

## SOME COMMON REVISED FUZZY FIXED POINT THEOREMS ON G-METRIC SPACE

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**Abstract:** The objective of this paper is to investigate and generalize fuzzy fixed point theorems within the framework of  $G$ -metric spaces a class of generalized metric spaces that provide a versatile structure for analyzing fixed point results in more flexible and abstract settings. By revisiting several classical fuzzy fixed point theorems, this study establishes new generalized versions that are specifically adapted to the inherent properties of  $G$ -metric spaces. These generalizations not only strengthen and extend existing results but also enhance their applicability to a broader spectrum of problems characterized by uncertainty and imprecision, which are frequently encountered in practical contexts. The outcomes of this research underscore the significance of integrating fuzzy logic with advanced structures in metric space theory, offering meaningful contributions to the mathematical foundations that support applications in engineering, applied sciences, and related fields.

**Keywords and Phrases:** Coincidence point, Revised fuzzy metric space,  $G$ -metric space,  $CLR_g$  property, weakly compatible maps.

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## 1. Introduction

The concept of fuzziness, introduced by Zadeh [22], laid the foundation for the systematic treatment of uncertainty and imprecision, leading to the development of fuzzy set theory as a significant area in mathematical analysis and applied sciences. Extending this framework, Kramosil and Michálek [7] proposed the notion of fuzzy metric spaces, providing a structure capable of integrating the concept of fuzziness within the topological framework of metric spaces. This extension has been instrumental in addressing problems where classical metrics fail to capture the inherent uncertainty of real-world phenomena. In the context of fixed point theory—a fundamental area in nonlinear analysis—the integration of fuzzy structures has received considerable attention. Matkowski [9] and Grabiec [4] made pioneering contributions by extending classical fixed point theorems to fuzzy metric spaces. Subsequent studies have focused on generalizing contractive conditions and exploring common fixed point results in various generalized spaces, as evident in the work of Aamri and El Moutawakil [1]. A significant advancement in the generalization of metric spaces was achieved with the introduction of  $G$ -metric spaces by Mustafa and Sims [14]. These spaces, characterized by a generalized triangular inequality involving three variables, offer a broader analytical framework compared to classical metric spaces. The flexibility of  $G$ -metric spaces has led to extensive investigations into fixed point results under diverse contractive conditions [3, 18], demonstrating their applicability in addressing convergence behaviors not adequately captured by traditional structures. Parallel developments in cone metric spaces [2, 5, 8], symmetric spaces [6], revised fuzzy metric spaces [11, 12, 20, 21], and modular revised fuzzy metric spaces [13] have further enriched the field of fixed point theory, offering alternative frameworks to handle generalized notions of distance and contraction. The work of Muraliraj and Thangathamizh [10, 11, 12, 13, 20] is particularly notable for advancing the theoretical underpinnings of revised fuzzy spaces and establishing their relevance in modeling imprecision more effectively than classical approaches. Additionally, recent research has extended fuzzy fixed point theory to more specialized structures, including fuzzy 2-normed linear spaces [16], 3-metric spaces [17], and relation-theoretic frameworks [12], thereby broadening its scope and enhancing its applicability to complex analytical problems. Motivated by these developments, the present study aims to investigate and extend fuzzy fixed point theorems within the framework of  $G$ -metric spaces. By systematically revisiting existing results in fuzzy fixed point theory [4, 19] and incorporating the unique structural characteristics of  $G$ -metric spaces, this work establishes new generalized theorems that not only refine classical outcomes but also address broader classes of problems involving uncertainty and imprecision. The

results presented herein underscore the significance of integrating fuzzy logic with advanced metric structures, offering theoretical insights with practical relevance to mathematics, engineering, and applied sciences.

The preliminary definitions needed for the primary results are listed below.

**Definition 1.1.** ([18]) *If  $X$  is an arbitrary non-empty set,  $*$  is a continuous  $t$ -norm, and  $G$  is a fuzzy set on  $X^3 \times (0, \infty)$  that satisfies the following conditions for any  $t, s > 0$  then the 3-tuple  $(X, G, *)$  is referred to as a  $G$ -fuzzy metric space.*

- (i)  $G(p, p, q, t) > 0$  for all  $p, q \in X$  with  $p \neq q$ .
- (ii)  $G(p, p, q, t) \geq G(p, q, r, t)$  for all  $p, q, r \in X$  with  $q \neq r$ .
- (iii)  $G(p, q, r, t) = 1$  if and only if  $p = q = r$ .
- (iv)  $G(p, q, r, t) = G(n(p, q, r), t)$ , where  $n$  is permutation function.
- (v)  $G(p, q, r, t + s) \geq G(x, q, c, t) * (p, x, x, s)$  for all  $p, q, c, x \in X$ .
- (vi)  $G(p, q, r, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.

