $S_n = \sum_{k=1}^{n} ak, t_n = \sum_{k=1}^{n} bk$.
Then $\{t_n\}$ is a sub-sequence of $\{S_n\}$. i.e.$t_n = S_{p(n)}$. [since $\sum b_n$ is obligated by inserting parentheses of$\sum a_n$ so is its parted sums]. If $\{S_n\}$ is convergent then $\{t_n\}$ also convergent. i.e.

$$\lim_{n \to \infty} S_n \to S \Rightarrow \lim_{n \to \infty} t_n \to S$$

1. Removing parenthesis may destroy convergence. Let observe this, consider $\sum b_n$ in which each term is o then obviously convergent. Let $p(n) = 2n$ & let $a_n = (-1)^n$ which satisfies $n = -1, 1, -1, ...$ oscillatory series.
   i) $p(n) < p(m), n < m$
   ii) $b_{n+1} = a_{p(n)+1} + ... + a_{p(n+1)}$. Holds but $\sum a_n$ is divergent.

2. Parenthesis can be removed if we further restrict$\sum b_n$ and p[6].

Rearrangements of Series

Let f be a function from Z^+ to Z^+ i.e. $f : Z^+ \to Z^+$ is a one to one & onto function and assume that f is one- to one on Z^+. Let $\sum a_n$ and $\sum b_n$ be two series such that $B_n = af(n)$ where $n = 1, 2, ...$
Then $\sum b_n$ is said to be a rearrangement of $\sum a_n$. Also $a_n = bf^{-(n)}$, then $\sum a_n$ is said to be arrangement of $\sum b_n$ [1].

Theorem 6. *Let $\sum a_n$ be an absolutely convergent series having sum s, then every rearrangement of $\sum a_n$ also converges absolutely and has sums.*
Proof. Define $\{b_n\}$ by $b_n = af(n)$... (1) Then,

$$|b_1| + |b_2| + ... + |b_n| = |af(1)| + |af(2)| + ... + |af(n)| \leq \sum_{n=1}^{\infty} |a_n|$$

$\Rightarrow \sum |b_n|$ is bounded by partial sums.
$\Rightarrow \sum b_n$ Converges absolutely.
Next we have to show that $\sum b_n = s$ for this. Let $t_n = b_1 + ... + b_n$ and $S_n = a_1 + ... + a_n$ Given $\epsilon > 0$, choose an integers N so that $|S_N - s| < \frac{\epsilon}{2}$ and

$$\sum_{n=1}^{\infty} |a_{N+k}| \leq \frac{\epsilon}{2}$$

Then by triangle inequality we have $|t_n - s| \leq |t_n - s_N| + |s_N - s| < |t_n s_N| \leq \frac{\epsilon}{2}$... (2)
Choose$M > 0$ so that $\{1, 2, ...N\} \leq \{f(1), f(2), ...f(M)\}$
Then $n > M \Rightarrow f(n) > N$ and we have $|t_n - s_n| = |(b_1 + b_2 + ... + b_n) - (a_1 + a_2 + ... + a_N|$
$= |af(1) + af(2) + ... + af(n)| - (a_1 + a_2 + ... + a_n)| \leq |a_{N+1}| + |a_{N+2}| + ... \leq \frac{\epsilon}{2}$.
Then from (2) we have,$|t_n - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$$\Rightarrow \lim_{n\to\infty} t_n = s$$

$\Rightarrow \sum b_n = s$. Since $\sum b_n$is arbitrary.

Theorem 7. *Let $\sum a_n$ be a conditional convergent series with real valued terms. Let x and y be given numbers in the$[-\infty, +\infty]$,with order relation $x \leq y$, then $\exists$ a rearrangement $\sum b_n$ of $\sum a_n$:*

$$\lim_{n\to\infty} f \cap t_n = x$$

and

$$\lim_{n\to\infty} supt_n = y,$$

where $t_n = b_1 + ... + b_n$ *(particle sums).*

Proof. Neglecting the terms having value zero. Assume that $\sum a_n$ has positive i.e.$a_1, a_2, ..., a_n \neq o$. Denote $p_n = n^(th)$ positive terms of $\sum a_n$ and $q_n = n^(th)$ negative terms of $\sum a_n$.

