

Some Problems Connected with Ramanujan's Hypergeometric Series

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Dedicated to Prof. Hari M. Srivastava on his 75th birth anniversary

Abstract: Starting from Gauss hypergeometric function, which Ramanujan studied in detail, limiting forms are considered and pathways are constructed to go from a Gauss hypergeometric series to confluent hypergeometric series to binomial series to Bessel series and finally to exponential series. The path from binomial to exponential is discussed in detail. It is shown that this path is connected to reaction-rate probability integrals, non-extensive statistical mechanics, Tsallis statistics, superstatistics, Kráttel integrals, inverse Gaussian density Bayesian procedures and generalizations of these. Products of binomials forms are shown to be connected to Mellin convolutions, densities of products and ratios and finally to fractional integrals. Some aspects of the corresponding theory of functions of matrix argument are also discussed.

Keywords Hypergeometric functions, pathway models, reaction-rate probability integrals, Kráttel integrals, fractional integrals, Mellin convolutions, functions of matrix argument, densities of product and ratios.

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1. Some Classical Limiting Properties of Hypergeometric Series

One of the areas of interest for Ramanujan was Gauss hypergeometric series ${}_2F_1(a, b; c; x)$. By specializing the parameters here he obtained several interesting relationships. Several people are still working in this area of specializing parameters and constructing results. One classical property of Gauss hypergeometric series is the following:

$$\lim_{a \rightarrow \infty} {}_2F_1(a, b; c; \frac{x}{a}) = {}_1F_1(b; c; x) \quad (1.1)$$

where ${}_1F_1$ is the confluent hypergeometric function. In a similar fashion

$$\lim_{b \rightarrow \infty} {}_1F_1(b; c; \frac{x}{b}) = {}_0F_1(-; c; x) \quad (1.2)$$

which is the Bessel series.

$$\lim_{c \rightarrow \infty} {}_1F_1(b; c; cx) = {}_1F_0(b; -; x) = (1-x)^{-b} \text{ for } |x| < 1 \quad (1.3)$$

which is the binomials series. From (1.2) and (1.3) we have the following properties:

$$\begin{aligned} \lim_{b \rightarrow \infty} {}_1F_0(b; -; -\frac{x}{b}) &= \lim_{q \rightarrow 1} {}_1F_0(\frac{1}{q-1}; -; -(q-1)x) = {}_0F_0(-; -; -x) \\ &= \lim_{q \rightarrow 1} [1 + (q-1)x]^{-\frac{1}{q-1}} = e^{-x} \end{aligned} \quad (1.4)$$

Hence in (1.4) one can take the binomial function $[1 + (q-1)x]^{-\frac{1}{q-1}}$ for $q > 1$ or equivalently $[1 - (1-q)x]^{\frac{1}{1-q}}$ for $q < 1$, and these can be taken as generalizations of the exponential function e^{-x} . This path of binomial series going to an exponential series created the whole new area of non-extensive statistical mechanics. It is stated that over 5000 articles are published in this area of non-extensive statistical mechanics, giving various aspects and applications into a large number of topics, during the period 1990 to 2010. The binomial function $x^\gamma [1 + (q-1)x^\delta]^{-\frac{1}{q-1}}$, $q > 1$ is superstatistics in statistical mechanics. Superstatistics was constructed in 2003 by using the following considerations. Suppose that a physical entity x is distributed according to a generalized gamma density

$$f(x) = \frac{\delta \theta^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})} x^{\gamma-1} e^{-\theta x^\delta}, \delta > 0, \gamma > 0, \theta > 0 \quad (1.5)$$

where θ may a parameter such as temperature. The model in (1.5) may hold good only when the temperature is a constant or at given θ . Hence we may write $f(x)$ as $f(x|\theta)$ or $f(x)$ at given value of θ . Note that θ may have its own distribution. Temperature can vary. Let it be a gamma density of the form

$$g(\theta) = \frac{a^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-a\theta}, a > 0, \alpha > 0, \theta > 0. \quad (1.6)$$

