

On Bailey pair and q-series transformations

Satya Prakash Singh, and Ashutosh Singh, Department of Mathematics,
 T.D.P.G. College, Jaunpur - 222002, (U.P.) India.
 E-mail: snsp39@yahoo.com

Address for correspondence: Dr. S.P. Singh, Department of Mathematics,
 T.D.P.G. College, Jaunpur - 222002 (U.P.) India.

Abstract: In this paper, making use of the definition of Bailey pair and theorems for constructing new Bailey pair from a known pair, we have established transformation formulae for q-series.

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1. Notations and Definitions As usual, for a and q complex numbers with $|q| < 1$, define

$$[a; q]_0 = 1,$$

$$[a; q]_n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

for $n \in N$,

$$[a_1, a_2, \dots, a_k; q]_n = [a_1; q]_n [a_2; q]_n \dots [a_k; q]_n.$$

and

$$[a; q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r)$$

A basic hypergeometric series is defined by,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots, b_s; q]_n} \left[(-)^n q^{n(n-1)/2} \right]^{1+s-r}. \quad (1.1)$$

we shall make use of following summation formulae in our analysis.

$${}_4\Phi_3 \left[\begin{matrix} a, -q\sqrt{a}, b, q^{-n}; q; \frac{a^{1/2}q^{1+n}}{b} \end{matrix} \right] = \frac{[aq, qa^{1/2}/b; q]_n}{[qa^{1/2}, aq/b; q]_n}. \quad (1.2)$$

[Gasper and Rahman 3; App II (II. 14)]

$${}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n}; q; \frac{aq^{1+n}}{bc} \end{matrix} \right] = \frac{[aq, aq/bc; q]_n}{[aq/c, aq/b; q]_n}. \quad (1.3)$$

[Gasper and Rahman 3; App II (II. 21)]

2. Introduction

A pair of sequences $\{\alpha_n, \beta_n\}$ satisfying the relation

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{[q; q]_{n-r} [aq; q]_{n+r}} = \frac{1}{[q, aq; q]_n} \sum_{r=0}^n \frac{[q^{-n}; q]_r (-)^r q^{nr}}{q^{r(r-1)/2} [aq^{n+1}; q]_r} \alpha_r. \quad (2.1)$$

is called Bailey pair relative to a .
Similarly a pair satisfying

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{[q; q]_{r-n} [aq; q]_{r+n}} = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{[q; q]_r [aq; q]_{r+2n}} \quad (2.2)$$

is called conjugate Bailey pair relative to a .

Following is a theorem to construct a new Bailey pair from a known pair.

Theorem [Andrews 1,2]

If $\{\alpha_n, \beta_n\}$ form a Bailey pair relative to a then so does $\{\alpha'_n, \beta'_n\}$ where

$$\alpha'_n = \frac{[\rho_1, \rho_2; q]_n}{[aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n \quad (2.3)$$

and

$$\beta'_n = \frac{\left[\frac{aq}{\rho_1 \rho_2}; q \right]_n}{[q; q]_n} \sum_{r=0}^n \frac{[\rho_1, \rho_2; q]_r [q^{-n}; q]_r q^r \beta_r}{[aq/\rho_1, aq/\rho_2; q]_r \left[\frac{\rho_1 \rho_2}{a} q^{-n}; q \right]_r}. \quad (2.4)$$

If we take

$$\alpha_r = (-)^r q^{r(r+1)/2} \left(\frac{\sqrt{a}}{b} \right)^r \frac{[a, -q\sqrt{a}, b; q]_r}{[q, -\sqrt{a}, aq/b; q]_r}. \quad (2.5)$$

in (2.1) and make use of the summation formula (1.2) we get,

$$\beta_n = \frac{[q\sqrt{a}/b; q]_n}{[q, q\sqrt{a}, aq/b; q]_n}. \quad (2.6)$$

Thus $\{\alpha_n, \beta_n\}$ given in (2.5) and (2.6) form a Bailey pair. Using Bailey pair (2.5) and (2.6) in (2.3) and (2.4) we get new Bailey pair as,

$$\begin{aligned} \alpha'_n &= \frac{[\rho_1, \rho_2; q]_n}{[aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \frac{[a, -q\sqrt{a}, b; q]_n (-)^n q^{n(n-1)/2}}{[q, -\sqrt{a}, aq/b; q]_n} \left(\frac{q\sqrt{a}}{b} \right)^n \\ &= \frac{[a, -q\sqrt{a}, b, \rho_1, \rho_2; q]_n}{[q, -\sqrt{a}, aq/b, aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{a^{3/2} q^2}{b \rho_1 \rho_2} \right)^n (-)^n q^{n(n-1)/2} \end{aligned} \quad (2.7)$$

and

$$\beta'_n = \frac{\left[\frac{aq}{\rho_1 \rho_2}; q \right]_n}{[q; q]_n} \sum_{r=0}^n \frac{[\rho_1, \rho_2, q\sqrt{a}/b, q^{-n}; q]_r q^r}{\left[q, q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1 \rho_2}{a} q^{-n}; q \right]_r}. \quad (2.8)$$

Again, choosing

$$\alpha_n = \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c; q]_n \left(\frac{aq}{bc} \right)^n (-)^n q^{n(n-1)/2}}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c; q]_n} \quad (2.9)$$

in (2.1) and using the summation formula (1.3) we get,

$$\beta_n = \frac{[aq/bc; q]_n}{[q, aq/b, aq/c; q]_n}. \quad (2.10)$$

Thus $\{\alpha_n, \beta_n\}$ given in (2.9) and (2.10) from a Bailey pair. Using Bailey pair (2.9) and (2.10) in (2.3) and (2.4) we get new Bailey pair as,

$$\alpha'_n = \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2; q]_n}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{a^2 q^2}{bc\rho_1\rho_2} \right)^n (-)^n q^{n(n-1)/2} \quad (2.11)$$

and

$$\beta'_n = \frac{\left[\frac{aq}{\rho_1\rho_2}; q \right]_n}{[q; q]_n} \sum_{r=0}^n \frac{[\rho_1, \rho_2, aq/bc, q^{-n}; q]_r q^r}{\left[q, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2}{a} q^{-n}; q \right]_r}. \quad (2.12)$$

We shall make use of these new Bailey pairs and following theorems due to Lovejoy [4] to establish transformation formulae for q -series.