Then $\sum p_n$ and $\sum q_n$ both are divergent series of positive terms (since the series $\sum a_n$ is conditionally convergent). Let $\{x_n\}$ and $\{y_n\}$ be two sequences: $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, Where $x_n < y_n$. Now, arrange the positive terms up to k_1 (just enough) i.e. $p_1 + p_2 + ... + p_(k_1) > y_1$, followed by just enough (let r_1) negative terms i.e. $p_1 + p_2 + ... + p_(k_1) - q_1 - q_2 - ... - q_(r_1) < x_1$. Again we take just enough further positive terms. i.e.$p_1 + ... + pk_1 q_1 - q_2 - ... - qr_1 + pk_1 + 1 + pk_1 + 2 + ... + pk_2 > y_2$, followed by just enough further negative terms i.e.$p_1 + ... + pk_1 q_1 - ... - qr_1 + pk_1 + 1 + ... - pk_2 - qr_1 + 1 - qr_1 + 2 - ... - qr_2 < x_2$. Since$\sum p_n$ and $\sum q_n$ are both divergent series of positive terms so the above steps are possible. If we continue their steps then we get a rearrangement of $\sum a_n$. Next we have to show that the partial sums of this arrangement have limit superior y and limit inferior x. Since $(p_1 + ... + pk_1) + (p_1 + ... + pk_1 - q_1 ... - qr_1 + pk_1 + 1 + ... + pk_2) + (p1 + ... + pk_1 - q_1 ... - qr_1 + pk + 1 + . + pk_2 - qr_1 + 1 . qr_2 + pk_2 + 1 + ... pk3) + ... > (y_1 + y_2 + y_3 + ...) = \sum_{k=1}^{\infty} y_k$

Thus $\{y_n\}$ is bounded above and hence it has limit superior.

$\lim_{n\to\infty} supy_n = y$. And $(p_1 + ... + pk_1 - q_1 - ... - qr1) + (p_1 + ... + pk_1 - q_1 ... qr_1 + pk_1 + pk_2 - qr_1 + 1 - - - - - - qr_2) + ... < x_1 + x_2 + ...$ Thus the sequences $\{x_n\}$ bounded from below and hence it has lim inferior.

i.e.

$$\lim_{n\to\infty} infx_n = x$$

Proved.

Sub Series

Let f be a function from infinite subsets of Z^+ to Z^+, assume that f is one to one on Z^+. Two series $\sum a_n$ and $\sum b_n$ are related as $b_n = a.f(n)$ if $n \in Z^+$, then $\sum b_n$ is said to be a sub series of $\sum a_n$ [3].

Theorem 8. *Let* $\sum a_n$ *be an absolutely convergent, every sub series* $\sum b_n$*,also converges absolutely. Moreover we have,*

$$|\sum_{n=1}^{\infty} b_n| \leq \sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

Proof. Since $\sum a_n$ converges absolutely, i.e. $|\sum b_n| \leq \sum_| a_n|$, Given $n \in Z^+$ and let N be the largest positive integer in the set $\{f_1, f_2, ...f_n\}$. Let $\sum b_n$ be a sub series of $\sum a_n$.
Then,

$$|\sum_{n=1}^{n} b_n| \leq \sum_{n=1}^{n} |b_n| \leq \sum_{n=1}^{N} |a_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

$\Rightarrow \sum b_n$ is absolutely convergent. Since $\sum b_n$ is arbitrary.

Theorem 9. *Let$\{f_1, f_2, ...f_n\}$ be a countable collection of functions, each of which is defined on Z^+ with following properties:-*

1. *Each f_n is one to one on Z^+.*
2. *The rang f_n is a sub set of Q_n of Z^+.*
3. *$\{Q_1, Q_2, ...\}$ is a collection of disjoint sets whose union is Z.*

OR

1. *Each f_n is one to on Z^+*
2. *$f_n(Z^+) = Q_n \subset Z^+$.*
3. *$Q = \{Q_1, Q_2, ...\}$, $Q_1 \cap Q_2 \cap ... = \phi$, & $Q_1 \cup Q_2 \cup Q_3 \cup ... = Z^+$.*