Then the joint density of x and θ is available from $f(x|\theta)g(\theta)$ or one can interpret it as super-imposing the density of θ over the density of x at given θ , and thus,

superstatistics. Then the density of x , over all θ , is given by

$$\begin{aligned} f_x(x) &= \int_{\theta} f(x|\theta)g(\theta)d\theta \\ &= \frac{\delta a^{\alpha}}{\Gamma(\alpha)\Gamma(\frac{\gamma}{\delta})} x^{\gamma-1} \int_0^{\infty} \theta^{\frac{\gamma}{\delta}+\alpha-1} e^{-\theta x^{\delta}-a\theta} \\ &= \frac{\delta \Gamma(\alpha + \frac{\gamma}{\delta})}{\Gamma(\alpha)\Gamma(\frac{\gamma}{\delta})} x^{\gamma-1} [a + x^{\delta}]^{-(\frac{\gamma}{\delta}+\alpha)} \end{aligned} \quad (1.7)$$

$$= \frac{\delta \Gamma(\alpha + \frac{\gamma}{\delta})}{a^{\frac{\gamma}{\delta}} \Gamma(\frac{\gamma}{\delta}) \Gamma(\alpha)} \left[1 + \frac{x^{\delta}}{a}\right]^{-(\frac{\gamma}{\delta}+\alpha)}. \quad (1.8)$$

The structure here is something like a constant times $x^{\gamma-1} [1 + \frac{x^{\delta}}{a}]^{-(\frac{\gamma}{\delta}+\alpha)}$ which is the superstatistics. Dozens of papers are available on superstatistics also. One observation can be made here. From the integral in (1.7) it is clear that in (1.8), $a > 0, x > 0$ and $1 + \frac{x^{\delta}}{a} > 0$ or in other words, only the type-2 beta form $x^{\gamma-1} [1 + (q-1)x^{\delta}]^{-\frac{1}{q-1}}$ for $q > 1$ is available here. But in Tsallis statistics, $\gamma = 1, \delta = 1$, both the forms $[1 - (1-q)x]^{\frac{1}{1-q}}$ for $q < 1$ and $[1 + (q-1)x]^{-\frac{1}{q-1}}$ for $q > 1$ are available, as well as the limiting form e^{-x} . Also, Tsallis statistics is a power function model in the sense

$$\frac{d}{dx} [1 - (1-q)x]^{\frac{1}{1-q}} = -[f(x)]^q \quad (1.9)$$

where $f(x) = [1 - (1-q)x]^{\frac{1}{1-q}}$ for $q < 1$ or $[1 + (q-1)x]^{-\frac{1}{q-1}}$ for $q > 1$.

From a statistical point of view, superstatistics is nothing but the unconditional density in a Bayes' procedure. If the conditional density of x , given θ , or $f(x|\theta)$ is of the form in (1.5) and the marginal density of θ , $g(\theta)$, is of the form in (1.6) then the unconditional density of x is available from $\int_{\theta} f(x|\theta)g(\theta)d\theta$, which is the case in (1.8). Thus, in Bayesian procedure it is the unconditional density of x . It still remains to be seen whether there is a superstatistics of the type-1 beta form $x^{\gamma} [1 - (1-q)x]^{\frac{1}{1-q}}, q < 1$?

A general binomial form going to exponential form can be written as follows:

$$\begin{aligned} \lim_{q \rightarrow 1+} c_2 x^{\gamma} [1 + a(q-1)x^{\delta}]^{-\frac{1}{q-1}} &= \lim_{q \rightarrow 1-} c_1 x^{\gamma} [1 - a(1-q)x^{\delta}]^{\frac{1}{1-q}} \\ &= c_3 x^{\gamma} e^{-ax^{\delta}}, a > 0, \delta > 0, \gamma > 0, x > 0 \end{aligned} \quad (1.10)$$

where c_1, c_2, c_3 are constants. If c_1, c_2, c_3 are normalizing constants in the sense

$$\int_x f_1(x) dx = \int_0^{[a(1-q)]^{-\frac{1}{\delta}}} c_1 x^{\gamma} [1 - a(1-q)x^{\delta}]^{\frac{1}{1-q}} dx = 1, \quad (1.11)$$

for $q < 1, a > 0, \delta > 0, 0 < x < [a(1 - q)]^{-\frac{1}{\delta}}$;

$$\int_x f_2(x) dx = \int_0^\infty c_1 x^\gamma [1 + a(q - 1)x^\delta]^{-\frac{1}{q-1}} dx = 1, \quad (1.12)$$

for $q > 1, a > 0, \delta > 0, x \geq 0$;

$$\int_x f_3(x) dx = \int_0^\infty c_3 x^\gamma e^{-ax^\delta} dx = 1, a > 0, \delta > 0, x \geq 0, \quad (1.13)$$

then $f_1(x), f_2(x), f_3(x)$ are statistical densities. In fact, f_2 and f_3 are available from f_1 . By writing $(1 - q) = -(q - 1)$ for $q > 1$ one gets f_2 from f_1 . By taking the limit $q \rightarrow 1$ in f_1 and f_2 one gets f_3 . This f_1 is a particular case of the pathway model introduced by this author in Mathai (2005). From (1.13) we can observe the following: a product of ${}_1F_0$'s in the limiting case is the following:

$$x^\gamma \prod_{j=1}^2 \lim_{q_j \rightarrow 1} [1 + a_j(q_j - 1)x^{\delta_j}]^{-\frac{1}{q_j-1}} = x^\gamma e^{-a_1 x^\delta - a_2 x^{-\rho}} \quad (1.14)$$

for $a_j > 0, q_j > 1, j = 1, 2, \delta_1 = \delta > 0, \delta_2 = -\frac{1}{\rho}, \rho > 0$.