**Definition 1.2.** ([18]) *If for every  $x, y \in X$  and every  $t > 0$ ,  $G(x, x, y, t) = G(x, y, y, t)$ , then a  $G$ -fuzzy metric space  $(X, G, *)$  is symmetric.*

**Definition 1.3.** ([18]) *If  $\lim_{n \rightarrow \infty} M(a_n, ga_n, t) = 1$  for some  $t \in X$ , then a pair of self-mapping  $(k, l)$  of a fuzzy metric space  $(X, M, *)$  is said to satisfy the E.A property.*

**Definition 1.4.** ([18]) *Given four self mappings defined on symmetric spaces  $(X, d)$ , let them be  $k, l, \chi$ , and  $\eta$ . If  $\exists$  two sequences  $\{a_n\}$  &  $\{b_n\}$  in  $X$  such that  $\lim_n ka_n = \lim_n \chi a_n = \lim_n kb_n = \lim_n \eta b_n = t$  with  $t = \chi u = \eta w$  for some  $t, u, w \in X$  then, the pairs  $(k, \chi)$  and  $(l, \eta)$  are said to have the common limit range property (w.r.t  $\chi$  and  $\eta$ ), sometimes indicated by  $CLR_{(\chi, \eta)}$  property.*

*If  $k = l$  and  $\chi = \eta$ , therefore,  $CLR_{(\chi, \eta)}$  property is implied by the preceding definition due to Sintunavarat et al. [19].*

**Definition 1.5.** ([20]) *If the following criteria are satisfied, the binary operation  $\oplus : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a  $t$ -conorm:*

- (i)  $\oplus$  is commutative and associative,
- (ii)  $\oplus$  is continuous,
- (iii)  $x \oplus 0 = x \forall x \in [0, 1]$ ,
- (iv)  $x \oplus y \leq z \oplus v$  whenever  $x \leq z$  and  $y \leq v \forall x, y, z, v \in [0, 1]$ .

**Example 1.1.** ([20])

- (i) Lukasiewicz  $t$  – conorm :  $x \oplus y = \max\{x, y\}$ ;
- (ii) Product  $t$  – conorm :  $x \oplus y = x + y - xy$ ;
- (iii) Minimum  $t$  – conorm :  $x \oplus y = \min(x + y, 1)$ .

**Definition 1.6.** ([20]) A revised fuzzy metric space is an ordered triple  $(X, \mu, \oplus)$  such that  $X$  is a non empty set,  $\oplus$  is continuous  $t$  – conorm and  $\mu$  is a Revised fuzzy set on  $\mu : X^2 \times R^+ \rightarrow [0, 1]$  satisfies the following conditions:

- (1)  $\mu(x, y, t) < 1$ ;
- (2)  $\mu(x, y, t) = 0$  if and only if  $x = y$ ;
- (3)  $\mu(x, y, t) = \mu(y, x, t)$ ;
- (4)  $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$ ;
- (5)  $\mu(x, y, -) : (0, \infty) \rightarrow [0, 1]$  is continuous for all  $x, y, z \in X$  and  $t, s \in R^+$ .  
Then  $\mu$  is called a Revised fuzzy metric on  $X$ .

**Example 1.2.** ([21]) Let  $(X, d)$  be a metric space. Define  $a \oplus b = \max\{a, b\}$  for all  $a, b \in [0, 1]$ , and define  $\mu : X \times X \times (0, \infty) \rightarrow [0, 1]$  as

$$\mu(x, y, t) = \frac{d(x, y)}{1 + d(x, y)}$$

$\forall x, y, z \in X$  and  $t > 0$ . Then  $(X, \mu, \oplus)$  is a Revised fuzzy metric space.

## 2. Main Results

**Definition 2.1.** A revised fuzzy  $G$  – metric space is an ordered triple  $(X, G, \oplus)$  such that  $X$  is a non empty set,  $\oplus$  is a continuous  $t$  – conorm and  $G$  is a Revised fuzzy set on  $X^3 \times (0, \infty)$  satisfying the following conditions for each  $t, s > 0$ .

- (i)  $G(p, p, q, t) > 0 \forall p, q \in X$  with  $p \neq q$ .
- (ii)  $G(p, p, q, t) \geq G(p, q, r, t) \forall p, q, r \in X$  with  $q \neq r$ .
- (iii)  $G(p, q, r, t) = 1$  if and only if  $p = q = r$ .
- (iv)  $G(p, q, r, t) = G(n(p, q, r), t)$ , where  $n$  is permutation function.

(v)  $G(p, q, r, t + s) \geq G(x, q, r, t) * (p, x, x, s)$  for all  $p, q, r, x \in X$ .

(vi)  $G(p, q, r, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.

Then  $G$  is called a Revised fuzzy  $G$  – metric on  $X$ .

**Example 2.1.** Let  $(X, d)$  be a metric space. Define  $G : X \times X \times X \times (0, \infty) \rightarrow [0, 1]$  as

$$G(p, q, r, t) = \frac{\max\{d(p, q), d(q, r), d(r, p)\}}{1 + \max\{d(p, q), d(q, r), d(r, p)\}}$$

$\forall p, q, r \in X$  and  $t > 0$ . Then  $(X, G, \oplus)$  is a Revised fuzzy  $G$  – metric space.

**Definition 2.2.** A Revised fuzzy  $G$  – metric space  $(X, G, \oplus)$  is said to be symmetric if  $G(x, x, y, t) = G(x, y, y, t)$  for all  $x, y \in X$  and for each  $t > 0$ .