Let $\sum a_n$ be an absolutely convergent and define $b_k(n) = af_k(n)$, if $n \in Z^+$ and $k \in Z^+$. Then (i) for each k,

$$\sum_{n=1}^{\infty} b_k(n)$$

is an absolutely convergent sub series of $\sum a_n$.
(ii) If

$$S_k = \sum_{n=1}^{\infty} b_k,$$

$$\sum_{n=1}^{\infty} s_k,$$

converges absolutely and has the same sum as

$$\sum_{n=1}^{\infty} a_k.$$

Proof. (i) follows from theorem (8).
(ii) Let

$$t_k = |s_1| + |s_2| + ... + |s_k|.$$

$$t_n \leq \sum_{n=1}^{\infty} |b_1(n)| + ... + \sum_{n=1}^{\infty} |b_1(k)|$$

$$= \sum_{n=1}^{\infty} (|b_1(n)| + ... + |b_k(n)|)$$

$$= \sum_{n=1}^{\infty} (|af_1(n)| + ... + |af_k(n)|$$

Since,

$$\sum_{n=1}^{\infty} (|af_1(n)| + ... + |af_k(n)| \leq \sum_{n=1}^{\infty} |a_n|.$$

$$\Rightarrow \sum_{n=1}^{\infty} |s_k|$$

has bounded partial sum and hence $\sum s_k$ converges absolutely.
Next we have to show that sum of

$$\sum_{k=1}^{\infty} |a_k| =$$

sum of

$$\sum_{k=1}^{\infty} |s_k|.$$

For this, let $\epsilon > 0$ be given and choose an integer $N, n \geq N$
$\Rightarrow \{\sum_{k=1}^{\infty} |a_k| - \sum_{k=1}^{n} |a_k|\} \ < \frac{\epsilon}{2}...(1)$
Choose sufficient functions $f_1, f_2..., f_r$ with appearing each terms $a_1, a_2, a_3, ..., a_n$ somewhere in $\sum_{n=1}^{\infty} af_1(n) + ... + \sum_{n=1}^{\infty} af_k(n)$ r depended on N and ϵ. Let $n > r, n > N$, then

$$|s_1 + s_2 + ... + s_n - \sum_{k=1}^{\infty} a_k| \leq |a_(n+1)| + a_(n+2)| + ... < \frac{\epsilon}{2}...(2)$$

From (1) we have

$$|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k| < \frac{\epsilon}{2}...(3)$$

Now combining (2) & (3) we have

$$|s_1 + s_2 + ... + s_n - \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k| \leq |s_1 + s_2 + ... + s_n$$

$$- \sum_{k=1}^{\infty} a_k| + |\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In other words we may expressed as,

$$|s_1 + s_2 + ... + s_n - \sum_{k=1}^{\infty} a_k| < |a_(N+1)| + |a_(N+2)| + ... + |\sum_{k=n+1}^{\infty} a_k| < \frac{\epsilon}{2}$$

$$+ \sum_{k=n+1}^{\infty} |a_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence this completes the proof.

Double Sequences

A function of whose domain $\{Z^+, Z^+\}$ is called a double sequences. $F : \{Z^+, Z^+\} \to Z^+$ Converges of a double sequences If $a \in C$, we write (complex no)

$$\lim_{(p,q)\to\infty} f(p, q) = a$$

and we say that the Double sequences$f$ converges to a, provided that the following condition is satisfied: for every $\epsilon > a$, $f(N) : |f(p, q)a| < \epsilon$, $p > N, q > N$ [7].

Theorem 10. *Assume that*

$$\lim_{(p,q)\to\infty} = a$$

for each fixed p and that the $lim f(p, q)$ *exists,then that*

$$\lim_{p\to\infty} (\lim_{q\to\infty} f(p, q))$$

.

Also exists and has the value a.

Proof. Let $f(p) = \lim_{q\to\infty} f(p, q)$. Given $\epsilon > 0$, choose integer N_1 So that $|f(p, q) - a| < \frac{\epsilon}{2}, p > N_1, q > N_1$.... (1) Now, for each p, choose N_2 so that $|f(p) - f(p, q)| < \frac{\epsilon}{2}, q > N_2$...(2) N_2 depends on $p\&\epsilon$. For each $p > N_1$ choose N_2, and then choose a fixed $q > N_1, q > N_2$. Then (1) and (2) both are held and we have $|f(p) - a| < \epsilon, p > N_1$.

