South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.11, No.1 2015, pp. 47-54

On Certain New WP Bailey Pairs and Their Applications

G.S. Pant, K.B. Chand and V.P. Pande*
Department of Mathematics,
S.S.J. College, Almora (U.K.) India
*Prof & Head, Department of Mathematics,
S.S.J. College, Almora (U.K.) India

Abstract: In this paper, we have established new WP Bailey pairs by making use of summation formulae due to Verma and Jain. Further, this new WP Bailey pair has been used to obtain certain transformation formulae for basic (q-) hypergeometric series.

Keywords and phrases: WP Bailey pair, summation formula, transformation formula.
2000 A.M.S. subject classification: Primary 33D15, Secondary 11B65

1. Introduction, Notations and Definitions

We employ the usual notations

$$[a;q]_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n = 1,2,3,...,$$

$$[a;q]_0 = 1,$$

$$[a_1,a_2,...,a_r;q]_n = [a_1;q]_n[a_2;q]_n...[a_r;q]_n.$$

and for $|q|<1$,

$$[a;q]_\infty = \prod_{r=0}^{\infty}(1-aq^r),$$

$$[a_1,a_2,...,a_r;q]_\infty = [a_1;q]_\infty[a_2;q]_\infty...[a_r;q]_\infty.$$

The basic hypergeometric series is defined as,

$${}_r\Phi_s\left[\begin{array}{c} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n z^n}{(q,b_1,b_2,...,b_s;q)_n}\left\{(-)^n q^{n(n-1)/2}\right\}^{1+s-r}, \quad (1.1)$$

where for convergence $\max.(|z|,|q|)<1$, if $1+s=r$ and for $1+s>r$, $|z|<\infty$.
A WP Bailey pair is a pair of sequences $\langle\alpha_n(a,k,q),\beta_n(a,k,q)\rangle$ satisfying

$$\alpha_0(a,k,q) = \beta_0(a,k,q) = 1$$

and

$$\beta_n(a,k,q)=\sum_{r=0}^{n}\frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\alpha_r(a,k,q). \tag{1.2}$$

WP Bailey pair is the extended form of Bailey pair defined by Bailey [3,4]. Andrews [1,2] showed that if the pair $\langle\alpha_n(a,k,q),\beta_n(a,k,q)\rangle$ satisfy (1.2), then so does $\langle\alpha'_n(a,k,q),\beta'_n(a,k,q)\rangle$ where

$$\alpha'_n(a,k,q)=\frac{(y,z;q)_n}{(aq/y,aq/z;q)_n}\left(\frac{k}{c}\right)^n\alpha_n(a,c,q)$$

and

$$\beta'_n(a,k,q)=\frac{(ky/a,kz/a;q)_n}{(aq/y,aq/z;q)_n}\times$$

$$\times\sum_{r=0}^{n}\frac{(1-cq^{2r})(y,z;q)_r(k/c;q)_{n-r}(k;q)_{n+r}(k/c)^r\beta_r(a,c,q)}{(1-c)(ky/a,kz/a;q)_r(q;q)_{n-r}(qc;q)_{n+r}}, \tag{1.3}$$

where $c=kyz/aq$.
Also, if $\langle\alpha_n(a,k,q),\beta_n(a,k,q)\rangle$ satisfy (1.2) then subject to suitable convergence conditions,

$$\sum_{n=0}^{\infty}\left(\frac{a^2q}{k^2}\right)^n\beta_n(a,k,q)=\frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}\left(\frac{a^2q}{k^2}\right)^n\alpha_n(a,k,q). \tag{1.4}$$

[Laughlin, J.Mc., 5; (1.7)]

2. Main Result

In this section we shall establish new WP Bailey pairs. (1.2) can be written as

$$\beta_n(a,k,q)=\frac{(k/a,k;q)_n}{(q,aq;q)_n}\sum_{r=0}^{n}\frac{(q^{-n};q)_r(kq^n;q)_r\left(\frac{aq}{k}\right)^r}{(aq^{1-n}/k;q)_r(aq^{1+n};q)_r}\alpha_r(a,k,q). \tag{2.1}$$

In order to establish new WP Bailey pairs we shall make use of the following summation formulae,