The first result is typical. Using a generalized strict contractive condition, Theorem 2.1 of ([18]) is extended to yield a revised fuzzy fixed point for a pair of self maps.

**Theorem 2.1.** Let  $(X, G, \oplus)$  be a symmetric Revised fuzzy  $G$  – metric space & let  $k, l$ , the two weakly compatible self maps on  $X$  satisfying

(i)  $CLR_{\chi, \eta}$  property.

$$(ii) \quad G(kx, ky, kz, ft) < \max \left\{ G(lx, ly, lz), \frac{G(kx, lx, lx, t) + G(ky, ly, ly, t) + G(kz, lz, lz, t)}{3} \right. \\ \left. \frac{G(ky, lx, lx, t) + G(kz, ly, ly, t)G(kx, lz, lz, t)}{3} \right\}$$

$\forall x, y, z$  in  $X$  with  $x$  not equals  $y$ . Thus  $k$  &  $l$  have a unique common Revised fuzzy fixed point.

**Proof.** As  $k$  and  $l$  satisfies  $CLR_{(\chi, \eta)}$  property,  $\exists$  a sequence  $\{a_n\}$  in  $X$  so that  $\lim_n ka_n = \lim_n la_n = la$  for some  $a \in X$ . Consider

$$G(ka_n, kx, kx, ft) < \max \left\{ G(la_n, lx, lx), \right. \\ \frac{G(ka_n, la_n, la_n, t) + G(kx, lx, lx, t) + G(kx, lx, lx, t)}{3}, \\ \left. \frac{G(kx, la_n, la_n, t) + G(kx, lx, lx, t) + G(kx, lz, lz, t)}{3} \right\}$$

Letting  $n \rightarrow \infty$ , we obtain  $G(lx, kx, kx) \leq \frac{2}{3}G(lx, kx, kx)$  which implies

$$lx = kx$$

As a result,  $x$  is where  $k$  and  $l$  coincide. Assume that  $z = kx = lx$ . Given the weak compatibility of  $(k, l)$ , we get  $kz = klz = lkz = lz$ . We are going to prove that  $kz$  equals  $z$ . If  $kz \neq z$ , then

$$\begin{aligned} G(kz, z, z, ft) &= G(kz, kx, kx, ft) \\ &< \max \left\{ G(kz, kx, kx, t), \frac{G(kx, lx, lx, t) + G(ky, ly, ly, t) + G(kz, lz, lz, t)}{3}, \right. \\ &\quad \left. \frac{G(ky, lx, lx, t) + G(kz, ly, ly, t) + G(kx, lz, lz, t)}{3} \right\} \\ &< G(kz, z, z, t). \end{aligned}$$

which is a contradiction. Hence  $kz = c = lz$ . Therefore the common Revised fixed point of  $k$  and  $l$  is  $z$ .

It is simple to prove the fixed point's uniqueness. This theorem can be demonstrated with the help of an example.

**Example 2.2.** Let  $X = [2, 20]$  and  $G : X^3 \times (0, \infty) \rightarrow (0, \infty)$  be defined as  $G(x, y, z, t) = \max\{x, y, z\}$  in all other situations and as  $G(x, y, z, t) = 0$  if  $x = y = z$ . Then a Revised fuzzy  $G$ -metric space is  $(X, G, \oplus)$ . Given two self maps on  $X$ ,  $k$  and  $l$ . Define

$$\begin{aligned} kx &= 5 \text{ if } x \leq 5, kx = 3 \text{ if } x > 5 \text{ and} \\ lx &= \frac{x+5}{2} \text{ if } x \leq 5, lx = 10 \text{ if } x > 5. \end{aligned}$$

Thus,  $k$  &  $l$  fulfill the common limit range property (w.r.t  $l$ ). Now let us consider a sequence

$$\{a_n\} = \left\{5 - \frac{1}{n}\right\} \text{ for all } n.$$

Then

$$ka_n = k\left(5 - \frac{1}{n}\right) \rightarrow 5 \text{ and } la_n = l\left(5 - \frac{1}{n}\right) = \frac{5 - \frac{1}{n} + 5}{2} \rightarrow 5$$

Therefore  $\lim_n ka_n = \lim_n la_n = 5 = l5$ . Further  $(k, l)$  are weakly compatible and

$$\begin{aligned} G(kx, ky, kz, ft) &< \max \left\{ G(lx, ly, lz, t), \frac{G(kx, lx, lx, t) + G(ky, ly, ly, t) + G(kz, lz, lz, t)}{3}, \right. \\ &\quad \left. \frac{G(ky, lx, lx, t) + G(kz, ly, ly, t) + G(kx, lz, lz, t)}{3} \right\} \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied by  $k$  and  $l$  and have a unique Common Revised fuzzy fixed point at  $x$  equals 5. Mathkowski [9] presented

the  $\Phi$  – map as the following in 1997. Assume that  $\Phi$  is the collection of auxiliary function  $\phi$  that satisfy  $\lim_n \phi^n(t) = 0$  for any  $t \rightarrow (0, \infty)$  and that  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a non decreasing function. A  $\Phi$  – map is defined as follows if  $\phi \in \Phi$ , then  $\phi(t) < t$  for every  $t \in (0, \infty)$  and if  $\phi(0) = 0$ . The following result is the extraction of unique common fixed point under Lipchitz kind of contraction for two self mapping involving  $\phi$  – map. Theorem 2 of [5] is extended and generalized by this conclusion.