(1) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) , then

$$\sum_{n=0}^{\infty} \beta_n q^n = \frac{1}{[aq; q]_{\infty} [q; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n(n+1)/2} q^{(2n+1)r} \alpha_r. \quad (2.13)$$

(2) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\sum_{n=0}^{\infty} [a; q]_n \beta_n q^n = \frac{[a; q]_{\infty}}{[a^2 q; q]_{\infty} [q; q]_{\infty}} \sum_{r,n=0}^{\infty} \frac{(-)^n a^{3n} q^{n(3n+1)/2} q^{(3n+1)r} (1 - a^2 q^{2n+2r+1})}{(1 - aq^r)} \alpha_r. \quad (2.14)$$

(3) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to $(a^2 q^2, q^2)$, then

$$\sum_{n=0}^{\infty} [aq; q]_{2n} \beta_n q^{2n} = \frac{[aq; q]_{\infty}}{[q^2, a^2 q^4; q^2]_{\infty}} \sum_{r,n=0}^{\infty} \frac{q^{n(n+1)/2 + (n+1)2r+n}}{(1 - aq^{2r+1})} \alpha_r. \quad (2.15)$$

(4) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to $(a^2 q, q)$, then

$$\sum_{n=0}^{\infty} [aq; q]_n \beta_n q^n = \frac{[aq; q]_{\infty}}{[q, a^2 q^2; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^{3n} q^{n(3n+5)/2 + 3nr+r} (1 + aq^{n+r+1}) \alpha_r. \quad (2.16)$$

(5) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) , then

$$\sum_{n=0}^{\infty} [aq; q^2]_n \beta_n q^n = \frac{1}{[aq^2; q^2]_{\infty} [q; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2 + 2nr+r+n} \alpha_r. \quad (2.17)$$

(6) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\sum_{n=0}^{\infty} \frac{[a^2 q; q]_{2n}}{[aq; q]_n} \beta_n q^n = \frac{1}{[aq; q]_{\infty} [q; q]_{\infty}} \sum_{r,n=0}^{\infty} a^{3n} q^{3n^2 + 2n + 3nr+r} (1 - aq^{2n+r+1}) \alpha_r. \quad (2.18)$$

(7) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\sum_{n=0}^{\infty} \frac{[a^2 q; q]_{2n}}{[aq, aq^2; q]_n} \beta_n q^n = \frac{1}{(1-a)[q, aq, aq^2; q]_{\infty}}$$

$$\times \left\{ \sum_{r=0}^{\infty} q^r \alpha_r + \sum_{r=0, n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+nr} (1 + aq^r) \alpha_r \right\}. \quad (2.19)$$

(8) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[a^2 q; q]_{2n}}{[a, aq; q]_n} \beta_n q^n &= \frac{1}{[q, aq, aq; q]_{\infty}} \\ &\times \sum_{n, r=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2+nr+r} (1+a)}{(1+aq^r)} \alpha_r. \end{aligned} \quad (2.20)$$

(9) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} [-aq; q]_n \beta_n q^n &= \frac{[-aq; q]_{\infty}}{[q^2, a^2 q; q]_{\infty}} \times \\ &\left\{ \sum_{r=0}^{\infty} q^r \alpha_r - \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} a^{3n-2} q^{n(3n-1)/2+3nr-r} (1+aq^r) (1-aq^{n+r}) \alpha_r \right\}. \end{aligned} \quad (2.21)$$

(10) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[a^2 q; q]_{2n}}{[aq; q]_n^2} \beta_n q^n &= \frac{1}{[q, aq, aq; q]_{\infty}} \times \\ &\left\{ \sum_{n_1, n_2, r=0}^{\infty} (-a)^{n_1+n_2} q^{n_1(n_1+1)/2+n_2(n_2+1)/2+(n_1+n_2)r+r} \alpha_r \right\}. \end{aligned} \quad (2.22)$$

(11) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n q^{2n} &= \frac{1}{[aq, q; q]_{\infty}} \\ &\times \left\{ \sum_{r=0}^{\infty} q^{2r} \alpha_r + \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+2nr} (1+aq^{2r}) \alpha_r \right\} \end{aligned} \quad (2.23)$$

(12) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} [aq; q^2]_n \beta_n q^{2n} &= \frac{1}{(1+q)[aq^2; q^2]_{\infty}[q; q]_{\infty}} \\ &\times \left\{ \sum_{r, n=0}^{\infty} (-a)^n q^{n^2+n+2nr+2r} (1-q^{2n+2}) \alpha_r \right\}. \end{aligned} \quad (2.24)$$

(13) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to $(a^2 q, q)$, then

$$\sum_{n=0}^{\infty} [-aq; q]_n \beta_n q^n = \frac{[-aq; q]_{\infty}}{[q, a^2 q^2; q]_{\infty}} \sum_{n, r=0}^{\infty} a^{3n} q^{n(3n+5)/2+r+3nr} (1-aq^{n+r+1}) \alpha_r \quad (2.25)$$

(14) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) , then

$$\sum_{n=0}^{\infty} [aq^2; q^2]_n \beta_n q^n = \frac{1}{(1+q)[aq; q^2]_{\infty} [q; q]_{\infty}} \sum_{n,r=0}^{\infty} (-a)^n q^{n^2+2nr+r} (1+q^{2n+1}) \alpha_r \quad (2.26)$$

