

On continued fraction representations of certain function of hypergeometric type

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Abstract: In the present work we have established a number of continued fraction representations for the function $F(a, b, c, d, t)$. Our results produce certain new and many known results in the literature as their special cases.

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1. Introduction, Notations and Definitions

Let, for $|q| < 1$

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}) \quad \& \quad (a; q)_0 = 1$$

Then ${}_3\Phi_2[a; b, c, d; t]$ series has been defined as,

$${}_3\Phi_2[a; b, c, d; t] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (d)_n} t^n$$

The function $F(a, b, c, d, t)$ has been extensively studied. It is easy to see that $F(a, b, c, d, t)$ satisfies the following relations.

$$F(a, b, c, d; t) = 1 + \frac{(1-a)(1-b)}{(1-c)(1-d)} t F(aq, bq, cq, dq; t) \quad (1.1)$$

$$\begin{aligned} q^2(1-t)F(a, b, c, d; t) + [(a+b)tq^2 - (c+d)q]F(a, b, c, d; tq) \\ + (cd - abtq^2)F(a, b, c, d; tq^2) = cd - (c+d-q)q \end{aligned} \quad (1.2)$$

$$\begin{aligned} (1-c)(1-d)(1-t)F(a, b, c, d; t) \\ = (1-c)(1-d) + (c+d-a-b)tF(a, b, cq, dq; tq) \\ + (ab - cd)tF(a, b, cq, dq, tq^2) \end{aligned} \quad (1.3)$$

In the present paper we have established many continued fraction representation for $F(a, b, c, d, t)$ by making use of relations (1.1), (1.2) and (1.3).

2. Main Results

In this section, we establish the following results:

$$\frac{F(a, b, c, d; t)}{F(aq, bq, cq, dq; t)} = A_0 + \frac{B_0}{A_1 \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \quad (2.1)$$

where

$$\begin{aligned} A_i &= \frac{(1 - aq^i)(1 - bq^i)}{(1 - cq^i)(1 - dq^i)}t + 1 \\ B_i &= -\frac{(1 - aq^{i+1})(1 - bq^{i+1})}{(1 - cq^{i+1})(1 - dq^{i+1})}t \\ \frac{F(a, b, c, d; t)}{F(a, b, c, d; tq)} &= A_0 + \frac{B_0}{A_1 \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} A_i &= S(tq^i) = \left[\frac{cd - (c + d - q)q}{q^2(1 - tq^i)F(a, b, c, d; tq^{i+1})} - \frac{[(a + b)tq^{i+2} - (c + d)q]}{q^2(1 - tq^i)} \right] \\ B_i &= \frac{(abtq^{i+2} - cd)}{q^2(1 - tq^i)} \\ \frac{F(a, b, cq, dq; t)}{F(a, b, cq, dq; tq)} &= A_0 + \frac{B_0}{A_1 \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} A_i &= S(tq^i) = \frac{(1 - c)(1 - d)[(atq^i + btq^i - c - d) + (tq^i - 1)(c + d)]}{tq^i(1 - tq^i)(a + b - c - d)F(a, b, cq, dq, tq^{i+1})} \\ &\quad + \frac{(c + d - atq^i - btq^i)}{(1 - tq^i)} \\ B_i &= \frac{(ab - cd)(atq^i + btq^i - c - d) - (1 - tq^i)[ac(d - b) + bd(c - a)]}{(1 - tq^i)(a + b - c - d)} \end{aligned}$$

3. Proof of (2.1)

We have

$$F(a, b, c, d; t) = 1 + \frac{(1-a)(1-b)}{(1-c)(1-d)} t F(aq, bq, cq, dq; t) \quad (3.1)$$

Replace a by aq, b by bq, c by cq and d by dq in (3.1) we have

$$F(aq, bq, cq, dq; t) = 1 + \frac{(1-aq)(1-bq)}{(1-cq)(1-dq)} t F(aq^2, bq^2, cq^2, dq^2; t) \quad (3.2)$$