**Theorem 2.2.** *Let  $(G, X, \oplus)$  be a symmetric Revised fuzzy  $G$  – metric space and the four self mappings  $k, l, \chi, \eta$  on  $X$  satisfy the following;*

- (i)  $(k, \chi)$  and  $(l, \eta)$  satisfies  $CLR_{(\chi, \eta)}$  property
- (ii)  $G(kx, ly, lz, ft) \leq \phi(\max\{G(\chi x, \eta y, \eta z, t), G(\chi x, \eta y, lz, t)G(\eta y, ly, lz, t), G(ly, \eta y, \eta z, t)\}) \forall x, y, z$  in  $X$ .
- (iii)  $(k, \chi)$  and  $(l, \eta)$  are compatibility weak.

Then, there is a unique Revised fuzzy common fixed point for  $k, l, \chi$  and  $\eta$ .

**Proof.** There are two sequences,  $\{a_n\}$  and  $\{b_n\}$ , since  $(k, \chi)$  and  $(l, \eta)$  satisfy the  $CLR_{(\chi, \eta)}$  property, such that

$$\lim_n ka_n = \lim_n \eta a_n = \lim_n a_n = \lim_n \eta a_n = t$$

with  $t = \chi x = \eta y$  for some  $t, x, y$  in  $X$

$$G(kx, lb_n, lb_n, ft) < \phi \left( \max \left\{ G(\chi x, \eta b_n, \eta b_n, t), G(\chi x, lb_n, lb_n, t), G(\eta b_n, lb_n, lb_n, t)G(lb_n, \eta b_n, \eta b_n, t) \right\} \right).$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} G(kx, v, v, t) &\leq \phi(0) = 0 \\ \implies kx &= v = \chi x \end{aligned}$$

Thus the coincidence point of  $k$  and  $\chi$  is  $x$ . As  $(k, \chi)$  are weakly compatible,

$$\implies k kx = k \chi x = \chi kx = \chi \chi x.$$

Now we prove that  $\eta y = ly$ . Consider

$$G(kx, ly, ly, ft) < \phi \left( \max \left\{ G(\chi a_n, \eta y, \eta y, t), G(\chi x, lb_n, lb_n, t)G(\eta b_n, lb_n, lb_n, t), G(lb_n, \eta b_n, \eta b_n, t) \right\} \right)$$

On letting  $n \rightarrow \infty$ , we have

$$G(v, ly, ly, t) \leq \phi(G(v, ly, ly, t))$$

which implies  $G(v, ly, ly, t) = 0$ . Therefore  $ly = v = \eta y$ , that is the coincidence point of  $l$  and  $\eta$  is  $y$ . As  $(l, \eta)$  are compatibility weak, we prove that  $lly = l\eta y = \eta ly = \eta\eta y$ . Also note that  $kx = \chi x = lx = \eta x = v$ . Now we prove that  $kkx = kx$ . Suppose  $kx \neq kkx$ , then

$$\begin{aligned} G(kkx, kx, kx, ft) &= G(kkx, ly, ly, t) \\ &\leq \phi\left(\max\left\{G(\chi kx, \eta y, \eta y, t), \right. \right. \\ &\quad \left. \left. G(\chi kx, ly, ly, t), G(\chi y, ly, ly, t), G(ly, \eta y, \eta y, t)\right\}\right) \\ &< G(kkx, kx, kx, t) \end{aligned}$$

which is a contradiction. Hence  $kkx = kx = \chi kx$ ,  $\Rightarrow kx$  is the common Revised fuzzy fixed point of  $k$  and  $\chi$ . Similarly it can be proved that  $ly$  is the common Revised fuzzy fixed point of  $l$  and  $\eta$ . The common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$  is  $kx = ly, z = kx$ . It is easy to see that the Revised fuzzy fixed point is unique.

**Corollary 2.1.** *Let  $(X, G, \oplus)$  be a symmetric Revised fuzzy  $G$  – metric space and  $k, l, \chi, \eta$  for four self mappings on  $X$  such that*

- (i)  $(k, \chi)$  and  $(l, \eta)$  satisfies  $CLR_{(\chi, \eta)}$  property.
- (ii)  $G(kx, ly, lz, ft) \leq \phi\left(\max\left\{G(\chi x, \eta y, \eta y, t), G(\chi x, ly, ly, t) \right. \right. \\ \left. \left. G(\eta y, ly, ly, t), G(ly, \eta y, \eta y, t)\right\}\right) \quad \forall x, y, z \text{ in } X.$
- (iii)  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible.

Then  $k, l, \chi$  and  $\eta$  have a unique Revised fuzzy common fixed point.

**Proof.** With appropriate restrictions on  $k, l, \chi, \eta$  the following corollaries involving two and three self mappings can be obtained.

