

On certain transformation formulae for q-series

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Abstract: In this paper, making use of Ramanujan's ${}_1\Psi_1$ summation formulae and some known transformation formulae for ${}_2\Psi_1$ series, we have established certain interesting transformation formulae for q-series.

Key words and phrases: q-series, basic hypergeometric series, basic bilateral hypergeometric series, transformation formula, summation formula, continued fraction and q-series identity.

2000 A.M.S. subject classification: 33A30, 33D15, 33D20.

1. Introduction, Notations and Definitions

Throughout this paper, we shall adopt following notations and definitions.

For any number α and q real or complex and $|q| < 1$,

Let

$$[\alpha; q]_n = (1 - \alpha)(1 - \alpha q) \dots (1 - \alpha q^{n-1}), \quad n > 0$$

$$[\alpha; q]_0 = 1,$$

and

$$[\alpha; q]_\infty = \prod_{r=0}^{\infty} (1 - \alpha q^r).$$

Following Gasper and Rahman [2], we define a basic hypergeometric series as,

$$\begin{aligned} & {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots, b_s; q]_n} [(-)^n q^{n(n-1)/2}]^{1+s-r}, \end{aligned} \quad (1.1)$$

where

$$[a_1, a_2, \dots, a_r; q]_n = [a_1; q]_n [a_2; q]_n \dots [a_r; q]_n$$

and $r \leq 1 + s$ and $\max. (|q|, |z|) < 1$ for convergence of the series (1.1)

The general bilateral basic hypergeometric series is defined by,

$${}_r\Psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] \\ = \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[b_1, b_2, \dots, b_s; q]_n} [(-)^n q^{n(n-1)/2}]^{s-r}, \quad (1.2)$$

where $r < s$ and $|q| < 1$. The series is convergent in the annulus $\left| \frac{b_1, b_2, \dots, b_s}{a_1, a_2, \dots, a_r} \right| < |z| < 1$.

We shall make use of following results in our analysis.

$${}_1\Psi_1 \left[\begin{matrix} a; q; z \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{[a; q]_n}{[b; q]_n} z^n = \frac{[az, q/az, b/a, q; q]_{\infty}}{[b, z, q/a, b/az; q]_{\infty}}. \quad (1.3)$$

[Gasper and Rahman 2; App. II (II.29)]

$${}_2\Phi_1 \left[\begin{matrix} a, q; q; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a; q]_n z^n}{[b; q]_n} \\ = \sum_{n=0}^{\infty} \frac{[a; q]_n [azq/b; q]_n (1 - azq^{2n}) b^n z^n q^{n(n-1)}}{[b; q]_n [z; q]_{n+1}}. \quad (1.4)$$

[Andrews and Berndt 1; (9.1.1) p. 223]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \end{matrix} \right] = \frac{[b, az; q]_{\infty}}{[c, z; q]_{\infty}} {}_2\Phi_1 \left[\begin{matrix} c/b, z; q; b \end{matrix} \right]. \quad (1.5)$$

[Srivastava and Karlsson 3; (275) p. 348]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \end{matrix} \right] = \frac{[abz/c; q]_{\infty}}{[z; q]_{\infty}} {}_2\Phi_1 \left[\begin{matrix} c/a, c/b; q; abz/c \end{matrix} \right]. \quad (1.6)$$

[Srivastava and Karlsson 3; (281) p. 348]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \end{matrix} \right] = \frac{[c/b, bz; q]_{\infty}}{[c, z; q]_{\infty}} {}_2\Phi_1 \left[\begin{matrix} abz/c, b; q; c/b \end{matrix} \right]. \quad (1.7)$$

[Srivastava and Karlsson 3; (276) p. 348]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] = \frac{[bz; q]_\infty}{[z; q]_\infty} {}_2\Phi_2 \left[\begin{matrix} b, c/b; q; az \\ c, bz \end{matrix} \right]. \quad (1.8)$$

[Srivastava and Karlsson 3; (280) p. 348]

2. Main Results

In this section we shall establish our main transformation formulae. It is easy to see that

$$\sum_{n=-\infty}^{\infty} \frac{[a; q]_n z^n}{[b; q]_n} = \sum_{n=0}^{\infty} \frac{[a; q]_n z^n}{[b; q]_n} + \frac{(b-q)}{z(a-q)} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n}{[q^2/a; q]_n} \left(\frac{b}{az} \right)^n, \quad (2.1)$$

where $|b/a| < |z| < 1$.