Subtracting (3.2) from (3.1)

$$\begin{aligned} F(a, b, c, d; t) &= \left[\frac{(1-a)(1-b)}{(1-c)(1-d)} t + 1 \right] F(aq, bq, cq, dq; t) \\ &\quad - \frac{(1-aq)(1-bq)}{(1-cq)(1-dq)} t F(aq^2, bq^2, cq^2, dq^2; t) \\ &= \frac{F(a, b, c, d; t)}{F(aq, bq, cq, dq; t)} = \\ &= \left[\frac{(1-a)(1-b)}{(1-c)(1-d)} t + 1 \right] - \frac{(1-aq)(1-bq)}{(1-cq)(1-dq)} \frac{t}{\frac{F(aq, bq, cq, dq; t)}{F(aq^2, bq^2, cq^2, dq^2; t)}} \end{aligned} \quad (3.3)$$

Again replacing a by aq, b by bq, c by cq and d by dq in (3.3) we have

$$\begin{aligned} &\frac{F(aq, bq, cq, dq; t)}{F(aq^2, bq^2, cq^2, dq^2; t)} = \\ &= \left[\frac{(1-aq)(1-bq)}{(1-cq)(1-dq)} t + 1 \right] - \frac{(1-aq^2)(1-bq^2)}{(1-cq^2)(1-dq^2)} \frac{t}{\frac{F(aq^2, bq^2, cq^2, dq^2; t)}{F(aq^3, bq^3, cq^3, dq^3; t)}} \end{aligned}$$

Similarly a repeated iterations yields, the following Continued fraction,

$$\frac{F(a, b, c, d; t)}{F(aq, bq, cq, dq; t)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$A_i = \frac{(1-aq^i)(1-bq^i)}{(1-cq^i)(1-dq^i)} t + 1$$

$$B_i = -\frac{(1 - aq^{i+1})(1 - bq^{i+1})}{(1 - cq^{i+1})(1 - dq^{i+1})}t$$

Proof of (2.2)

We have

$$\begin{aligned} & q^2(1-t)F(a, b, c, d; t) + [(a+b) tq^2 - (c+d)q]F(a, b, c, d; tq) \\ & + (cd - abtq^2)F(a, b, c, d; tq^2) = cd - (c+d-q)q \\ & \frac{F(a, b, c, d; t)}{F(a, b, c, d; tq)} = \\ & = \left[\frac{cd - (c+d-q)q}{q^2(1-t)F(a, b, c, d; tq)} - \frac{[(a+b) tq^2 - (c+d)q]}{q^2(1-t)} \right] \\ & \quad - \frac{(cd - abtq^2)}{q^2(1-t)} \frac{1}{\frac{F(a, b, c, d; tq)}{F(a, b, c, d; tq^2)}} \\ & \frac{F(a, b, c, d; t)}{F(a, b, c, d; tq)} = S(t) + \frac{(abtq^2 - cd)}{q^2(1-t)} \frac{1}{\frac{F(a, b, c, d; tq)}{F(a, b, c, d; tq^2)}} \end{aligned} \quad (3.4)$$

where

$$S(t) = \left[\frac{cd - (c+d-q)q}{q^2(1-t)F(a, b, c, d; tq)} - \frac{[(a+b) tq^2 - (c+d)q]}{q^2(1-t)} \right]$$

Replacing t by tq in equation (3.4)

$$\frac{F(a, b, c, d; tq)}{F(a, b, c, d; tq^2)} = S(tq) + \frac{(abtq^3 - cd)}{q^2(1-tq)} \frac{1}{\frac{F(a, b, c, d; tq^2)}{F(a, b, c, d; tq^3)}}$$

Similarly, a repeated iterations yields, the following Continued Fraction,

$$\frac{F(a, b, c, d; t)}{F(a, b, c, d; tq)} = A_0 + \frac{B_0}{A_1 \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$A_i = S(tq^i) = \left[\frac{cd - (c+d-q)q}{q^2(1-tq^i)F(a, b, c, d; tq^{i+1})} - \frac{[(a+b) tq^{i+2} - (c+d)q]}{q^2(1-tq^i)} \right]$$

$$B_i = \frac{(abtq^{i+2} - cd)}{q^2(1 - tq^i)}$$

Proof of (2.3)