(15) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2q, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[a^2q^2; q]_{2n} \beta_n q^n}{[aq; q]_n} &= \frac{1}{[aq, q; q]_{\infty}} \sum_{n,r=0}^{\infty} a^{3n-1} q^{3n^2+3nr+n-1} (1-q^n) \alpha_r \\ &+ \frac{1}{[aq; q; q]_{\infty}} \sum_{n,r=0}^{\infty} a^{3n} q^{3n^2+3nr+4n+1} (1-aq^{n+r+1}) \alpha_r. \end{aligned} \quad (2.27)$$

(16) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2q, q) , then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{[a^2q^2; q]_{2n} \beta_n q^n}{[aq, aq^2; q]_n} &= \frac{1}{[aq, q, aq^2; q]_{\infty}} \times \\ &\sum_{n_1, n_2, r=0}^{\infty} (-a)^{n_1+n_2} q^{n_1(n_1-1)/2+n_2(n_2+1)/2+(r+1)n_1+(r+1)n_2+r} \alpha_r. \end{aligned} \quad (2.28)$$

(17) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a^2, q^2) , then

$$\begin{aligned} \sum_{n=0}^{\infty} [-aq; q]_{2n} \beta_n q^{2n} &= \frac{1}{[aq; q]_{\infty} [q^2; q^2]_{\infty} (1-q)} \times \\ &\times \sum_{n,r=0}^{\infty} (-a)^n q^{n(n+1)/2+2n+2r} (1-q^{n+1}) \alpha_r. \end{aligned} \quad (2.29)$$

(18) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) and $\{\alpha'_n, \beta'_n\}$ is another Bailey pair relative to (z, q) , then

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} a^r z^n q^{2nr+r} \beta_n \beta'_r = \frac{1}{[aq, z; q]_{\infty}} \sum_{n,r=0}^{\infty} a^n z^r q^{2nr+n} \frac{(1-z)}{(1-zq^{2n})} \alpha_r \alpha'_n. \quad (2.30)$$

(19) If $\{\alpha_n, \beta_n\}$ is a Bailey pair relative to (a, q) and $\{\alpha'_n, \beta'_n\}$ is another Bailey pair relative to (z, q) , then

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} [aq; q]_{2n} a^r z^{n+r} q^{r^2+r+2nr} \beta_n \beta'_r = \frac{1}{[z; q]_{\infty}} \sum_{n,r=0}^{\infty} a^n z^{r+n} q^{n^2+n+2nr} \frac{(1-z)}{(1-zq^{2n})} \alpha_r \alpha'_n. \quad (2.31)$$

3. Main Results

In this section we shall make use of new Bailey pairs (2.7) - (2.8) and (2.11) - (2.12) and identities (2.13) - (2.31) to establish the transformation formulae for q -series.

(i) Using the Bailey pair (2.7) - (2.8) in (2.13) we get,

$$\sum_{n=0}^{\infty} \frac{[aq/\rho_1\rho_2; q]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, q\sqrt{a}/b, q^{-n}, 0, 0; q; q \\ q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a} \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{1}{[q, aq; q]_\infty} \sum_{r,n=0}^{\infty} (-)^{r+n} a^n q^{n(n-1)/2+r(r-1)/2} q^{2nr+n+r} \\
&\quad \times \frac{[\rho_1, \rho_2, a, -q\sqrt{a}, b; q]_r q^r}{[q, -\sqrt{a}, aq/b, aq/\rho_1, aq/\rho_2; q]_r} \left(\frac{q\sqrt{a}}{b} \right)^r.
\end{aligned} \tag{3.1}$$

(ii) Using the Bailey pair (2.11) - (2.12) in (2.13) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[aq/\rho_1\rho_2; q]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, aq/bc, q^{-n}, 0, 0; q; q \\ aq/c, aq/b, aq/\rho_1, aq/\rho_2, \rho_1\rho_2q^{-n}/a \end{matrix} \right] \\
&= \frac{1}{[q, aq; q]_\infty} \sum_{r,n=0}^{\infty} (-)^{r+n} a^n q^{n(n+1)/2+r(r-1)/2+(2n+1)r} \\
&\quad \times \frac{[\rho_1, \rho_2, a, b, c, q\sqrt{a}, -q\sqrt{a}; q]_r q^r}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/\rho_1, aq/\rho_2; q]_r} \left(\frac{a^2q^2}{bc\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.2}$$

(iii) Using the Bailey pair (2.7) - (2.8) in (2.14) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a; q]_n \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{b}, q^{-n}, 0, 0; q; q \\ aq, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
&= \frac{[a; q]_\infty}{[q, a^2q; q]_\infty} \sum_{r,n=0}^{\infty} (-)^n a^{3n} q^{n(3n+1)/2+(3n+1)r} \frac{(1 - a^2q^{2n+2r+1})}{(1 - aq^r)} \\
&\quad \times \frac{[\rho_1, \rho_2, a^2, b, -aq; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^3q^2}{b\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.3}$$

(iv) Using the Bailey pair (2.11) - (2.12) in (2.14) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a; q]_n \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{c}, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
&= \frac{[a; q]_\infty}{[q, a^2q; q]_\infty} \sum_{r,n=0}^{\infty} (-)^{r+n} a^{3n} q^{n(3n+1)/2+r(r-1)/2+(3n+1)r} \frac{(1 - a^2q^{2n+2r+1})}{(1 - aq^r)} \\
&\quad \times \frac{[\rho_1, \rho_2, a^2, b, c, aq, -aq; q]_r}{\left[a, -a, \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^4q^2}{bc\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.4}$$