**Corollary 2.2.** *Consider a symmetric Revised fuzzy  $G$  – metric space,  $(X, G, \oplus)$  and consider three self mappings  $k, l, \chi$  on  $X$  such that*

- (i)  $(k, \chi)$  and  $(l, \chi)$  satisfies  $CLR_{\chi}$  property.
- (ii)  $G(kx, ly, lz, ft) \leq \phi(\max\{G(\chi x, \chi y, \chi z, t), G(\chi x, ly, lz, t)G(\chi y, ly, lz, t), \\ G(ly, \chi y, \chi z, t)\}) \quad \forall x, y, z \text{ in } X.$

(iii)  $(k, \chi)$  and  $(l, \eta)$  are compatibility weak.

**Proof.** By following from Theorem 2.2 and setting  $\chi = \eta$  follows from Theorem 2.2, we prove this theorem with the example as follows.

**Example 2.3.** Let  $X = [0, 6]$  and  $G : X^3 \times (0, \infty) \rightarrow (0, \infty)$  be defined as  $G(x, y, z, t) = \max\{x, y, z\}$  in all other situations and as  $G(x, y, z, t) = 0$  if  $x = y = z$ .  $(X, G, \oplus)$  is thus obviously a symmetric Revised fuzzy  $G$ -metric space. Let  $k, l, \chi, \eta$  be four self mapping on  $X$  defined by:

$$\begin{aligned} kx &= 3 \text{ if } x \leq 3, \quad kx = 4 \text{ if } x > 3 \text{ and} \\ \chi x &= 6 - x \text{ if } x \leq 3, \quad \chi x = 5 \text{ if } x > 3, \\ lx &= 4 \text{ if } x < 3, \\ \eta x &= \frac{2x + 3}{3} \text{ if } x \geq 3 \end{aligned}$$

The  $CLR_{(\chi, \eta)}$  property is now satisfied by  $(k, \chi)$  and  $(l, \eta)$ . Choose two sequences to view this

$$\{a_n\} = \{3 - \frac{1}{n}\} \text{ for all } n.$$

$$\text{Then } ka_n = k(3 - \frac{1}{n}) \text{ for all } n.$$

$$\text{Then } ka_n = k(3 - \frac{1}{n}) \rightarrow 3$$

$$\chi a_n = \chi(3 - \frac{1}{n}) = 6 - (3 - \frac{1}{n}) \rightarrow 3$$

$$la_n = l(3 - \frac{1}{n}) = \frac{3 + \frac{1}{n} + 3}{2} \rightarrow 3 \text{ and}$$

$$\eta a_n = \eta(3 - \frac{1}{n}) = \frac{2(3 - \frac{1}{n}) + 3}{2} \rightarrow 3.$$

Therefore  $\lim_n ka_n = \lim_n \chi a_n = \lim la_n = \lim_n \eta a_n = t = 3$  with  $3 = \chi 3 = \eta 3$ . Also  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible. Let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a function defined by  $\phi(v) = \frac{v}{2}$ , for all  $v \in [0, \infty)$ . Then  $\phi(0) = 0$  and  $0 < \phi(v) < v$ , for all  $v \in (0, \infty)$ . Further,

$$\begin{aligned} G(kx, ly, lz, ft) &< \phi \left( \max \left\{ G(\chi x, \eta y, \eta y, t), \right. \right. \\ &\quad \frac{G(\chi x, ly, lz, t) + G(\eta y, lb, lz, t) + G(lz, \eta y, \eta z, t)}{3}, \\ &\quad \left. \left. \frac{G(\eta y, ly, lz, t) + G(kz, ly, ly, t) + G(kx, lz, lz, t)}{3} \right\} \right) \end{aligned}$$

Therefore, all of the requirements of Theorem 2.2 are found, and  $x = 3$  is the unique common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$ .

The following two theorems generalize the findings in Theorems 2.8 [2], Theorem 2.2 [2], and Theorem 3.11 [5] and deal with Hardy Roger's type of contractive condition for two pairs of self maps.

**Theorem 2.3.** *Let  $(X, G, \oplus)$  be a Revised fuzzy  $G$  – metric space & let  $k, l, \chi, \eta$ , the four self map on  $X$  such that*

(i)  $(k, \chi) \& (l, \eta)$  satisfies the common limit range property (w.r.t  $\chi$  and  $\eta$ ).

(ii)  $G(kx, ly, lz, ft) \leq pG(kx, \chi x, \chi x, t) + qG(\chi x, \eta y, \eta y, t) + rG(\eta y, lz, lz, t) + w[G(kx, \eta y, \eta y, t) + G(\chi x, ly, lz, t)] \quad \forall x, y, z \text{ in } X$

where  $p, q, r, w \in [0, 1)$  which satisfy sum of  $p, q, r$  &  $2w$  is less than 1. Then in  $X$ ,  $(k, \chi)$  and  $(l, \eta)$  have unique point of coincidence. In addition,  $(k, \chi)$  and  $(l, \eta)$  have a weak compatibility, then Revised fuzzy common fixed point is unique for  $k, l, \chi$  and  $\eta$ .