Now, using (1.3) on the right hand side of (2.1) we have

$$\begin{aligned} {}_2\Phi_1 \left[\begin{matrix} a, q; q; z \\ b \end{matrix} \right] + \frac{(b-q)}{z(a-q)} {}_2\Phi_1 \left[\begin{matrix} q^2/b, q; q; b/az \\ q^2/a \end{matrix} \right] \\ = \frac{[az, q/az, b/a, q; q]_\infty}{[b, z, q/a, b/az; q]_\infty}. \end{aligned} \quad (2.2)$$

Again, applying (1.4) on the second part on the left hand side of (2.2) we get the transformation,

$$\begin{aligned} {}_2\Phi_1 \left[\begin{matrix} a, q; q; z \\ b \end{matrix} \right] &= \frac{[az, q/az, b/a, q; q]_\infty}{[b, z, q/a, b/az; q]_\infty} \\ &- \frac{(b-q)}{z(a-q)} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n [q/z; q]_n \left(1 - \frac{q^{2n+2}}{az} \right) \left(\frac{b}{a^2 z} \right)^n q^{n(n+1)}}{[q^2/a; q]_n [b/az; q]_{n+1}} \end{aligned} \quad (2.3)$$

Applying (1.4) on the left hand side of (2.3) we get another transformation, viz.;

$$\begin{aligned} {}_5\Phi_6 \left[\begin{matrix} a, azq/b, q\sqrt{az}, -q\sqrt{az}, q; q; bz \\ b, zq, \sqrt{az}, -\sqrt{az}, 0, 0 \end{matrix} \right] &= \frac{[azq, q/az, b/a, q; q]_\infty}{[b, zq, q/a, b/az; q]_\infty} \\ &- \frac{(1-z)(b-q)(1-q^2/az)}{z(a-q)(1-b/az)(1-az)} {}_5\Phi_6 \left[\begin{matrix} q/z, q^2/b, q^2/\sqrt{az}, -q^2/az, q; q; bq^2/a^2 z \\ q^2/a, bq/az, q/\sqrt{az}, -q/\sqrt{az}, 0, 0 \end{matrix} \right] \end{aligned} \quad (2.4)$$

Applying (1.6) on the left hand side of (2.2) we get,

$${}_2\Phi_1 \left[\begin{matrix} b/a, b/q; q; azq/b \\ b \end{matrix} \right] = \frac{[az, q/az, b/a, q; q]_\infty}{[b, azq/b, q/a, b/az; q]_\infty}$$

$$-\frac{(b-q)}{z(a-q)} \frac{[q/z; q]_{\infty} [z; q]_{\infty}}{[b/az; q]_{\infty} [azq/b; q]_{\infty}} {}_2\Phi_1 \left[\begin{matrix} b/a, q/a; q; q/z \\ q^2/a \end{matrix} \right]. \quad (2.5)$$

Applying (1.6) on the left hand side of (2.3) we get

$$\begin{aligned} \frac{[azq/b; q]_{\infty}}{[z; q]_{\infty}} {}_2\Phi_1 \left[\begin{matrix} b/a, b/q; q; azq/b \\ b \end{matrix} \right] &= \frac{[az, q/az, b/a, q; q]_{\infty}}{[b, z, q/a, b/az; q]_{\infty}} \\ &-\frac{(b-q)}{z(a-q)} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n [q/z; q]_n \left(1 - \frac{q^{2n+2}}{az}\right) \left(\frac{bq^2}{a^2z}\right)^n q^{n(n-1)}}{[q^2/a; q]_n [b/az; q]_{n+1}}, \end{aligned} \quad (2.6)$$

which can be written as

$$\begin{aligned} {}_2\Phi_1 \left[\begin{matrix} b/a, b/q; q; azq/b \\ b \end{matrix} \right] &= \frac{[az, q/az, b/a, q; q]_{\infty}}{[b, azq/b, q/a, b/az; q]_{\infty}} \\ &-\frac{(b-q)[z; q]_{\infty}}{z(a-q)[azq/b; q]_{\infty}} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n [q/z; q]_n \left(1 - \frac{q^{2n+2}}{az}\right) \left(\frac{bq^2}{a^2z}\right)^n q^{n(n-1)}}{[q^2/a; q]_n [b/az; q]_{n+1}}. \end{aligned} \quad (2.7)$$

Applying (1.8) on the left hand side of (2.3) we get,

$$\begin{aligned} {}_2\Phi_2 \left[\begin{matrix} q, b/q; q; az \\ b, zq \end{matrix} \right] &= \frac{[az, q/az, b/a, q; q]_{\infty}}{[b, zq, q/a, b/az; q]_{\infty}} \\ &-\frac{(b-q)(1-z)}{z(a-q)} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n [q/z; q]_n \left(1 - \frac{q^{2n+2}}{az}\right) \left(\frac{bq^2}{a^2z}\right)^n q^{n(n-1)}}{[q^2/a; q]_n [b/az; q]_{n+1}}. \end{aligned} \quad (2.8)$$

