

A note on confluent hypergeometric series

K.B. Yadav, Sunil Singh* and Vinod Yadav

Department of Mathematics

T.D.P.G. College, Jaunpur-222002 (U.P.) India.

*Department of Mathematics,

Sydenham College of Commerce and Economics, Churchgate, Mumbai

Abstract: In this paper, using second order differential equation and some contiguous relations of ordinary hypergeometric series we have established Kummer's transformations for confluent hypergeometric series.

Keywords and Phrases: Ordinary hypergeometric series, Identities and transformation formulae .

Mathematics Subject Classification: Primary 33D15, 33D90, 11A55; Secondary 11F20.

1. Introduction, Notations and Definitions

We know that the second order differential equation

$$z(1-z) \frac{d^2w}{dz^2} + \{c - (1+a+b)z\} \frac{dw}{dz} - abw = 0 \quad (1.1)$$

has three regular singularities at 0, 1 and ∞ . If we put z/b for z in (1.1) we get the new form of this equation as,

$$b^2 \frac{z}{b} \left(1 - \frac{z}{b}\right) \frac{d^2w}{dz^2} + \left\{c - (1+a+b) \frac{z}{b}\right\} b \frac{dw}{dz} - abw = 0$$

which on simplification gives,

$$z \left(1 - \frac{z}{b}\right) \frac{d^2w}{dz^2} + \left\{c - (1+a+b) \frac{z}{b}\right\} \frac{dw}{dz} - aw = 0. \quad (1.2)$$

It has three regular singularities at 0, b , ∞ .

Now, let $b \rightarrow \infty$ in (1.2), then it takes the form

$$z \frac{d^2w}{dz^2} + (c-z) \frac{dw}{dz} - aw = 0. \quad (1.3)$$

In (1.3), there is the confluence of two singularities, so it is called confluent hypergeometric equation.

Equation (1.1) has two solutions,

$${}_2F_1[a, b; c; z] \text{ and } z^{1-c} {}_2F_1[1 + a - c, 1 + b - c; 2 - c; z], \quad (1.4)$$

where c is not an integer.

Putting z/b for z in these expressions and then taking $b \rightarrow \infty$ we get that expressions takes the form ${}_1F_1[a; c; z]$ and $z^{1-c} {}_1F_1[1 + a - c; 2 - c; z]$,

which are solutions of confluent hypergeometric equation (1.3) around $z = 0$. These solutions are valid over the whole complex plane, since ${}_1F_1$ is an entire function. The hypergeometric function ${}_1F_1[a; c; z]$ is called confluent hypergeometric function. Thus the differential equation (1.3) has the complete solution as,

$$w = A {}_1F_1[a; c; z] + B z^{1-c} {}_1F_1[1 + a - c; 2 - c; z]. \quad (1.5)$$

There are following three relations between the ${}_1F_1$ and pairs of its contiguous functions.

$$(a - c + 1) {}_1F_1[a; c; z] = a {}_1F_1[a + 1; c; z] - (c - 1) {}_1F_1[a; c - 1; z], \quad (1.6)$$

$$c(a + z) {}_1F_1[a; c; z] = ac {}_1F_1[a + 1; c; z] - (a - c) z {}_1F_1[a; c + 1; z], \quad (1.7)$$

$$c {}_1F_1[a; c; z] = c {}_1F_1[a - 1; c; z] + z {}_1F_1[a; c + 1; z]. \quad (1.8)$$

2. Kummer's Transformations

(a) Kummer's first transformation

If c is neither zero nor a negative integer

$${}_1F_1[a; c; z] = e^z {}_1F_1[c - a; c; -z]. \quad (2.1)$$

Proof

$$\begin{aligned} e^{-z} {}_1F_1[a; c; z] &= \sum_{r=0}^{\infty} \frac{(-)^r z^r}{r!} \sum_{s=0}^{\infty} \frac{(a)_s z^s}{(c)_s s!} \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^r (a)_s z^{r+s}}{(c)_s s! r!} \end{aligned}$$

Now, taking $r + s = n$, ie $r = n - s$ then

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{s=0}^n \frac{(-)^{n-s} (a)_s z^n}{(c)_s s! (1)_{n-s}} \\ &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{s=0}^n \frac{(-n)_s (a)_s}{(c)_s s!} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} {}_2F_1 \left[\begin{matrix} -n, a; 1 \\ c \end{matrix} \right] \\
&= \sum_{n=0}^{\infty} \frac{(c-a)_n (-z)^n}{(c)_n n!} \\
&= {}_1F_1 [c-a; c; -z].
\end{aligned} \tag{2.2}$$

For the convergence of ${}_1F_1 [a; c; z]$ it is essential that neither c is zero nor negative integer. Also, from (2.2) we have

$${}_1F_1 [a; c; z] = e^z {}_1F_1 [c-a; c; -z], \tag{2.3}$$

which is precisely (2.1).