**Proof.** As  $(k, \chi)$  and  $(l, \eta)$  satisfies  $CLR_{(\chi, \eta)}$  property, there are two sequences  $\{a_n\}$  and  $\{b_n\}$  such that

$$\lim_n ka_n = \lim_n \chi a_n = t = \lim_n la_n = \lim_n \eta a_n = t$$

and also  $t = \chi x = \eta y$  for some  $t, x$  in  $X$ . Let

$$G(kx, lb_n, lb_n, ft) < pG(kx, \chi x, \chi x, t) + qG(\chi x, \eta b_n, \eta b_n, t) + rG(\eta b_n, lb_n, lb_n, t) + w[G(kx, \eta b_n, \eta b_n, t) + G(\chi x, lb_n, lb_n, t)]$$

On letting  $n \rightarrow \infty$ , we get

$$[1 - (p + w)]G(kx, e, e, t) \leq 0$$

which gives  $kx = e = \chi x$ , since  $p + q + r + 2w < 1$ . Thus,  $X$  is where  $k$  and  $\chi$  coincide. In the same way,  $y$  is where  $l$  and  $\eta$  coincide. Thus  $e = kx = \chi x = ly = \eta y$ .

**Uniqueness of coincidence point.**

Let us consider the two points of coincidence of  $(k, \chi)$  and  $(l, \eta)$  be  $e_1$  and  $e_2$ .

$$e_1 = kx_1 = \chi x_1 = ly_1 = \eta y_1$$

and

$$e_2 = kx_2 = \chi x_2 = ly_2 = \eta y_2.$$

Consider

$$\begin{aligned} G(e_1, e_1, e_2, t) &= G(kx_1, ly_2, ly_2, t) \\ &\leq pG(kx_1, \chi x_1, \chi x_1, t) + qG(\chi x_1, \eta y_2, \eta y_2, t) + rG(\eta y_2, ly_2, ly_2, y, t) \\ &\quad + w[G(kx_1, \eta y_2, ly_2, t) + G(\chi x_1, ly_2, ly_2, t)] \end{aligned}$$

which implies  $[1 - (q + 2w)]G(e_1, e_1, e_2, t) \leq 0$  that is  $e_1 = e_2$ . Since  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible, we have

$$kkx = k\chi x = \chi kx = \chi\chi x$$

and

$$lly = l\eta y = \eta ly = \eta\eta y.$$

Now we prove that  $kkx = kx$ . Consider,

$$\begin{aligned} G(kkx, , kx, kx, t) &= G(kkx, ly, ly, t) \\ &\leq pG(kkx, \chi kx, \chi kx, t) + qG(\chi kx, \eta y, \eta y, t) + rG(\eta y, ly, ly, t) \\ &\quad + w[G(kkx, \eta y, \eta y, t) + G(\chi kx, ly, ly, t)] \end{aligned}$$

Which gives

$$[1 - (q + 2w)]G(kkx, kx, kx, t) \leq 0.$$

Hence  $kx = kkx = \chi kx$ . That is the common Revised fuzzy fixed point of  $k$  and  $\chi$  is  $kx$ . Likewise we can prove that the common Revised fuzzy fixed point of  $l$  and  $\eta$  is  $ly$ . Therefore the common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$  is  $z = kz$ . We can prove the uniqueness of the Revised fuzzy fixed point easily.

**Example 2.4.** Consider  $X = [0, 4]$  and  $G : X^3 \times (0, \infty) \rightarrow [0, \infty)$  defined as

$$G(x, y, z, t) = \max \left\{ d(x, y, t), d(y, z, t), d(z, x, y) \right\},$$

where  $d$  is the usual metric on  $X$ . Then obviously  $(X, G, \oplus)$  is a Revised fuzzy  $G$ -metric space. Let  $k, l, \chi, \eta$  be four self mapping on  $X$  defined by

$$kx = 2, \chi x = 4 - x, lx = \frac{5x + 12}{11}$$

and

$$\eta x = \frac{x + 2}{2}$$

for all  $x \in X$ . The  $CLR_{(\chi, \eta)}$  property now is satisfied by  $(k, \chi)$  and  $(l, \eta)$ . To see this, consider two sequence  $\{a_n\} = \{2 + \frac{1}{n}\}$  and  $\{b_n\} = \{2 + \frac{1}{n}\}$  for all  $n$ . Then

$$ka_n = k(2 + \frac{1}{n}) \rightarrow 2, \chi a_n = \chi(2 + \frac{1}{n}) = 4 - (2 + \frac{1}{n}) \rightarrow 2,$$

$$lb_n = l(2 - \frac{1}{n}) = \frac{5(2 - \frac{1}{n}) + 12}{11} \rightarrow 2,$$

$$\eta b_n = \eta(2 - \frac{1}{n}) = \frac{2 - \frac{1}{n} + 2}{2} \rightarrow 2.$$

Therefore

$$\lim_n ka_n = \lim_n \chi a_n = \lim_n la_n = \lim_n \eta a_n = t = 2$$

with

$$2 = \chi 2 = \eta 2.$$

Further,  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible and  $k, l, \chi, \eta$  satisfy the contractive condition of equation (ii) for

$$p = \frac{1}{8}, q = \frac{1}{3}, r = \frac{1}{8}, w = \frac{1}{7}.$$

Thus, all conditions of Theorem 2.3 are satisfied and the unique common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$  is  $x = 2$ .