(v) Replacing q by q^2 and a by a^2q^2 in (2.7), (2.8) and then, using this new form of Bailey pair in (2.15) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[aq; q]_{2n} \left[\frac{a^2q^4}{\rho_1\rho_2}; q^2 \right]_n q^{2n}}{[q^2; q^2]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq^3}{b}, q^{-2n}, 0, 0; q^2; q^2 \\ aq^3, \frac{a^2q^4}{b}, \frac{a^2q^4}{\rho_1}, \frac{a^2q^4}{\rho_2}, \frac{\rho_1\rho_2q^{-2n-2}}{a^2} \end{matrix} \right] \\ &= \frac{[aq; q]_{\infty}}{[q^2, a^2q^4; q^2]_{\infty}} \sum_{r,n=0}^{\infty} \frac{q^{n(n+1)/2+2(n+1)r+n}}{(1-aq^{2r+1})} \\ & \quad \times \frac{[\rho_1, \rho_2, a^2q^2, b, -aq^3; q^2]_r (-)^r q^{r(r-1)}}{\left[q^2, -aq, \frac{a^2q^4}{b}, \frac{a^2q^4}{\rho_1}, \frac{a^2q^4}{\rho_2}; q^2 \right]_r} \left(\frac{a^3q^7}{b\rho_1\rho_2} \right)^r. \end{aligned} \quad (3.5)$$

(vi) Replacing q by q^2 and a by a^2q^2 in (2.11), (2.12) and then putting this new form of Bailey pair in (2.15) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[aq; q]_{2n} \left[\frac{a^2q^4}{\rho_1\rho_2}; q^2 \right]_n q^{2n}}{[q^2; q^2]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q^4}{bc}, q^{-2n}, 0, 0; q^2; q^2 \\ \frac{a^2q^4}{b}, \frac{a^2q^4}{c}, \frac{a^2q^4}{\rho_1}, \frac{a^2q^4}{\rho_2}, \frac{\rho_1\rho_2q^{-2n-2}}{a^2} \end{matrix} \right] \\ &= \frac{[aq; q]_{\infty}}{[q^2, a^2q^4; q^2]_{\infty}} \sum_{r,n=0}^{\infty} \frac{q^{n(n+1)/2+2(n+1)r+n}}{(1-aq^{2r+1})} \\ & \quad \times \frac{[\rho_1, \rho_2, a^2q^2, b, c, aq^3, -aq^3; q^2]_r (-)^r q^{r(r-1)}}{\left[q^2, aq, -aq, \frac{a^2q^4}{b}, \frac{a^2q^4}{c}, \frac{a^2q^4}{\rho_1}, \frac{a^2q^4}{\rho_2}; q^2 \right]_r} \left(\frac{a^4q^8}{bc\rho_1\rho_2} \right)^r. \end{aligned} \quad (3.6)$$

(vii) Replacing a by a^2q in (2.7), (2.8) and then putting this Bailey pair in (2.16) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[aq; q]_n \left[\frac{a^2q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq^{3/2}}{b}, q^{-n}, 0, 0; q; q \\ aq^{3/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}, \frac{\rho_1\rho_2q^{-n-1}}{a^2} \end{matrix} \right] \\ &= \frac{[aq; q]_{\infty}}{[q^2, a^2q^2; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^{3n} q^{n(3n+5)/2+(3n+1)r} (1+aq^{n+r+1}) \\ & \quad \times \frac{[\rho_1, \rho_2, a^2q, b, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -aq^{1/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}; q \right]_r} \left(\frac{a^3q^{7/2}}{b\rho_1\rho_2} \right)^r. \end{aligned} \quad (3.7)$$

(viii) Replacing a by a^2q in (2.11), (2.12) and then putting this Bailey pair in (2.16) we get,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[aq; q]_n \left[\frac{a^2q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q^2}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}, \frac{\rho_1\rho_2q^{-n-1}}{a^2} \end{matrix} \right] \\
&= \frac{[aq; q]_{\infty}}{[q, a^2q^2; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^{3n} q^{n(3n+5)/2+(3n+1)r} (1 + aq^{n+r+1}) \\
&\quad \times \frac{[\rho_1, \rho_2, a^2q, b, aq^{3/2}, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, aq^{1/2}, -aq^{1/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}; q \right]_r} \left(\frac{a^4q^4}{bc\rho_1\rho_2} \right)^r. \tag{3.8}
\end{aligned}$$

(ix) Using the Bailey pair (2.7) - (2.8) in (2.17) we get,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[aq; q^2]_n \left[\frac{aq}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{a}}{b}, q^{-n}, 0, 0; q; q \\ \frac{aq}{b}, q\sqrt{a}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a} \end{matrix} \right] \\
&= \frac{1}{[q; q]_{\infty} [aq^2; q^2]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+2nr+n+r} \\
&\quad \times \frac{[a, \rho_1, \rho_2, -q\sqrt{a}, b; q]_r (-)^r q^{r(r-1)/2}}{[q, -\sqrt{a}, aq/b, aq/\rho_1, aq/\rho_2; q]_r} \left(\frac{a^{3/2}q^2}{b\rho_1\rho_2} \right)^r. \tag{3.9}
\end{aligned}$$

(x) Using the Bailey pair (2.11) - (2.12) in (2.17) we get,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[aq; q^2]_n \left[\frac{aq}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{bc}, q^{-n}, 0, 0; q; q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a} \end{matrix} \right] \\
&= \frac{1}{[q; q]_{\infty} [aq^2; q^2]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+2nr+n+r} \\
&\quad \times \frac{[a, \rho_1, \rho_2, q\sqrt{a}, -q\sqrt{a}, b, c; q]_r (-)^r q^{r(r-1)/2}}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/\rho_1, aq/\rho_2; q]_r} \left(\frac{a^2q^2}{bc\rho_1\rho_2} \right)^r. \tag{3.10}
\end{aligned}$$