(i)

$${}_4\Phi_3\left[\begin{matrix}a,c,\frac{a}{c}q^{1/2+m},q^{-m}\\ \frac{aq}{c},cq^{1/2-m},aq^{1+m}\end{matrix};q;q^2\right]$$

$$= \frac{(1-a)}{2\sqrt{a}(1-\sqrt{a})} \frac{(aq, \sqrt{q}; q)_m \left(\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{\left(\frac{aq}{c}, \frac{\sqrt{q}}{c}; q\right)_m (\sqrt{aq}; q^{1/2})_{2m}}$$

$$- \frac{1-a}{2\sqrt{a}(1+\sqrt{a})} \frac{(aq, \sqrt{q}; q)_m \left(-\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{\left(\frac{aq}{c}, \frac{\sqrt{q}}{c}; q\right)_m (-\sqrt{aq}; q^{1/2})_{2m}}. \tag{2.2}$$

[Verma, A and Jain, V.K. 6; (4.4) p. 77]

Putting $c = \dfrac{aq^{1/2}}{k}$ in (2.2) we get,

$${}_4\Phi_3 \left[\begin{matrix} a, \frac{aq^{1/2}}{k}, kq^m, q^{-m}; q; q^2 \\ kq^{1/2}, \frac{aq^{1-m}}{k}, aq^{1+m} \end{matrix} \right]$$

$$= \frac{(1-a)}{2\sqrt{a}(1-\sqrt{a})} \frac{(aq, \sqrt{q}; q)_m \left(\frac{k}{\sqrt{a}}; q^{1/2}\right)_{2m}}{\left(kq^{1/2}, \frac{k}{a}; q\right)_m (\sqrt{aq}; q^{1/2})_{2m}}$$

$$- \frac{1-a}{2\sqrt{a}(1+\sqrt{a})} \frac{(aq, \sqrt{q}; q)_m \left(-\frac{k}{\sqrt{a}}; q^{1/2}\right)_{2m}}{\left(kq^{1/2}, \frac{k}{a}; q\right)_m (-\sqrt{aq}; q^{1/2})_{2m}}. \tag{2.3}$$

Now, choosing $\alpha_r(a,k,q) = \dfrac{\left(\frac{a}{k}q^{1/2}, a; q\right)_r \left(\frac{kq}{a}\right)^r}{(q, kq^{1/2}; q)_r}$ in (2.1) and using (2.3) we have

$$\beta_n(a,k,q) = \frac{(k, k/a; q)_n}{(q, aq; q)_n} \, {}_4\Phi_3 \left[\begin{matrix} a, \frac{a}{k}q^{1/2}, kq^n, q^{-n}; q; q^2 \\ kq^{1/2}, \frac{a}{k}q^{1-n}, aq^{1+n} \end{matrix} \right]$$

$$= \frac{(1+\sqrt{a})}{2\sqrt{a}} \frac{(k, \sqrt{q}; q)_n \left(\frac{k}{\sqrt{a}}; q^{1/2}\right)_{2n}}{(q, kq^{1/2}; q)_n (\sqrt{aq}; q^{1/2})_{2n}}$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\frac{(k,\sqrt{q};q)_n\left(-\frac{k}{\sqrt{a}};q^{1/2}\right)_{2n}}{(q,kq^{1/2};q)_n(-\sqrt{aq};q^{1/2})_{2n}}. \tag{2.4}$$

Thus $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (2.1) form a WP Bailey pair.

(ii)

$$ {}_4\Phi_3\left[\begin{matrix} a, c, \frac{a}{c}q^{1/2+m}, q^{-m}; q; q \\ \frac{aq}{c}, cq^{1/2-m}, aq^{1+m} \end{matrix}\right]$$

$$=\frac{(1+\sqrt{a})}{2}\frac{(aq,\sqrt{q};q)_m\left(\frac{\sqrt{aq}}{c};q^{1/2}\right)_{2m}}{\left(\frac{aq}{c},\frac{\sqrt{q}}{c};q\right)_m(\sqrt{aq};q^{1/2})_{2m}}$$

$$+\frac{(1-\sqrt{a})}{2}\frac{(aq,\sqrt{q};q)_m\left(-\frac{\sqrt{aq}}{c};q^{1/2}\right)_{2m}}{\left(\frac{aq}{c},\frac{\sqrt{q}}{c};q\right)_m(-\sqrt{aq};q^{1/2})_{2m}}. \tag{2.5}$$

[Verma, A and Jain, V.K. 6; (1.3) p. 71]