We have

$$\begin{aligned} & (1-c)(1-d)(1-t)F(a, b, c, d; t) \\ &= (1-c)(1-d) + (c+d-a-b)tF(a, b, cq, dq; tq) \\ & \quad + (ab-cd)tF(a, b, cq, dq; tq^2) \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} & (1-c)(1-d)(at+bt-c-d)F(a, b, c, d; t) \\ &= (c+d)(c-1)(1-d) + (a+b-c-d)tF(a, b, cq, dq, t) \\ & \quad + [ac(d-b) + bd(c-a)]tF(a, b, cq, dq, tq^2) \end{aligned} \quad (3.6)$$

Multiplying equation (3.5) by $(at+bt-c-d)$ and equation (3.6) by $(1-t)$ and then subtracting (3.6) from (3.5)

$$\begin{aligned} & (at+bt-c-d)(1-c)(1-d) + (at+bt-c-d)(c+d-a-b)tF(a, b, cq, dq, tq) \\ & \quad + (ab-cd)t(at+bt-c-d)F(a, b, cq, dq, tq^2) \\ & \quad - (1-t)t[ac(d-b) + bd(c-a)]F(a, b, cq, dq, tq^2) \\ & - (c+d)(c-1)(1-d)(1-t) = (1-t)t(a+b-c-d)F(a, b, cq, dq; t) \\ & \quad \frac{F(a, b, cq, dq, t)}{F(a, b, cq, dq, tq)} = \\ &= \frac{(1-c)(1-d)[(at+bt-c-d) + (t-1)(c+d)]}{t(1-t)(a+b-c-d)F(a, b, cq, dq, tq)} + \frac{(c+d-at-bt)}{(1-t)} \\ & \quad + \frac{(ab-cd)(at+bt-c-d) - (1-t)[ac(d-b) + bd(c-a)]}{(1-t)(a+b-c-d)\frac{F(a, b, cq, dq, tq)}{(a, b, cq, dq, tq^2)}} \\ & \quad \frac{F(a, b, cq, dq, t)}{F(a, b, cq, dq, tq)} \\ &= S(t) + \frac{(ab-cd)(at+bt-c-d) - (1-t)[ac(d-b) + bd(c-a)]}{(1-t)(a+b-c-d)\frac{F(a, b, cq, dq, tq)}{(a, b, cq, dq, tq^2)}} \end{aligned}$$

where

$$S(t) = \frac{(1-c)(1-d)[(at+bt-c-d) + (t-1)(c+d)]}{t(1-t)(a+b-c-d)F(a, b, cq, dq, tq)} + \frac{(c+d-at-bt)}{(1-t)}$$

Now replacing t by tq in above equation,

$$\begin{aligned} & \frac{F(a, b, cq, dq, tq)}{F(a, b, cq, dq, tq^2)} \\ &= S(tq) + \frac{(ab-cd)(atq+btq-c-d) - (1-tq)[ac(d-b) + bd(c-a)]}{(1-tq)(a+b-c-d) \frac{F(a, b, cq, dq, tq^2)}{(a, b, cq, dq, tq^3)}} \end{aligned}$$

Similarly, a repeated iterations yields

$$\frac{F(a, b, cq, dq; t)}{F(a, b, cq, dq; tq)} = A_0 + \frac{B_0}{A_1 \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$\begin{aligned} A_i = S(tq^i) &= \frac{(1-c)(1-d)[(atq^i+btq^i-c-d) + (tq^i-1)(c+d)]}{tq^i(1-tq^i)(a+b-c-d)F(a, b, cq, dq, tq^{i+1})} \\ &+ \frac{(c+d-atq^i-btq^i)}{(1-tq^i)} \\ B_i &= \frac{(ab-cd)(atq^i+btq^i-c-d) - (1-tq^i)[ac(d-b) + bd(c-a)]}{(1-tq^i)(a+b-c-d)} \end{aligned}$$

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