(xi) Replacing a by a^2 in (2.7)-(2.8) and then using this pair in (2.18) we have,

$$\sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, aq, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{1}{[aq; q]_\infty} \sum_{r,n=0}^{\infty} a^{3n} q^{3n^2+2n+3nr+r} (1 - aq^{2n+r+1}) \\
&\quad \times \frac{[a^2, \rho_1, \rho_2, -aq, b; q]_r (-)^r q^{r(r-1)/2}}{[q, -a, a^2q/b, a^2q/\rho_1, a^2q/\rho_2; q]_r} \left(\frac{a^3 q^2}{b\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.11}$$

(xii) Replacing a by a^2 in (2.11)-(2.12) and then using this pair in (2.18) we have,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[aq; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
&= \frac{1}{[aq; q]_\infty} \sum_{r,n=0}^{\infty} a^{3n} q^{3n^2+2n+3nr+r} (1 - aq^{2n+r+1}) \\
&\quad \times \frac{[a^2, \rho_1, \rho_2, aq, -aq, b, c; q]_r (-)^r q^{r(r-1)/2}}{[q, a, -a, a^2q/b, a^2q/c, a^2q/\rho_1, a^2q/\rho_2; q]_r} \left(\frac{a^4 q^2}{bc\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.12}$$

(xiii) Replacing a by a^2 in (2.7)-(2.8) and then using this pair in (2.19) we have,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[aq, aq^2; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{b}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, aq, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
&= \frac{1}{(1-a)[q, aq, aq^2; q]_\infty} {}_5\Phi_5 \left[\begin{matrix} a^2, -aq, b, \rho_1, \rho_2; q; \frac{a^3 q^3}{b\rho_1\rho_2} \\ -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, 0 \end{matrix} \right] \\
&+ \frac{1}{(1-a)[q, aq, aq^2; q]_\infty} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+nr} (1 + aq^r) \\
&\quad \times \frac{[a^2, -aq, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^3 q^2}{b\rho_1\rho_2} \right)^r.
\end{aligned} \tag{3.13}$$

(xiv) Replacing a by a^2 in (2.11)-(2.12) and then using them in (2.19) we have,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[aq, aq^2; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
&= \frac{1}{(1-a)[q, aq, aq^2; q]_\infty} {}_7\Phi_7 \left[\begin{matrix} a^2, aq, -aq, b, c, \rho_1, \rho_2; q; \frac{a^4 q^3}{bc\rho_1\rho_2} \\ a, -a, \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, 0 \end{matrix} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(1-a)[q, aq, aq^2; q]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+nr} (1+aq^r) \\
& \times \frac{[a^2, aq, -aq, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^4 q^2}{bc\rho_1\rho_2} \right)^r}{\left[q, a, -a, \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}; q \right]_r}. \tag{3.14}
\end{aligned}$$

(**xv**) Replacing a by a^2 in (2.7) - (2.8) and (2.11) - (2.12) and then using them in (2.20) we get following results,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[a^2; q]_{2n} \left[\frac{a^2 q}{\rho_1 \rho_2}; q \right]_n q^n}{[a, aq; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{b}, q^{-n}, 0, 0; q; q \\ \frac{a^2 q}{b}, aq, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{1}{[q, aq, aq; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n(n+1)/2+nr+r} \frac{(1+a)}{(1+aq^r)} \\
& \times \frac{[a^2, -aq, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^3 q^2}{b\rho_1\rho_2} \right)^r}{\left[q, -a, \frac{a^2 q}{b}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}; q \right]_r}, \tag{3.15}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[a^2; q]_{2n} \left[\frac{a^2 q}{\rho_1 \rho_2}; q \right]_n q^n}{[q, a, aq; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2 q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{1}{[q, aq, aq; q]_{\infty}} \sum_{r,n=0}^{\infty} (-a)^n q^{n(n+1)/2+nr+r} \frac{(1+a)}{(1+aq^r)} \\
& \times \frac{[a^2, aq, -aq, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^4 q^2}{bc\rho_1\rho_2} \right)^r}{\left[q, a, -a, \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}; q \right]_r}, \tag{3.16}
\end{aligned}$$

respectively.

(**xvi**) Similarly, replacing a by a^2 in (2.11) - (2.12) and then using them in (2.21) we have,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[-aq; q]_n \left[\frac{a^2 q}{\rho_1 \rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2 q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{[-aq; q]_{\infty}}{[q, a^2 q; q]_{\infty}} {}_7\Phi_7 \left[\begin{matrix} a^2, aq, -aq, b, c, \rho_1, \rho_2; q; \frac{a^4 q^3}{bc\rho_1\rho_2} \\ a, -a, \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}, 0 \end{matrix} \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{[-aq; q]_{\infty}}{[q, a^2q; q]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} a^{3n-2} q^{n(3n-1)/2+3nr-r} (1+aq^r)(1-aq^{n+r}) \\
& \times \frac{[a^2, aq, -aq, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, a, -a, \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^4 q^2}{bc\rho_1\rho_2} \right)^r. \tag{3.17}
\end{aligned}$$

(xvii) Replacing a by a^2 in (2.7)-(2.8) and then using them in (2.21) we get,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[-aq; q]_n \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{b}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, aq, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{[-aq; q]_{\infty}}{[q, a^2q; q]_{\infty}} {}_5\Phi_5 \left[\begin{matrix} a^2, -aq, b, \rho_1, \rho_2; q; \frac{a^3q^3}{b\rho_1\rho_2} \\ -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, 0 \end{matrix} \right] \\
& - \frac{[-aq; q]_{\infty}}{[q, a^2q; q]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} a^{3n-2} q^{n(3n-1)/2+3nr-r} (1+aq^r)(1-aq^{n+r}) \\
& \times \frac{[a^2, -aq, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^3q^2}{b\rho_1\rho_2} \right)^r. \tag{3.18}
\end{aligned}$$