1.1. The pathway model

The pathway model introduced in Mathai (2005) is for rectangular matrix-variate case. This for the real scalar positive variable $x > 0$ is the following, denoted again by $f_1(x)$.

$$f_1(x) = c_1 x^\gamma [1 - a(1 - q)x^\delta]^{\frac{1}{1-q}}, q < 1. \quad (1.15)$$

This for $q > 1$ and $q \rightarrow 1$ will go to (1.12) and (1.13) and hence all the forms of (1.11), (1.12) and (1.13) are contained in (1.15).

In physical sciences and engineering, usually a concept is introduced by optimizing an entropy measure. Entropy is a measure of uncertainty or its complement is information content. This author has shown that (1.15) can be obtained by optimizing Mathai's entropy, subject to two moment-type constraints. Mathai's entropy for the continuous case is the following:

$$M_\alpha(f) = \frac{\int_x [f(x)]^{2-\alpha} dx - 1}{\alpha - 1}, \alpha \neq 1, \alpha < 2 \quad (1.16)$$

where $f(x)$ is a density and x could be scalar, vector or matrix variable. Let us consider the real scalar variable case and let us optimize (1.16) subject to the conditions

$$\int_x x^{\gamma(1-\alpha)} f(x) dx = \text{fixed and } \int_x x^{\gamma(1-\alpha)+\delta} f(x) dx = \text{fixed.} \quad (1.17)$$

Then, if we are using calculus of variations to optimize (1.16) over all functional f , subject to (1.17), then the Euler equation reduces to the following:

$$\frac{\partial}{\partial f} [f^{2-\alpha} - \lambda_1 x^{\gamma(1-\alpha)} f + \lambda_2 x^{\gamma(1-\alpha)+\delta} f] = 0$$

for some Lagrangian multipliers λ_1 and λ_2 . This gives

$$f^{1-\alpha} = \delta_1 x^{\gamma(1-\alpha)} - \delta_2 x^{\gamma(1-\alpha)+\delta} = x^{\gamma(1-\alpha)} [\delta_1 - \delta_2 x^\delta]$$

for some δ_1 and δ_2 . This yields

$$f = x^\gamma [\delta_1 - \delta_2 x^\delta]^{\frac{1}{1-\alpha}} = \lambda x^\gamma [1 - a(1-q)x^\delta]^{\frac{1}{1-q}} \quad (1.18)$$

for $\frac{\delta_2}{\delta_1} = a(1-\alpha)$, $\delta_1^{\frac{1}{1-\alpha}} = \lambda$. This (1.18) is the pathway model for $\delta > 0, a > 0, 1 - a(1-q)x^\delta > 0, q < 1, x > 0$. All the three forms are available from (1.18) for $q < 1, q > 1$, and $q \rightarrow 1$. Particular cases of (1.18) are available by taking particular values of the parameters in (1.18). For example, for $\gamma = 0, \delta = 1$ the two moment type restrictions become

$$\int_x f(x) dx = 1 \text{ and } \int_x x f(x) dx = \text{fixed.} \quad (1.19)$$

The first condition, together with $f(x) \geq 0$ for all x , indicates that $f(x)$ is a density and the second condition says that the first moment is fixed. This in terms of energy distribution is the principle of conservation of total energy. Under (1.19), Mathai's entropy (1.16) gives Tsallis statistics of non-extensive statistical mechanics for all the cases of $q < 1, q > 1, q \rightarrow 1$ directly, without going through any escort density as done by workers in statistical mechanics.

Through the entropy measure (1.16) one can go from $f_1(x)$ to $f_2(x)$ and $f_3(x)$. This is called the entropic pathway. When the constants c_1, c_2, c_3 in $f_1(x), f_2(x), f_3(x)$ are the normalizing constants then $f_j(x), j = 1, 2, 3$ are statistical densities. We can go from any one form to the other two forms. This is the distributional pathway.

We can construct the corresponding differential equations which will give a differential pathway. Thus, we have entropic, distributional and differential pathways in (1.15).