Applying (1.5) on the left hand side of (2.3) we get,

$$\begin{aligned} {}_2\Phi_1 \left[\begin{matrix} b/q, z; q; q \\ az \end{matrix} \right] &= \frac{[q/az, b/a; q]_{\infty}}{[q/a, b/az; q]_{\infty}} \\ &-\frac{(b-q)[b; q]_{\infty} [z; q]_{\infty}}{z(a-q)[q; q]_{\infty} [az; q]_{\infty}} \sum_{n=0}^{\infty} \frac{[q^2/b; q]_n [q/z; q]_n \left(1 - \frac{q^{2n+2}}{az}\right) \left(\frac{bq^2}{a^2z}\right)^n q^{n(n-1)}}{[q^2/a; q]_n [b/az; q]_{n+1}}. \end{aligned} \quad (2.9)$$

Applying (1.7) on the left hand side of (2.3) we get,

$${}_2\Phi_1 \left[\begin{matrix} azq/b, q; q; b/q \\ zq \end{matrix} \right] = \frac{[az, q/az, b/a, q; q]_{\infty}}{[b/q, zq, q/a, b/az; q]_{\infty}}$$

$$+\frac{q(1-z)}{z(a-q)}\sum_{n=0}^{\infty}\frac{[q^2/b;q]_n[q/z;q]_n\left(1-\frac{q^{2n+2}}{az}\right)\left(\frac{bq^2}{a^2z}\right)^nq^{n(n-1)}}{[q^2/a;q]_n[b/az;q]_{n+1}}. \quad (2.10)$$

3. Special Cases

In this section we shall discuss certain special cases of the results establish in previous section.

As $z \rightarrow 1$, (2.4) yields the summation formula,

$${}_4\Phi_5 \left[\begin{matrix} a, aq/b, q\sqrt{a}, -q\sqrt{a}; q; b \end{matrix} \right] = \frac{[aq;q]_{\infty}}{[b;q]_{\infty}}. \quad (3.1)$$

Also taking $z=q$ in (2.4) we get,

$${}_5\Phi_6 \left[\begin{matrix} a, aq^2/b, q\sqrt{aq}, -q\sqrt{aq}, q; q; bq \end{matrix} \right] = \frac{(1-q)(b-q)}{(b-aq)(1-aq)} + \frac{(1-q)(1-a)q[aq^2;q]_{\infty}}{(b-aq)[b;q]_{\infty}}. \quad (3.2)$$

As $z \rightarrow q$, yields the summation formula,

$${}_2\Phi_1 \left[\begin{matrix} a, q; q; q \end{matrix} \right] = \frac{q[a;q]_{\infty}}{(b-aq)[b;q]_{\infty}} - \frac{(b-q)}{q(a-q)}. \quad (3.3)$$

Taking $z=q$ in (2.5) we get the summation formula,

$${}_2\Phi_1 \left[\begin{matrix} b/a, b/q; q; aq^2/b \end{matrix} \right] = \frac{[q, aq; q]_{\infty}q(1-a)}{(b-aq)[b, aq^2/b; q]_{\infty}}. \quad (3.4)$$

For $z=1$, (2.8) yields

$${}_1\Phi_1 \left[\begin{matrix} b/q; q; a \end{matrix} \right] = \frac{[a;q]_{\infty}}{[b;q]_{\infty}}. \quad (3.5)$$

For $z=q$, (2.8) yields

$${}_2\Phi_2 \left[\begin{matrix} q, b/q; q; aq \end{matrix} \right] = \frac{[a;q]_{\infty}q(1-q)}{(b-aq)[b;q]_{\infty}}. \quad (3.6)$$

For $z=q$, (2.9) yields

$${}_2\Phi_1 \left[\begin{matrix} b/q, q; q; q \end{matrix} \right] = \frac{q(1-a)}{(b-aq)} - \frac{(b-q)[b;q]_{\infty}}{q(a-q)[aq;q]_{\infty}}. \quad (3.7)$$

For $z=q$, (2.10) yields

$${}_2\Phi_1 \left[\begin{matrix} aq^2/b, q; q; b/q \end{matrix} \right] = \frac{q(1-q)[a;q]_{\infty}}{(b-aq)[b/q;q]_{\infty}} + \frac{q(1-q)}{q(a-q)}. \quad (3.8)$$

Acknowledgement

The first author is thankful to University Grants Commission, New Delhi for sanctioning major research project no. F.38-122/2009 (SR) dated 19-12-2009 under this work has been done and We are thankful to Dr. S.N. Singh, Department of Mathematics, T.D.P.G. College, Jaunpur, for his valuable guidance in the preparation of this paper.

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