(xviii) Replacing a by a^2 in (2.7) - (2.8) and then using them in (2.22) we have,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[aq; q]_n^2 [q; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{b}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, aq, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{1}{[q, aq, aq; q]_{\infty}} \sum_{n_1, n_2, r=0}^{\infty} (-a)^{n_1+n_2} q^{n_1(n_1+1)/2+n_2(n_2+1)/2+(n_1+n_2)r+r} \\
& \times \frac{[a^2, -aq, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -a, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^3q^2}{b\rho_1\rho_2} \right)^r. \tag{3.19}
\end{aligned}$$

(xix) Replacing a by a^2 in (2.11) - (2.12) and then using them in (2.22) we have,

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{[a^2q; q]_{2n} \left[\frac{a^2q}{\rho_1\rho_2}; q \right]_n q^n}{[aq; q]_n^2 [q; q]_n [q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q}{b}, \frac{a^2q}{c}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}, \frac{\rho_1\rho_2q^{-n}}{a^2} \end{matrix} \right] \\
& = \frac{1}{[q, aq, aq; q]_{\infty}} \sum_{n_1, n_2, r=0}^{\infty} (-a)^{n_1+n_2} q^{n_1(n_1+1)/2+n_2(n_2+1)/2+(n_1+n_2)r+r}
\end{aligned}$$

$$\times \frac{[a^2, aq, -aq, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, a, -a, \frac{a^2 q}{b}, \frac{a^2 q}{c}, \frac{a^2 q}{\rho_1}, \frac{a^2 q}{\rho_2}; q \right]_r} \left(\frac{a^4 q^2}{bc\rho_1\rho_2} \right)^r. \quad (3.20)$$

(xx) Using the Bailey pair (2.7) - (2.8) in (2.23) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left[\frac{aq}{\rho_1\rho_2}; q \right]_{2n}}{[q; q]_n} q^{2n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{a}}{b}, q^{-n}, 0, 0; q; q \\ q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2 q^{-n}}{a} \end{matrix} \right] \\ &= \frac{1}{[q, aq; q]_{\infty}} {}_5\Phi_5 \left[\begin{matrix} a, -q\sqrt{a}, b, \rho_1, \rho_2; q; \frac{a^{3/2}q^4}{b\rho_1\rho_2} \\ -\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, 0 \end{matrix} \right] \\ &+ \frac{1}{[q, aq; q]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+2nr} (1 + aq^{2r}) \\ &\times \frac{[a, -q\sqrt{a}, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r} \left(\frac{a^2 q^{3/2}}{b\rho_1\rho_2} \right)^r. \end{aligned} \quad (3.21)$$

(xxi) Using the Bailey pair (2.11) - (2.12) in (2.23) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left[\frac{aq}{\rho_1\rho_2}; q \right]_n}{[q; q]_n} q^{2n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{bc}, q^{-n}, 0, 0; q; q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2 q^{-n}}{a} \end{matrix} \right] \\ &= \frac{1}{[q, aq; q]_{\infty}} {}_7\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2; q; \frac{a^2 q^4}{bc\rho_1\rho_2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, 0 \end{matrix} \right] \\ &+ \frac{1}{[q, aq; q]_{\infty}} \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} (-)^n a^{n-1} q^{n(n+1)/2+2nr} (1 + aq^{2r}) \\ &\times \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2}}{\left[q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r} \left(\frac{a^2 q^2}{bc\rho_1\rho_2} \right)^r. \end{aligned} \quad (3.22)$$

(xxii) Using the Bailey pair (2.7) - (2.8) in (2.24) we get,

$$\sum_{n=0}^{\infty} \frac{[aq; q^2]_n \left[\frac{aq}{\rho_1\rho_2}; q \right]_n}{[q; q]_n} q^{2n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{a}}{b}, q^{-n}, 0, 0; q; q \\ q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2 q^{-n}}{a} \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{1}{[q; q]_{\infty} [aq^2; q^2]_{\infty} (1+q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+n+2nr+2r} (1-q^{2n+2}) \\
&\quad \times \frac{[a, -q\sqrt{a}, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^{3/2} q^2}{b\rho_1\rho_2} \right)^r}{\left[q, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r}. \tag{3.23}
\end{aligned}$$

(xxiii) Using the Bailey pair (2.11) - (2.12) in (2.24) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[aq; q^2]_n \left[\frac{aq}{\rho_1\rho_2}; q \right]_n q^{2n}}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{bc}, q^{-n}, 0, 0; q; q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2 q^{-n}}{a} \end{matrix} \right] \\
&= \frac{1}{[q; q]_{\infty} [aq^2; q^2]_{\infty} (1+q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+n+2nr+2r} (1-q^{2n+2}) \\
&\quad \times \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^2 q^2}{bc\rho_1\rho_2} \right)^r}{\left[q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r}. \tag{3.24}
\end{aligned}$$

(xxiv) Replacing a by $a^2 q$ in Bailey pair (2.7) - (2.8) then using them in (2.25) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[-aq; q]_n \left[\frac{a^2 q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^3 q^{3/2}}{b}, q^{-n}, 0, 0; q; q \\ aq^{3/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}, \frac{\rho_1\rho_2 q^{-n-1}}{a^2} \end{matrix} \right] \\
&= \frac{[-aq; q]_{\infty}}{[q, a^2 q^2; q]_{\infty}} \sum_{r,n=0}^{\infty} a^{3n} q^{n(3n+5)/2+(3n+1)r} (1-aq^{n+r+1}) \\
&\quad \times \frac{[\rho_1, \rho_2, a^2 q, b, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^3 q^{7/2}}{b\rho_1\rho_2} \right)^r}{\left[q, -aq^{1/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q \right]_r}. \tag{3.25}
\end{aligned}$$