2. Connections to Various Problems in Different Areas

The left-side function in (1.14), before taking limits, is a generalization of the right side of (1.14). If the right side is the ideal situation or the stable situation in a physical system then the left side gives the unstable neighborhoods or the path leading to the stable situation. Consider the integral

$$A = c \int_0^{\infty} x^{\gamma} e^{-a_1 x^{\delta} - a_2 x^{-\rho}} dx \quad (2.1)$$

where c is a constant.

Reaction-rate probability integral For $\delta = 1, \rho = \frac{1}{2}$, (2.1) is the reaction-rate probability integral in nuclear reaction-rate theory. Hence (2.1) is the generalized reaction-rate probability integral. No physical interpretation may be currently available for this generalized integral but the theory is there so that new physics or new interpretation can be worked out. In a series of papers Haubold and Mathai, starting from 1984, have shown that various types of reactions, in the resonant case, non-resonant case, depleted case, high energy tail cut off case etc, are all associated with various special cases of (2.1). A summary of the work until 1988 may be seen from Mathai and Haubold (1988).

Krätzel integral and Krätzel transform For $\delta = 1, \rho = 1$, (2.1) gives the basic form of Krätzel integral in applied analysis and hence (2.1) gives the generalized Krätzel integral. There is a Krätzel transform associated with Krätzel integral, see Mathai (2012) for some details.

Inverse Gaussian density For $\delta = 1, \rho = 1$ the integrand in (2.1), normalized, is the inverse Gaussian density. Hence (2.1) will correspond to arbitrary moments in inverse Gaussian density. Inverse Gaussian is a very popular model in stochastic process and statistics.

Bayesian analysis Let $a_2 = y$, then y^n times (2.1) has the following structure: Let the conditional density of y , at given x , be of the form

$$f_1(y|x) = c_1 y^n e^{-\frac{y}{x^{\rho}}}, y > 0, x > 0, \rho > 0$$

and zero elsewhere. and the marginal density of x be of the form

$$f_2(x) = c_2 x^{\gamma} e^{-a_1 x^{\delta}}, a_1 > 0, \delta > 0, x > 0$$

and zero elsewhere, where c_1 and c_2 are the normalizing constants. Then the joint density of x and y is of the form $f_1(y|x)f_2(x)$ and the unconditional density of y is available by integrating out x , which is nothing but (2.1) multiplied by y^η . Hence (2.1) can also be taken as the unconditional density in a Bayesian structure or in Bayesian analysis.

Mellin convolution of a product One can take (2.1) as the Mellin convolution of a product also. Consider the functions $g_1(x_1) = d_1 e^{-a_1 x_1^\delta}$, $a_1 > 0, x_1 > 0, \delta > 0$ and let $g_2(x_2) = d_2 e^{-a_2 x_2^\rho}$, $x_2 > 0, \rho > 0, a_2 > 0$ where d_1 , and d_2 are constants. Then, the Mellin convolution of a product of g_1 and g_2 , namely

$$\int_0^\infty u^{s-1} \left[\int_{v=0}^\infty \frac{1}{v} g_1(v) g_2\left(\frac{u}{v}\right) dv \right] = M_{g_1}(s) M_{g_2}(s) \quad (2.2)$$

where $M_{g_1}(s)$ and $M_{g_2}(s)$ are the Mellin transforms of g_1 and g_2 respectively, or

$$M_{g_1}(s) = \int_0^\infty x_1^{s-1} g_1(x_1) dx_1 \text{ and } M_{g_2}(s) = \int_0^\infty x_2^{s-1} g_2(x_2) dx_2 \quad (2.3)$$

whenever the integrals exist, where s is the Mellin parameter, which is usually a complex variable. When g_1 and g_2 are generalized gamma functions, as described above, we have the integrand on the left side of (2.2) or the integral in (2.1).

Density of a product The above idea of Mellin convolution of a product can be easily interpreted in terms of a statistical problem. Let $x_1 > 0, x_2 > 0$ be two real scalar random variables, statistically independently distributed, with densities $h_1(x_1)$ and $h_2(x_2)$ respectively. Let $u = x_1 x_2, v = x_2$. The Jacobian is $\frac{1}{v}$ and the density of u , through transformation of variables, denoted by $g(u)$ is given by

$$g(u) = \int_v \frac{1}{v} h_1(v) h_2\left(\frac{u}{v}\right) dv = \int_v \frac{1}{v} h_1\left(\frac{u}{v}\right) h_2(v) dv. \quad (2.4)$$

When h_1 and h_2 are generalized gamma densities then we have the exact form in (2.1). Hence (2.1), a generalized form of which is the left side of (1.14) without the limits, is appearing in all sorts of problems in different areas.