Again, if we consider the Paff's transformation

$${}_2F_1 \left[\begin{matrix} a, b; z \\ c \end{matrix} \right] = (1-z)^{-b} {}_2F_1 \left[\begin{matrix} b, c-a; \frac{z}{1-z} \\ c \end{matrix} \right], \tag{2.4}$$

Then putting z/b for z in (2.4) and then taking $b \rightarrow \infty$ we get

$${}_1F_1 \left[\begin{matrix} a; z \\ c \end{matrix} \right] = e^z {}_1F_1 \left[\begin{matrix} c-a; -z \\ c \end{matrix} \right], \tag{2.5}$$

which is (2.3).

(b) Kummer's second transformation

$${}_1F_1 \left[\begin{matrix} a; 4z \\ 2a \end{matrix} \right] = e^{2z} {}_0F_1 \left[\begin{matrix} -; z^2 \\ a + \frac{1}{2} \end{matrix} \right]. \tag{2.6}$$

Proof; If we take $c = 2a$ in Kummer's first transformation (2.1) we find,

$${}_1F_1 \left[\begin{matrix} a; z \\ 2a \end{matrix} \right] = e^z {}_1F_1 \left[\begin{matrix} a; -z \\ 2a \end{matrix} \right]$$

or

$$e^{-\frac{1}{2}z} {}_1F_1 \left[\begin{matrix} a; z \\ 2a \end{matrix} \right] = e^{\frac{1}{2}z} {}_1F_1 \left[\begin{matrix} a; -z \\ 2a \end{matrix} \right]. \tag{2.7}$$

(2.7) shows that the function $e^{-z} {}_1F_1 \left[\begin{matrix} a; 2z \\ 2a \end{matrix} \right]$ is an even function of z .

By expanding $e^{-z} {}_1F_1 \left[\begin{matrix} a; 2z \\ 2a \end{matrix} \right]$ in the form of series we have,

$$e^{-z} {}_1F_1 \left[\begin{matrix} a; 2z \\ 2a \end{matrix} \right] = \sum_{r=0}^{\infty} \frac{(-)^r z^r}{r!} \sum_{s=0}^{\infty} \frac{(a)_s 2^s z^s}{(2a)_s s!}$$

Taking $r + s = n$ we have

$$\begin{aligned} e^{-z} {}_1F_1 \left[\begin{matrix} a; 2z \\ 2a \end{matrix} \right] &= \sum_{n=0}^{\infty} \sum_{s=0}^n \frac{(-)^{n-s} (a)_s 2^s z^n}{(1)_{n-s} (2a)_s s!} \\ &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} {}_2F_1 \left[\begin{matrix} -n, a; 2 \\ 2a \end{matrix} \right]. \end{aligned} \quad (2.8)$$

Since, left hand function of (2.8) is an even function so

$${}_2F_1 \left[\begin{matrix} -2n - 1, a; 2 \\ 2a \end{matrix} \right] = 0,$$

Thus we have

$$e^{-z} {}_1F_1 \left[\begin{matrix} a; 2z \\ 2a \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(z)^{2n}}{(1)_{2n}} {}_2F_1 \left[\begin{matrix} -2n, a; 2 \\ 2a \end{matrix} \right]. \quad (2.9)$$

Let us now determine a differential equation whose solution is ${}_1F_1 [a, 2a; 2z]$. We know that the differential equation

$$z \frac{d^2 w}{dz^2} + (c - z) \frac{dw}{dz} - aw = 0. \quad (2.10)$$

has a solution $w = {}_1F_1 [a, c; z]$.

Putting $c = 2a$ and replacing z by $2z$ and w by $e^z w$ we find that (2.10) takes the form,

$$z \frac{d^2 w}{dz^2} + 2a \frac{dw}{dz} - zw = 0. \quad (2.11)$$

whose one solution is $e^{-z} {}_1F_1 [a, 2a; 2z]$.