(xxv) Replacing a by $a^2 q$ in Bailey pair (2.11) - (2.12) then using them in (2.25) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[-aq; q]_n \left[\frac{a^2 q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2 q^2}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2 q^2}{b}, \frac{a^2 q^2}{c}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}, \frac{\rho_1\rho_2 q^{-n-1}}{a^2} \end{matrix} \right] \\
&= \frac{[-aq; q]_{\infty}}{[q, a^2 q^2; q]_{\infty}} \sum_{r,n=0}^{\infty} a^{3n} q^{n(3n+5)/2+(3n+1)r} (1-aq^{n+r+1}) \\
&\quad \times \frac{[\rho_1, \rho_2, a^2 q, b, c, aq^{3/2}, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^4 q^4}{bc\rho_1\rho_2} \right)^r}{\left[q, aq^{1/2}, -aq^{1/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{c}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q \right]_r}. \tag{3.26}
\end{aligned}$$

(xxvi) Using the Bailey pair (2.7) - (2.8) in (2.26) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[aq^2; q^2]_n \left[\frac{aq}{\rho_1 \rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{a}}{b}, q^{-n}, 0, 0; q; q \\ q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a} \end{matrix} \right] \\ &= \frac{1}{[q; q]_{\infty} [aq; q^2]_{\infty} (1+q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+2nr+r} (1+q^{2n+1}) \\ & \quad \times \frac{[a, -q\sqrt{a}, b, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^{3/2} q^2}{b \rho_1 \rho_2} \right)^r}{\left[q, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r}. \end{aligned} \quad (3.27)$$

(xxvii) Using the Bailey pair (2.11) - (2.12) in (2.26) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[aq^2; q^2]_n \left[\frac{aq}{\rho_1 \rho_2}; q \right]_n q^n}{[q; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq}{bc}, q^{-n}, 0, 0; q; q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a} \end{matrix} \right] \\ &= \frac{1}{[q; q]_{\infty} [aq; q^2]_{\infty} (1+q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n^2+2nr+r} (1+q^{2n+1}) \\ & \quad \times \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, \rho_1, \rho_2; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^2 q^2}{bc \rho_1 \rho_2} \right)^r}{\left[q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r}. \end{aligned} \quad (3.28)$$

(xxviii) Replacing a by $a^2 q$ in Bailey pair (2.7) - (2.8) and using them in (2.27) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[a^2 q^2; q]_{2n} \left[\frac{a^2 q^2}{\rho_1 \rho_2}; q \right]_n q^n}{[q; q]_n [aq; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq^{3/2}}{b}, q^{-n}, 0, 0; q; q \\ aq^{3/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n-1}}{a^2} \end{matrix} \right] \\ &= \frac{1}{[q, aq; q]_{\infty}} \sum_{r,n=0}^{\infty} a^{3n-1} q^{3n^2+3nr+n-1} (1-q^n) \frac{[\rho_1, \rho_2, a^2 q, b, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^3 q^{7/2}}{b \rho_1 \rho_2} \right)^r}{\left[q, -aq^{1/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q \right]_r} \\ &+ \frac{1}{[q, aq; q]_{\infty}} \sum_{r,n=0}^{\infty} a^{3n} q^{3n^2+3nr+4n+1} (1-aq^{n+r+1}) \frac{[\rho_1, \rho_2, a^2 q, b, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2} \left(\frac{a^3 q^{7/2}}{b \rho_1 \rho_2} \right)^r}{\left[q, -aq^{1/2}, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q \right]_r}. \end{aligned} \quad (3.29)$$

(xxix) Replacing a by $a^2 q$ in Bailey pair (2.11) - (2.12) then using them in (2.27) we get,

$$\sum_{n=0}^{\infty} \frac{[a^2 q^2; q]_{2n} \left[\frac{a^2 q^2}{\rho_1 \rho_2}; q \right]_n q^n}{[q; q]_n [aq; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2 q^2}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2 q^2}{b}, \frac{a^2 q^2}{c}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n-1}}{a^2} \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{1}{[q, aq; q]_\infty} \sum_{r,n=0}^{\infty} a^{3n-1} q^{3n^2+3nr+n-1} (1 - q^n) \\
&\times \frac{[\rho_1, \rho_2, a^2q, b, c, aq^{3/2}, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, aq^{1/2}, -aq^{1/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}; q \right]_r} \left(\frac{a^4q^4}{bc\rho_1\rho_2} \right)^r \\
&+ \frac{1}{[q, aq; q]_\infty} \sum_{r,n=0}^{\infty} a^{3n} q^{3n^2+3nr+4n+1} (1 - aq^{n+r+1}) \\
&\times \frac{[\rho_1, \rho_2, a^2q, b, c, aq^{3/2}, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, aq^{1/2}, -aq^{1/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}; q \right]_r} \left(\frac{a^4q^4}{bc\rho_1\rho_2} \right)^r. \tag{3.30}
\end{aligned}$$

(xxx) Replacing a by a^2q in Bailey pair (2.7) - (2.8) and then using them in (2.28) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a^2q^2; q]_{2n} \left[\frac{a^2q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n [aq, aq^2; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{aq^{3/2}}{b}, q^{-n}, 0, 0; q; q \\ aq^{3/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}, \frac{\rho_1\rho_2q^{-n-1}}{a^2} \end{matrix} \right] \\
&= \frac{1}{[q, aq, aq^2; q]_\infty} \sum_{n_1, n_2, r=0}^{\infty} (-)^{n_1+n_2} q^{n_1(n_1-1)/2+n_2(n_2+1)/2+n_1(r+1)+n_2(r+1)+r} \\
&\times \frac{[\rho_1, \rho_2, a^2q, b, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, -aq^{1/2}, \frac{a^2q}{b}, \frac{a^2q}{\rho_1}, \frac{a^2q}{\rho_2}; q \right]_r} \left(\frac{a^3q^{7/2}}{b\rho_1\rho_2} \right)^r. \tag{3.31}
\end{aligned}$$