**Theorem 2.4.** Let  $(X, G, \oplus)$  be a Revised fuzzy  $G$ -metric space & let  $k, l, \chi, \eta$ , the four self maps on  $X$  such that

(i)  $(k, \chi) \& (l, \eta)$  satisfies  $CLR_{(\chi, \eta)}$  property.

(ii)  $G(kx, ly, lz, ft) \leq jr(x, y, z, t), j \in (0, 1)$  for all  $x, y, z \in X$  and  $r(x, y, z) \in \left\{ G(kx, \chi x, \chi x, t), G(\chi x, \eta y, \eta y, t), G(\eta y, lz, lz, t), \frac{G(kx, \eta x, \eta x, t) + G(\chi x, \eta y, \eta y, t)}{2} \right\}$ .

(iii)  $(k, \chi) \& (l, \eta)$  are compatibility weak.

Then  $k, l, \chi$  and  $\eta$  have a unique Revised fuzzy common fixed point.

**Proof.** As  $(k, \chi) \& (l, \eta)$  satisfies  $CLR_{(\chi, \eta)}$  property then there are 2 sequences  $\{a_n\}$  &  $\{b_n\}$  so that

$$\lim_n ka_n = \lim_n \chi a_n = \lim_n lb_n = \lim_n \eta b_n = t, t = \chi x = \eta x$$

for some  $t, x, y$  in  $X$ .

Let,

$$G(kx, lb_n, lb_n, ft) \leq jr(x, lb_n, lb_n, y)$$

where

$$r(x, b_n, b_n, ft) \in \left\{ G(kx, \chi x, \chi x, t), G(\chi x, \eta b_n, \eta b_n, t), G(\eta b_n, lb_n, lb_n, t), \right. \\ \left. \frac{G(kx, \eta b_n, \eta b_n, t) + G(\chi x, lb_n, lb_n, t)}{2} \right\}$$

On letting  $n \rightarrow \infty$ ,

$$G(kx, r, r, t) \leq jr(x, lb_n, lb_n, t)$$

Where

$$r(x, b_n, b_n, ft) \in \left\{ G(kx, r, r, t), \frac{G(kx, r, r, t)}{2} \right\},$$

If

$$r(x, b_n, b_n, ft) = G(kx, r, r, t)$$

then

$$G(kx, r, r, t) \leq jG(kx, r, r, t)$$

which also gives

$$kx = r = \chi x.$$

If

$$r(x, b_n, b_n, ft) = \frac{G(kx, r, r, t)}{2}$$

then

$$G(kx, r, r, t) \leq \frac{G(kx, r, r, t)}{2}$$

which also gives

$$kx = r = \chi x.$$

Hence in both cases  $kx = \chi x = t$  and therefore the coincidence point of  $k$  and  $l$  is  $x$ . Likewise the coincidence point of  $l$  and  $\eta$  is  $y$ , thus

$$r = kx = \chi x = ly = \eta y.$$

Since  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible, we have

$$kkx = k\chi x = \chi kx = \chi\chi x$$

and

$$lly = l\eta y = \eta ly = \eta\eta y.$$

Now we prove that  $kkx = kx$ . Consider

$$G(kkx, kx, kx, t) = G(kkx, ly, ly, t) \leq jr(kx, y, y, t),$$

where

$$r(kx, y, y, ft) \in \left\{ G(kkx, \chi kx, \chi kx, t), G(\chi kx, \eta y, \eta y, t), G(\eta y, ly, ly, t), \frac{G(kkx, \eta y, \eta y, t) + G(\chi kx, ly, ly, t)}{2} \right\}$$

That is

$$r(kx, y, y, ft) = G(kkx, kx, kx, t).$$

Therefore

$$G(kkx, kx, kx, t) \leq jG(kkx, kx, kx, t)$$

this implies

$$kx = kkx = \chi kx.$$

Hence the common Revised fuzzy fixed point of  $k$  and  $\chi$  is  $kx$ . Likewise we can prove that the common Revised fuzzy fixed point of  $l$  and  $\eta$  is  $ly$ . As  $kx = ly, z = kx$  is the common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$ .

We can prove the uniqueness of the Revised fuzzy fixed point easily.

**Example 2.5.** Consider  $X = [0, 4]$  and

$$G : X^3 \times (0, \infty) \rightarrow [0, \infty)$$

defined as

$$G(u, v, w, t) = \max \left\{ d(u, v, t), d(v, w, t), d(w, u, t) \right\},$$

where  $d$ , the usual metric on  $X$ . Therefore  $(X, G, \oplus)$  is a Revised fuzzy  $G$ -metric space. Consider the four self mapping on  $X$  be  $k, l, \chi, \eta$  which are defined by

$$ku = 3, \chi u = u, lu = \frac{u + 24}{9}$$

and

$$\eta u = \frac{u + 3}{2} \text{ for all } u \in X$$

Now the  $CLR_{(\chi, \eta)}$  property are satisfied by  $(k, \chi)$  and  $(l, \eta)$ .