3. Connection to Fractional Integrals

So far we have considered product of ${}_1F_0$ functions and integrals over them. Let x_1 have a type-1 beta density with parameters $(\gamma + 1, \alpha)$, $\alpha > 0, \gamma > -1$ or with the density

$$P_1(x_1) = \frac{\Gamma(\gamma + 1 + \alpha)}{\Gamma(\gamma + 1)\Gamma(\alpha)} x_1^\gamma (1 - x_1)^{\alpha-1}, 0 \leq x_1 \leq 1 \quad (3.1)$$

for $\alpha > 0, \gamma > -1$ and zero elsewhere, which is again a ${}_1F_0$ function. Let $x_2 > 0$ have an arbitrary density $f(x_2)$. Then the density of the product $u = x_1x_2$, where x_1 and x_2 are statistically independently distributed, denoted again as $g(u)$, is available from (2.4) as

$$g(u) = \int_v \frac{1}{v} P_1\left(\frac{u}{v}\right) f(v) dv \quad (3.2)$$

$$\begin{aligned} &= \frac{\Gamma(\gamma + 1 + \alpha)}{\Gamma(\gamma + 1)\Gamma(\alpha)} \int_v \frac{1}{v} \left(\frac{u}{v}\right)^\gamma \left(1 - \frac{u}{v}\right)^{\alpha-1} f(v) dv \\ &= \frac{\Gamma(\gamma + 1 + \alpha)}{\Gamma(\gamma + 1)} K_{2,u,\gamma}^{-\alpha} f \end{aligned} \quad (3.3)$$

where

$$K_{2,u,\gamma}^{-\alpha} f = \frac{u^\gamma}{\Gamma(\alpha)} \int_{v>u} v^{-\alpha-\gamma} (v-u)^{\alpha-1} f(v) dv, \Re(\alpha) > 0 \quad (3.4)$$

is Kober fractional integral of the second kind of order α and parameter γ . Hence this fractional integral, namely the Kober fractional integral of the second kind, is a constant multiple of a statistical density or

$$K_{2,u,\gamma}^{-\alpha} f = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 + \alpha)} g(u) \quad (3.5)$$

where $g(u)$ is the density of $u = x_1x_2$ with $x_1 > 0$ and $x_2 > 0$ being independently distributed real scalar random variables having a type-1 beta density and an arbitrary density respectively.

Fractional calculus is revived during the last twenty years due to its applicability in various disciplines. Solutions coming from fractional order differential equations are seen to describe real-life situations much better compared to the solutions coming from the corresponding integer order differential equations. Some aspects of fractional calculus may be seen from Gorenflo and Mainardi (1997), Mainardi (1996), Kiryakova (1994), Mathai (2010, 2013, 2014, 2015), Mathai and Haubold (2008, 2013), Mathai, Saxena and Haubold (2010). In Mathai (2014) a geometrical interpretation of fractional integral and a general definition of fractional integral are given. Let

$$f_1(x_1) = \phi_1(x_1) \frac{(1-x_1)^{\alpha-1}}{\Gamma(\alpha)}, \Re(\alpha) > 0 \text{ and } f_2(x_2) = \phi_2(x_2)f(x_2) \quad (3.6)$$

where ϕ_1 and ϕ_2 are pre-fixed functions and $f(x_2)$ is an arbitrary function. Consider the Mellin convolution of a product. Then the product of the Mellin transforms of f_1 and f_2 is the Mellin transform of the integral

$$\begin{aligned} g(u) &= \int_v \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v) dv \\ &= \frac{1}{\Gamma(\alpha)} \int_v \frac{1}{v} \phi_1\left(\frac{u}{v}\right) \left[1 - \frac{u}{v}\right]^{\alpha-1} \phi_2(v) f(v) dv. \end{aligned} \quad (3.7)$$

Let $\phi_1 = 1$ and $\phi_2(x_2) = x_2^\alpha$. Then (3.7) becomes

$$g(u) = \frac{1}{\Gamma(\alpha)} \int_{v>u} (v-u)^{\alpha-1} f(v) dv, \Re(\alpha) > 0, \quad (3.8)$$

where (3.8) is Weyl fractional integral of the second kind of order α . If v is bounded above by a constant b then (3.8) is Riemann-Liouville fractional integral of order α and of the second kind for the real scalar variable case. If $\phi_1(x_1) = x_1^\gamma$ and $\phi_2 = 1$ in (3.8) then

$$g(u) = \frac{u^\gamma}{\Gamma(\alpha)} \int_{v>u} v^{-\alpha-\gamma} (v-u)^{\alpha-1} f(v) dv \quad (3.9)$$

which is Kober fractional integral of the second kind of order α and parameter γ for the real scalar variable case. Thus, by specializing ϕ_1 and ϕ_2 one can obtain various fractional integrals of the second kind of order α , introduced by various authors from time to time.