(xxxi) Replacing a by a^2q in Bailey pair (2.11) - (2.12) and then using them in (2.28) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[a^2q^2; q]_{2n} \left[\frac{a^2q^2}{\rho_1\rho_2}; q \right]_n q^n}{[q; q]_n [aq, aq^2; q]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q^2}{bc}, q^{-n}, 0, 0; q; q \\ \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}, \frac{\rho_1\rho_2q^{-n-1}}{a^2} \end{matrix} \right] \\
&= \frac{1}{[q, aq, aq^2; q]_\infty} \sum_{n_1, n_2, r=0}^{\infty} (-)^{n_1+n_2} q^{n_1(n_1-1)/2+n_2(n_2+1)/2+n_1(r+1)+n_2(r+1)+r} \\
&\times \frac{[\rho_1, \rho_2, a^2q, b, c, aq^{3/2}, -aq^{3/2}; q]_r (-)^r q^{r(r-1)/2}}{\left[q, aq^{1/2}, -aq^{1/2}, \frac{a^2q^2}{b}, \frac{a^2q^2}{c}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}; q \right]_r} \left(\frac{a^4q^4}{bc\rho_1\rho_2} \right)^r. \tag{3.32}
\end{aligned}$$

(xxxii) Replacing q by q^2 and a by a^2 in Bailey pair (2.7) - (2.8) and then using them in (2.29) we get,

$$\sum_{n=0}^{\infty} \frac{[-aq; q]_{2n} \left[\frac{a^2q^2}{\rho_1\rho_2}; q \right]_n q^{2n}}{[q^2; q^2]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2q}{b}, q^{-2n}, 0, 0; q^2; q^2 \\ aq^2, \frac{a^2q^2}{b}, \frac{a^2q^2}{\rho_1}, \frac{a^2q^2}{\rho_2}, \frac{\rho_1\rho_2q^{-2n}}{a^2} \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{1}{[aq; q]_{\infty} [q^2; q^2]_{\infty} (1-q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n(n+1)/2+2n+2r} (1-q^{n+1}) \\
&\quad \times \frac{[\rho_1, \rho_2, a^2, b, -aq^2; q^2]_r (-)^r q^{r(r-1)/2} \left(\frac{a^3 q^4}{b \rho_1 \rho_2} \right)^r}{\left[q^2, -a, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q^2 \right]_r}. \quad (3.33)
\end{aligned}$$

(xxxiii) Replacing q by q^2 and a by a^2 in Bailey pair (2.11) - (2.12) and then using them in (2.29) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[-aq; q]_{2n} \left[\frac{a^2 q^2}{\rho_1 \rho_2}; q \right]_n q^{2n}}{[q^2; q^2]_n} {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{a^2 q^2}{bc}, q^{-2n}, 0, 0; q^2; q^2 \\ \frac{a^2 q^2}{b}, \frac{a^2 q^2}{c}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}, \frac{\rho_1 \rho_2 q^{-2n}}{a^2} \end{matrix} \right] \\
&= \frac{1}{[aq; q]_{\infty} [q^2; q^2]_{\infty} (1-q)} \sum_{r,n=0}^{\infty} (-a)^n q^{n(n+1)/2+2n+2r} (1-q^{n+1}) \\
&\quad \times \frac{[\rho_1, \rho_2, aq^2, -aq^2, a^2, b, c; q^2]_r (-)^r q^{r(r-1)/2} \left(\frac{a^4 q^4}{bc \rho_1 \rho_2} \right)^r}{\left[q^2, a, -a, \frac{a^2 q^2}{b}, \frac{a^2 q^2}{c}, \frac{a^2 q^2}{\rho_1}, \frac{a^2 q^2}{\rho_2}; q^2 \right]_r}. \quad (3.34)
\end{aligned}$$

(xxxiv) Using the Bailey pairs (2.7) - (2.8) in (2.30) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} a^r z^n q^{2nr+r} \frac{\left[\frac{aq}{\rho_1 \rho_2}; q \right]_n [zq/\rho_1 \rho_2; q]_r}{[q; q]_n [q; q]_r} \\
&\quad \times {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{a}}{b}, q^{-n}, 0, 0; q; q \\ q\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1 \rho_2 q^{-n}}{a} \end{matrix} \right] {}_6\Phi_5 \left[\begin{matrix} \rho_1, \rho_2, \frac{q\sqrt{z}}{b}, q^{-r}, 0, 0; q; q \\ q\sqrt{z}, \frac{zq}{b}, \frac{zq}{\rho_1}, \frac{zq}{\rho_2}, \frac{\rho_1 \rho_2 q^{-r}}{z} \end{matrix} \right] \\
&= \frac{1}{[aq, z; q]_{\infty}} \sum_{n,r=0}^{\infty} a^n z^r q^{2nr+n} \left(\frac{1-z}{1-zq^{2n}} \right) \frac{[a, -q\sqrt{a}, b, \rho_1, \rho_2; q]_r}{\left[q, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q \right]_r} \left(\frac{a^{3/2} q^2}{b \rho_1 \rho_2} \right)^r \\
&\quad \times \frac{(-)^r q^{r(r-1)/2} [z, -q\sqrt{z}, b, \rho_1, \rho_2; q]_n}{\left[q, -\sqrt{z}, \frac{zq}{b}, \frac{zq}{\rho_1}, \frac{zq}{\rho_2}; q \right]_n} \left(\frac{z^{3/2} q^2}{b \rho_1 \rho_2} \right)^n (-)^n q^{n(n-1)/2}. \quad (3.35)
\end{aligned}$$

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