Now taking $\sigma = \frac{z^2}{4}$ in (2.11) the differential equation becomes

$$\sigma^2 \frac{d^2 w}{d\sigma^2} + \left(a + \frac{1}{2} \right) \sigma \frac{dw}{d\sigma} - \sigma w = 0 \quad (2.12)$$

If $2a$ is not an odd integer, the general solution of (2.12) is

$$w = A {}_0F_1 \left[-; \left(a + \frac{1}{2} \right); \frac{z^2}{4} \right] + B z^{1-2a} {}_0F_1 \left[-; \frac{3}{2} - a; \frac{z^2}{4} \right]. \quad (2.13)$$

But solution of (2.12) is also $e^{-z} {}_1F_1 [a, 2a; 2z]$. Therefore there exist constant A and B such that right hand side of (2.13) becomes $e^{-z} {}_1F_1 [a, 2a; 2z]$. It is easy to see that B = 0 and A = 1. Hence

$$e^{-z} {}_1F_1 [a, 2a; 2z] = {}_0F_1 \left[-; \left(a + \frac{1}{2} \right); \frac{z^2}{4} \right]. \quad (2.14)$$

If we compare (2.14) and (2.9) we find

$${}_2F_1 \left[\begin{matrix} -2n, a; 2 \\ 2a \end{matrix} \right] = \frac{(1/2)_n}{(a + \frac{1}{2})_n}. \quad (2.15)$$

Again, if we consider the transformation

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} a, b; \frac{4x}{(1+x)^2} \\ 2a \end{matrix} \right] &= (1+x)^{2a} {}_2F_1 \left[\begin{matrix} a, a + \frac{1}{2} - b; x^2 \\ b + \frac{1}{2} \end{matrix} \right] \\ &= (1+x)^{2b} {}_2F_1 \left[\begin{matrix} b, b + \frac{1}{2} - a; x^2 \\ a + \frac{1}{2} \end{matrix} \right], \end{aligned} \quad (2.16)$$

Putting $x/2b$ for x in (2.16) and then taking $b \rightarrow \infty$ we get

$${}_1F_1 [a, 2a; 2x] = e^x {}_0F_1 \left[-; \left(a + \frac{1}{2} \right); \frac{z^2}{4} \right], \quad (2.17)$$

which is Kummer's second transformation.

References

- [1] Agarwal R.P; Generalized hypergeometric series and its application to the theory of combinatorial analysis and partition (*unpublished monograph*)
- [2] Agarwal R.P; Resonance of Ramanujans Mathematics, Vol. 1 (1996), New Age International (P) Limited Publishers, New Delhi.
- [3] Agarwal R.P; Resonance of Ramanujans Mathematics, Vol. 2 (1996), New Age International (P) Limited Publishers, New Delhi.

- [4] Andrews, G.E. and Berndt, B.C., Ramanujan's Lost Notebook, Part I, *Springer (2005)*.
- [5] Andrews, G.E. and Berndt, B.C., Ramanujans lost Note books, Part II, *Springer - New York (2009)*.
- [6] Bailey W.N., Identities of the Rogers-Ramanujan type, *Proc. London Math. Soc 1949; 50 (2): 1-10*.
- [7] Bailey W.N., Generalized hypergeometric series, *Cambridge University Press, Cambridge (1935)*.
- [8] Bailey W.N., Some theorems concerning products of hypergeometric series, *Proc. London, Math. Soc.(2) 38 (1935), 377-84*.
- [9] Denis R.Y., Singh S.N., Singh S. P.; On Extended Bailey's transform, *South East Asian J. of Math. And Math. Sc. 8(2009)59-66*.
- [10] Denis R.Y., Singh S.N., Singh S. P.; On certain transformation formulae for abnormal q- series, *South East Asian J. of Math. And Math. Sc. 3(2003)7-19*.
- [11] Gasper G & Rahman M. Basic hypergeometric series, Cambridge University Press, *New York*, 1991; pp; 4,125,137,239.
- [12] Joshi C.M. and Vyas Y.; Extensions of Bailey's transformation and application, *Inter. J. of Math. And Math. Sc.12(2005)1990-1923*.
- [13] Ramanujan, S., The 'Lost' Notebook and other unpublished papers, *Narosa, New Delhi, 1988*.
- [14] Singh, S.N., Basic Hypergeometric series and continued fractions, *Math. Student, 56 (1988), Nos. 1-4, 91-96*.
- [15] Slater, L.J., Further identities of the Roger-Ramanujan type, *Proce. London Math. Soc. 54 (1951-52), 147-167*.
- [16] Slater, L.J., Generalized hypergeometric functions, *Cambridge University Press, London, 1966*.
- [17] Srivasatva, H. M. and Manocha, H. L., A treatise on generating functions, *Ellishorwood limited, Halsted Press; a division of John Wiley and Sons, New York 1984*.