3.1. Mellin convolution of a ratio and density of a ratio

Instead of the Mellin convolution of a product, let us consider the Mellin convolution of a ratio. This can be easily interpreted in terms of statistical densities. Let $x_1 > 0$ and $x_2 > 0$ be real scalar positive random variables, independently distributed with densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Let $u = \frac{x_2}{x_1}, v = x_2$. Then the Jacobian is $-\frac{v}{u_1^2}$ and then the density of u_1 , denoted by $g_1(u_1)$, is the following:

$$g_1(u_1) = \int_v \frac{v}{u_1^2} f_1\left(\frac{v}{u_1}\right) f_2(v) dv. \quad (3.10)$$

Let $f_1(x_1)$ be a type-1 beta density with parameters (γ, α) and f_2 be an arbitrary density. Then

$$\begin{aligned} g_1(u_1) &= \frac{\Gamma(\gamma + \alpha)}{\Gamma(\gamma)\Gamma(\alpha)} \int_v \frac{v}{u_1^2} \left(\frac{v}{u_1}\right)^{\gamma-1} \left(1 - \frac{v}{u_1}\right)^{\alpha-1} f(v) dv, \Re(\alpha) > 0, \Re(\gamma) > 0 \\ &= \frac{\Gamma(\gamma + \alpha)}{\Gamma(\gamma)} K_{1, u_1, \gamma}^{-\alpha} f \end{aligned} \quad (3.11)$$

where

$$K_{1,u_1,\gamma}^{-\alpha} f = \frac{u_1^{-\gamma-\alpha}}{\Gamma(\alpha)} \int_{v < u_1} v^\gamma (u_1 - v)^{\alpha-1} f(v) dv \quad (3.12)$$

is Kober fractional integral of the first kind of order α and parameter γ . The general definition for fractional integral of the first kind of order α introduced by Mathai (2014) is the following: Let

$$f_1(x_1) = \phi_1(x_1) \frac{(1-x_1)^{\alpha-1}}{\Gamma(\alpha)} \text{ and } f_2(x_2) = \phi_2(x_2) f(x_2), \quad (3.13)$$

where ϕ_1 and ϕ_2 are preassigned functions and f is arbitrary. Then consider the Mellin convolution of a ratio of the type in (3.10). [Note that when it comes to Mellin convolution of a ratio then we can have four different forms corresponding to (3.1) and two different forms for Mellin convolution of a product]. We have

$$g_1(u_1) = \frac{1}{\Gamma(\alpha)} \int_v \frac{v}{u_1^2} \phi_1\left(\frac{v}{u_1}\right) \left[1 - \frac{v}{u_1}\right]^{\alpha-1} \phi_2(v) f(v) dv \quad (3.14)$$

for $\Re(\alpha) > 0$. Let $\phi_1(x_1) = x_1^{-\alpha-1}$ and $\phi_2(x_2) = x_2^\alpha$. Then (3.14) becomes

$$g_1(u_1) = \frac{1}{\Gamma(\alpha)} \int_{v < u_1} (u_1 - v)^{\alpha-1} f(v) dv \quad (3.15)$$

which is Weyl fractional integral of the first kind of order α . If v is bounded below by a constant a then (3.15) is Riemann-Liouville fractional integral of the first kind of order α .

4. Matrix-variate Cases

All the matrices appearing here are $p \times p$ real symmetric and positive definite unless stated otherwise. If $X = (x_{ij})$ is $m \times n$ then the wedge product $dX = \prod_{i=1}^m \prod_{j=1}^n \wedge dx_{ij}$ and if $X = X'$ (symmetric) and $p \times p$ then $dX = \prod_{i \geq 1}^p \wedge dx_{ij}$. The following Jacobians are needed in our discussions to follow. These will be stated here as lemmas. For proofs and for similar results see Mathai (1997).

Lemma 4.1. *Let $Y = AXB$ where X and Y are $m \times n$ matrices of distinct real variables as elements, A is $m \times m$ and B is $n \times n$ nonsingular constant matrices. Then*

$$Y = AXB \Rightarrow dY = |A|^n |B|^m dX \quad (4.1)$$

where $|\cdot|$ denotes the determinant of $(\cdot)$.