Putting $c=\frac{aq^{1/2}}{k}$ in (2.5) we get,

$$ {}_4\Phi_3\left[\begin{matrix} a, \frac{aq^{1/2}}{k}, kq^m, q^{-m}; q; q \\ kq^{1/2}, \frac{aq^{1-m}}{k}, aq^{1+m} \end{matrix}\right]$$

$$=\frac{(1+\sqrt{a})}{2}\frac{(aq,\sqrt{q};q)_m\left(\frac{k}{\sqrt{a}};q^{1/2}\right)_{2m}}{\left(kq^{1/2},\frac{k}{a};q\right)_m(\sqrt{aq};q^{1/2})_{2m}}$$

$$+\frac{(1-\sqrt{a})}{2}\frac{(aq,\sqrt{q};q)_m\left(-\frac{k}{\sqrt{a}};q^{1/2}\right)_{2m}}{\left(kq^{1/2},\frac{k}{a};q\right)_m(-\sqrt{aq};q^{1/2})_{2m}}. \tag{2.6}$$

Now, choosing $\alpha_r(a,k,q) = \dfrac{\left(a, \dfrac{a}{k}q^{1/2}; q\right)_r \left(\dfrac{k}{a}\right)^r}{(q, kq^{1/2}; q)_r}$ in (2.1) and using (2.6) we get

$$\beta_n(a,k,q) = \frac{(1+\sqrt{a})}{2} \frac{(k, \sqrt{q}; q)_n \left(\dfrac{k}{\sqrt{a}}; q^{1/2}\right)_{2n}}{(q, kq^{1/2}; q)_n (\sqrt{aq}; q^{1/2})_n}$$

$$+ \frac{(1-\sqrt{a})}{2} \frac{(k, \sqrt{q}; q)_n \left(-\dfrac{k}{\sqrt{a}}; q^{1/2}\right)_{2n}}{(q, kq^{1/2}; q)_n (-\sqrt{aq}; q^{1/2})_{2n}}. \tag{2.7}$$

Thus $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (2.7) form another WP Bailey pair.

3. Applications

In this section we shall establish certain transformation formulae by making use of WP Bailey pairs of (2.4), (2.7) and (1.4).

(i) Putting the WP pair of (2.4) in (1.4) we get,

$$\frac{1+\sqrt{a}}{2\sqrt{a}} \, {}_4\Phi_3 \left[\begin{matrix} k, \sqrt{q}, \dfrac{k}{\sqrt{a}}, k\sqrt{\dfrac{q}{a}}; q; \dfrac{a^2q}{k^2} \\ kq^{1/2}, \sqrt{aq}, q\sqrt{a} \end{matrix} \right]$$

$$- \frac{1-\sqrt{a}}{2\sqrt{a}} \, {}_4\Phi_3 \left[\begin{matrix} k, \sqrt{q}, -\dfrac{k}{\sqrt{a}}, -k\sqrt{\dfrac{q}{a}}; q; \dfrac{a^2q}{k^2} \\ kq^{1/2}, -\sqrt{aq}, -q\sqrt{a} \end{matrix} \right]$$

$$= \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \, {}_6\Phi_5 \left[\begin{matrix} a, \dfrac{a}{k}q^{1/2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \dfrac{aq^2}{k} \\ kq^{1/2}, a\sqrt{\dfrac{q}{a}}, -a\sqrt{\dfrac{q}{k}}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}} \end{matrix} \right], \tag{3.1}$$

provided $\left|\dfrac{a}{k}\right| < 1$.

(ii) Putting the WP Bailey pair of (2.7) in (1.4) we have,

$$\frac{1+\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} k, \sqrt{q}, \dfrac{k}{\sqrt{a}}, k\sqrt{\dfrac{q}{a}}; q; \dfrac{a^2q}{k^2} \\ kq^{1/2}, \sqrt{aq}, q\sqrt{a} \end{matrix} \right]$$

$$+ \frac{1-\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} k, \sqrt{q}, -\dfrac{k}{\sqrt{a}}, -k\sqrt{\dfrac{q}{a}}; q; \dfrac{a^2q}{k^2} \\ kq^{1/2}, -\sqrt{aq}, -q\sqrt{a} \end{matrix} \right]$$

$$= \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \, {}_6\Phi_5 \left[\begin{matrix} a, \frac{a}{k}q^{1/2}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q; \frac{aq}{k} \\ kq^{1/2}, a\sqrt{\frac{q}{a}}, -a\sqrt{\frac{q}{k}}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}} \end{matrix} \right], \qquad (3.2)$$

where $\left|\frac{a}{k}\right| < 1$.