Therefore,$\lim_{p\to\infty} f(p) = a$
Thus the existences of the double limit $\lim_{p\to\infty} f(p,q)$ & $\lim_{q\to\infty} f(p,q)$ Implies the existences of the iterated limit.
i.e.

$$\lim_{p\to\infty}(\lim_{q\to\infty}) = a.$$

Double Limit

Limit $f(p,q)$ is called double,

$$\lim_{(p,q)\to\infty} limit.$$

Iterated Limit

$$\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$$

is called the iterated limit. $f(p,q) = \frac{pq}{p^2+q^2}$, $p = 1, 2, ...$ and $q = 1, 2, ...$,then,

$$\lim_{q\to\infty} = \lim_{q\to\infty}\left(\frac{\frac{p}{q}}{1+\frac{p^2}{q^2}}\right) = 0$$

and hence

$$\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0,$$

(i.e. iterated limit) But $f(p,q = \frac{1}{2})$,if $p = q \& f(p,q) = \frac{2}{3}$, if $p = 2q$
Thus it is clear that the double limit cannot exists in this care.

Double Series

Let f be a double sequences and let s be the double sequences defined by $S(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n)$.
The order pair (f,s) is said to be a double series and is denoted by the symbol $\sum_{m,n} f(m,n)$ or more briefly, by $\sum f(m,n)$. The double series is said to converges to the sum a if $\lim_{(p,q)\to\infty} S(p,q) = a$. The each term $f(m,n)$ is said to be a term of the double series and each $S(p,q)$ is a partial sum. If $\sum f(m,n)$ has only +ve terms, it is easy to show that it converges if, and only if, the set of partial sums is bounded. We say that $\sum f(m,n)$ converges absolutely if $\sum f(m,n)$ converges.

Iterated Series

The series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$ & the series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$ are said to be the iterated series converges of both iterated series does not imply their equality:-

Example. Let a function

$$F(m,n) = \begin{cases} 1 & if \quad m = n+1, n = 1,2,3,... \\ -1 & if \quad m = n-1, n = 1,2,3,... \\ 0 & \quad otherwise \end{cases}$$

Then,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = -1,$$

but [3]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = 1.$$

Theorem 11. *Let f be a complex valued double sequences, Assume that*

$$\sum_{n=1}^{\infty} f(m,n)$$

converges absolutely for each fixed m and that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$$

converges. Then,
(i) The double series

$$\sum_{m,n} f(m,n)$$

converges absolutely.
(ii)The series

$$\sum_{m=1}^{\infty} f(m,n)$$

converges absolutely for each n. Both iterated series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$$

&

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$$

converges absolutely & we have,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n).$$

Proof. Let a be an arrangement of the double sequence f into a sequences G defined by $G(n) = f[g(n)]$ if$n\epsilon Z^+$.
Then $\sum G(n)$ is absolutely convergent because of all the partial sums of the series. The series $\sum |G(n)|$ is bounded by the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(m,n)|.$$

Since $\sum G(n)$ converges absolutely then the double series $\sum_{m,n} f(m,n)$ converges absolutely.
Next we have to prove that

$$\sum_{m=1}^{\infty} f(m,n)$$

converges absolutely for each n.
Since,

$$\sum_{m,n} f(m,n)$$

converges absolutely and

$$\sum_{n=1}^{\infty} G(n) = S$$

Thus,

$$\sum_{m=1}^{\infty} f(m,n)$$

also converges absolutely.
Finally, the series

$$\sum_{n=1}^{\infty} f(m,n)$$

converges absolutely as the same (before) manner. Thus clearly the iterated series,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n), \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$$

converges and we have,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n).$$

Theorem 12. *Let $\sum a_m$ and $\sum b_n$ be two absolutely converges series with sums A and B respectively. Define double sequences as, $F(m,n) = a_m b_n$, if $(m,n) \in Z^+.Z^+$. Then $\sum_{m,n} f(m,n)$ converges absolutely and has the sum AB.*

Proof. Since $\sum_{m=1}^{\infty} |a_m| \sum_{n=1}^{\infty} |b_n| = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m||b_n|$. Then by theorem 11, $\sum_{m,n}(a_m b_n)$ converges absolutely and has sum AB.

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