Lemma 4.2. *Let $X = X'$ and $p \times p$ and let A be a nonsingular $p \times p$ constant matrix. Then*

$$Y = AXA' \Rightarrow dY = |A|^{p+1} dX. \quad (4.2)$$

Lemma 4.3. *Let X be $p \times p$ real and positive definite. Let $T = (t_{ij})$ be a lower triangular matrix with distinct nonzero elements $t_{ij}, i \geq j, t_{ij} = 0, i < j$ and positive diagonal elements, $t_{jj} > 0, j = 1, \dots, p$. Then*

$$X = TT' \Rightarrow dX = 2^p \left\{ \prod_{j=1}^p t_{jj}^{p+1-j} \right\} dT. \quad (4.3)$$

Lemma 4.4. *Let the $p \times p$ matrix A be nonsingular with the regular inverse X^{-1} . Then*

$$Y = X^{-1} \Rightarrow dY = \begin{cases} |X|^{-2p} dX & \text{for a general } X \\ |X|^{-(p+1)} dX & \text{for } X = X'. \end{cases} \quad (4.4)$$

By using (4.3) we can evaluate the following matrix-variate integral and the result will be written as $\Gamma_p(\alpha)$ which has the expression

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}. \quad (4.5)$$

This is known as the real matrix-variate gamma function. It has the integral representation

$$\Gamma_p(\alpha) = \int_{X>O} |X|^{\alpha - \frac{p+1}{2}} e^{-\text{tr}(X)} dX, \Re(\alpha) > \frac{p-1}{2}, \quad (4.6)$$

where $X > O$ means X is positive definite and $\text{tr}(\cdot)$ means the trace of $(\cdot)$.

By using (4.6) and (4.2) we can show that the following real-valued scalar function $f_1(X_1)$ of the $p \times p$ matrix argument X_1 is a density function called the real matrix-variate gamma density.

$$f_1(X_1) = \frac{|B|^\alpha}{\Gamma_p(\alpha)} |X_1|^{\alpha - \frac{p+1}{2}} e^{-\text{tr}(BX_1)}, X_1 > O, B > O, \Re(\alpha) > \frac{p-1}{2}, \quad (4.7)$$

and zero elsewhere.

A real matrix-variate type-1 beta density $f_2(X_2)$, with parameters α and β is defined as the real-valued scalar function of X_2 of the following form:

$$f_2(X_2) = \frac{\Gamma_p(\alpha + \beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)} |X_2|^{\alpha - \frac{p+1}{2}} |I - X_2|^{\beta - \frac{p+1}{2}} \quad (4.8)$$

for $X_2 > O, I - X_2 > O, \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}$ and zero elsewhere. The range of X_2 is also written as $O < X_2 < I$.

4.1. Limiting properties

The limiting properties considered in Section 1 will go through for functions of matrix argument also. There are three approaches available in the literature for defining functions of matrix argument or real-valued scalar functions where the argument is a $p \times p$ matrix. Details of the definitions may be seen from Mathai (1997). Let us denote Gauss hypergeometric function of matrix argument by ${}_2F_1(a, b; c; X)$ which is available as an integral or as a series involving zonal polynomials. For details of zonal polynomials see Mathai, Provost and Hayakawa (1995). All the limiting properties will go through. That is,

$$\lim_{a \rightarrow \infty} {}_2F_1(a, b; c; \frac{1}{a}X) = {}_1F_1(b; c; X)$$

where a, b, c are scalars and X is a $p \times p$ matrix.

$$\lim_{b \rightarrow \infty} {}_1F_1(b; c; \frac{1}{b}X) = {}_0F_1(; c; X).$$

$$\lim_{c \rightarrow \infty} {}_1F_1(b; c; cX) = {}_1F_0(b; ; X) = {}_1F_0(b; ; X) \text{ for } \|X\| < 1$$

where $\|(\cdot)\|$ represents a norm of the matrix $(\cdot)$. This ${}_1F_0$ has an explicit representation in terms of a determinant:

$${}_1F_0(b; ; X) = |I - X|^{-b} \text{ for } \|X\| < 1. \quad (4.9)$$

Also note that

$$\lim_{q \rightarrow 1} {}_1F_0(\frac{1}{q-1}; ; -(q-1)X) = \lim_{q \rightarrow 1} |I + (q-1)X|^{-\frac{1}{q-1}} = e^{-\text{tr}(X)}. \quad (4.10)$$