To see this, consider two sequences

$$\{a_n\} = \left\{3 + \frac{1}{n}\right\} \text{ and } \{b_n\} = \left\{3 - \frac{1}{n}\right\} \text{ for all } n.$$

$$\text{Then } ka_n = k\left(3 + \frac{1}{n}\right) \rightarrow 3, \chi a_n = \chi\left(3 + \frac{1}{n}\right) = 3 + \frac{1}{n} \rightarrow 3$$

$$lb_n = l\left(3 - \frac{1}{n}\right) = \frac{3 - \frac{1}{n} + 24}{9} \rightarrow 3$$

$$\eta b_n = \eta\left(3 - \frac{1}{n}\right) = \frac{3 - \frac{1}{n} + 3}{2} \rightarrow 3.$$

Therefore

$$\lim_n ka_n = \lim_n \chi a_n = \lim_n la_n = \lim_n \eta a_n = t = 3$$

with

$$3 = \chi 3 = \eta 3.$$

Further,  $(k, \chi)$  and  $(l, \eta)$  are weakly compatible &  $k, l, \chi$  &  $\eta$  satisfy the contractive condition (ii) for  $j = \frac{1}{2}$ . Thus all the conditions of Theorem 2.4 are satisfied and the unique common Revised fuzzy fixed point of  $k, l, \chi$  and  $\eta$  is  $x = 2$ .

Our final end results is likewise a common revised fuzzy fixed factor theorem however expansive sort of contractive condition, which increase the effects Theorem 3.1 of [???].

**Theorem 2.5.** *Let the 2 self mapping of a Revised fuzzy  $G$ -metric space  $(X, G, \oplus)$  which satisfy the property of  $CLR_l$  be  $k, l$  and*

$$\begin{aligned} G(lx, ly, lz, ft) &\geq PG(kx, ky, kz, t) + QG(kx, lx, lx, t) \\ &+ RG(ky, ly, ly, t) + SG(kz, lz, lz, t) \end{aligned}$$

$\forall x, y, z$  in  $X$ . Where  $0 < P, Q, R, S$  then there is a coincidence point of  $k$  and  $l$ . The coincidence point is unique, if  $P > 1$ . Moreover, a unique common Revised fuzzy fixed point exists for  $k$  and  $l$  if they are weakly compatible.

**Proof.** As  $k$  and  $l$  satisfies  $CLR_l$  property, there is a sequence  $\{a_n\}$  in  $X$  such that

$$\lim_n ka_n = \lim_n la_n = lx$$

for some  $x \in X$ . Consider,

$$\begin{aligned} G(ka_n, lx, lx, ft) &\geq PG(ka_n, kx, kx, t) + PG(ka_n, la_n, la_n, t) + RG(kx, lx, lx, t) \\ &+ SG(kx, lx, lx, t) \end{aligned}$$

On letting  $n \rightarrow \infty$ , We obtain

$$\left(\frac{P}{2} + Q + R\right)G(kx, lx, lx, t) \leq 0$$

which implies  $kx = lx$ . Hence the point of coincidence of  $k$  and  $l$  is  $x$ .

**Uniqueness of coincidence point.**

Consider the of coincidence points of  $k$  and  $l$  be

$$z = kx = lx$$

and

$$r = ky = ly.$$

Then,

$$\begin{aligned} G(z, r, r, t) &= G(lx, ly, ly, t) \\ &\geq PG(kx, ky, ky, t) + QG(kx, lx, lx, t) + RG(ky, ly, ly, t) + SG(ky, ly, ly, t) \end{aligned}$$

That is

$$(P - 1)G(z, r, r, t) \leq 0.$$

As  $P > 1$ , we obtain  $z = r$ . Now  $(k, l)$  are weakly compatible

$$\Rightarrow kx = kx = lx = lx.$$

To prove that  $kx = kx$ . Consider

$$\begin{aligned} G(kx, kx, kx, t) &= G(lx, lx, lx, t) \\ &\geq PG(kx, kx, kx, t) + QG(kx, lx, lx, t) + RG(kx, lx, lx, t) + SG(kx, lx, lx, t) \end{aligned}$$

which implies

$$kx = kx = lx.$$

Hence  $kx$  is the common Revised fuzzy fixed point of  $k$  and  $l$ .

### 3. Conclusion

In this paper, the study of revised fuzzy fixed point theorems in  $G$ -metric spaces significantly advances the understanding and application of these mathematical concepts. By adapting traditional fuzzy fixed point theorems to the more complex structure of  $G$ -metric spaces, this research bridges a crucial gap in the literature, providing a more comprehensive framework for addressing problems involving uncertainty and imprecision. The enhanced theorems extend the utility of fuzzy fixed point results, making them more applicable to a wide range of fields, including mathematics, engineering, and the applied sciences. This work not only broadens the theoretical foundation but also offers practical tools for researchers and practitioners dealing with complex systems. The integration of fuzzy logic with  $G$ -metric spaces thus emerges as a powerful approach, fostering further exploration and innovation in the study of fixed point theory.

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