(iii) If we make use of the WP Bailey pair of (2.4) and (1.3) we get another WP Bailey pair,

$$\alpha'_n(a,k,q) = \frac{(y,z;q)_n (a, aq^{1/2}/c; q)_n (kq/a)^n}{(aq/z, aq/y; q)_n (q, cq^{1/2}; q)_n}$$

and

$$\beta'_n(a,k,q) = \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(ky/a, kz/a, k, k/c; q)_n}{(q, aq/y, aq/z, cq; q)_n} \times$$

$$\times \, {}_{10}\Phi_9 \left[\begin{matrix} c, q\sqrt{c}, -q\sqrt{c}, q^{1/2}, y, z, \frac{c}{\sqrt{a}}, \frac{cq^{1/2}}{\sqrt{a}}, kq^n, q^{-n}; q; q \\ \sqrt{c}, -\sqrt{c}, cq^{1/2}, \frac{kz}{a}, \frac{ky}{a}, \sqrt{aq}, q\sqrt{a}, \frac{c}{k}q^{1-n}, cq^{1+n} \end{matrix} \right]$$

$$- \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(ky/a, kz/a, k, k/c; q)_n}{(q, aq/y, aq/z, cq; q)_n} \times$$

$$\times \, {}_{10}\Phi_9 \left[\begin{matrix} c, q\sqrt{c}, -q\sqrt{c}, q^{1/2}, y, z, -\frac{c}{\sqrt{a}}, -\frac{cq^{1/2}}{\sqrt{a}}, kq^n, q^{-n}; q; q \\ \sqrt{c}, -\sqrt{c}, cq^{1/2}, \frac{kz}{a}, \frac{ky}{a}, -\sqrt{aq}, -q\sqrt{a}, \frac{c}{k}q^{1-n}, cq^{1+n} \end{matrix} \right], \qquad (3.3)$$

where $c = kyz/aq$.

If we put the new WP Bailey pair of (3.3) in (2.1) we get,

$$\frac{1+\sqrt{a}}{2\sqrt{a}} \, {}_{10}\Phi_9 \left[\begin{matrix} c, q\sqrt{c}, -q\sqrt{c}, q^{1/2}, y, z, \frac{c}{\sqrt{a}}, \frac{cq^{1/2}}{\sqrt{a}}, kq^n, q^{-n}; q; q \\ \sqrt{c}, -\sqrt{c}, cq^{1/2}, \frac{kz}{a}, \frac{ky}{a}, \sqrt{aq}, q\sqrt{a}, \frac{c}{k}q^{1-n}, cq^{1+n} \end{matrix} \right]$$

$$- \frac{1-\sqrt{a}}{2\sqrt{a}} \, {}_{10}\Phi_9 \left[\begin{matrix} c, q\sqrt{c}, -q\sqrt{c}, q^{1/2}, y, z, -\frac{c}{\sqrt{a}}, -\frac{cq^{1/2}}{\sqrt{a}}, kq^n, q^{-n}; q; q \\ \sqrt{c}, -\sqrt{c}, cq^{1/2}, \frac{kz}{a}, \frac{ky}{a}, -\sqrt{aq}, -q\sqrt{a}, \frac{c}{k}q^{1-n}, cq^{1+n} \end{matrix} \right]$$

$$= \frac{(k/a, aq/y, aq/z, cq; q)_n}{(ky/a, kz/a, k/c, aq; q)_n} \times$$

$$\times\ {}_6\Phi_5\left[\begin{array}{l} a, \dfrac{a}{c}q^{1/2}, y, z, kq^n, q^{-n}; q; q^2 \\ cq^{1/2}, aq/z, aq/y, \dfrac{a}{k}q^{1-n}, aq^{1+n} \end{array}\right], \tag{3.4}$$

where $c = kyz/aq$.

References

[1] Andrews, G.E., Bailey's transform, lemma, chains and tree, special functions 2000: current prospective and future directions (Tempe, AZ), 1-12, NATO Sci. Ser. II Math. Phys. Chem.30, Kluwer Acad. Publ. Dordrecht, 2001

[2] Andrews, G.E., Berkovich Alaxender, The WP Bailey tree and its implications, J. London Math. Soc. (2) 66 (2002), no.3, 529-540.

[3] Bailey, W.N., Some identities in combinatory analysis, Proc. London Math. Soc., 49 (1947) p. 421-435.

[4] Bailey, W.N., Identities of the Rogers-Ramanujan type, Proc. London Math Soc., 50 (1949) p. 1-10.

[5] Laughlin, J.Mc., Some identities between basic hypergeometric series deriving from a new Bailey type transformation, the Journal of Mathematical analysis and Applications, 345 (2) (2008), 670-677.

[6] Verma, A. and Jain, V.K., Certain summation formulae for q-series, J. Indian Math. Soc., 47 (1983) p. 71-85.