This is the path of a binomial function ${}_1F_0$ going to an exponential function ${}_0F_0$ where X is a $p \times p$ matrix. Then

$$\begin{aligned} \lim_{q \rightarrow 1+} |X|^\gamma |I + a(q-1)X|^{-\frac{1}{q-1}} &= \lim_{q \rightarrow 1-} |X|^\gamma |I - a(1-q)X|^{\frac{1}{1-q}} \\ &= |X|^\gamma e^{-a\text{tr}(X)} \end{aligned} \quad (4.11)$$

where $a > 0$ is a real scalar. Here (4.11) is one form of the pathway model introduced by Mathai (2005). In Mathai (2005) a rectangular matrix-variate case is considered. Let X be $p \times q$, $p \leq q$ of rank p . Consider the models

$$\begin{aligned} f_1(X) &= C_1 |AXBX'|^\gamma |I - a(1-q)AXBX'|^{-\frac{1}{1-q}}, q < 1, a > 0 \\ f_2(X) &= C_2 |AXBX'|^\gamma |I + a(q-1)AXBX'|^{-\frac{1}{q-1}}, q > 1, a > 0, \text{ and} \\ f_3(X) &= C_3 |AXBX'|^\gamma e^{-\text{atr}(AXBX')} \end{aligned} \quad (4.12)$$

where A is $p \times p$ and B is $q \times q$ constant positive definite matrices. Note that from $f_1(X)$ we can obtain $f_2(X)$ and $f_3(X)$. Also observe that

$$\lim_{1-} f_1(X) = \lim_{q \rightarrow 1+} f_2(X) = f_3(X).$$

The advantage of the models in (4.12) is that the theory of quadratic forms, bilinear forms, generalized quadratic form etc can be set in a very general structure of the pathway model in (4.12). It is shown in Mathai (2005) that almost all densities in current use in various disciplines are either special cases of (4.12) or one-to-one functions of special cases of (4.12).

Fractional calculus of matrix-variate functions, in the real and complex domains, is available from Mathai (2013, 2014, 2015). For the sake of illustration let us extend Kober fractional integral of the second kind to matrix-variate functions. Let X_1 and X_2 be $p \times p$ real positive definite matrix-variate random variables and statistically independently distributed with densities $g_1(X_1)$ and $g_2(X_2)$ respectively. Let $U = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$, $V = X_2$ be a symmetric product where $X_2^{\frac{1}{2}}$ is the unique positive definite square root of the positive definite matrix X_2 . This means, $X_2 = V$, $X_1 = V^{-\frac{1}{2}} U V^{-\frac{1}{2}}$ and the Jacobian, from the above lemmas, can be shown to be $|V|^{-\frac{p+1}{2}}$. Then from transformation of variables, the density of U , denoted by $g(U)$, is given by the following:

$$g(U) = \int_V |V|^{-\frac{p+1}{2}} g_1(V^{-\frac{1}{2}} U V^{-\frac{1}{2}}) g_2(V) dV. \quad (4.13)$$

Let g_1 be a type-1 beta density with parameters $(\gamma + \frac{p+1}{2}, \alpha)$ for $\Re(\alpha) > \frac{p-1}{2}$, $\Re(\gamma) > -1$ or with density

$$g_1(X_1) = \frac{\Gamma_p(\gamma + \frac{p+1}{2} + \alpha)}{\Gamma_p(\gamma + \frac{p+1}{2}) \Gamma_p(\alpha)} |X_1|^\gamma |I - X_1|^{\alpha - \frac{p+1}{2}}$$

for $0 < X_1 < I$, $\Re(\gamma) > -1$, $\Re(\alpha) > \frac{p-1}{2}$. Then from transformation of variables the density of U is given by the following:

$$\begin{aligned} g(U) &= D_1 \int_V |V|^{-\frac{p+1}{2}} |V^{-\frac{1}{2}} U V^{-\frac{1}{2}}|^\gamma |I - V^{-\frac{1}{2}} U V^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} f(V) dV \\ &= D_2 K_{2,U,\gamma}^{-\alpha} f \end{aligned} \quad (4.14)$$

where

$$D_1 = \frac{\Gamma_p(\alpha + \gamma + \frac{p+1}{2})}{\Gamma_p(\alpha) \Gamma_p(\gamma + \frac{p+1}{2})}, D_2 = \frac{\Gamma_p(\alpha + \gamma + \frac{p+1}{2})}{\Gamma_p(\gamma + \frac{p+1}{2})}$$

and

$$K_{2,U,\gamma}^{-\alpha} f = \frac{|U|^\gamma}{\Gamma_p(\alpha)} \int_{V>U} |V|^{-\gamma-\alpha} |V - U|^{\alpha - \frac{p+1}{2}} f(V) dV \quad (4.15)$$

for $\Re(\alpha) > \frac{p-1}{2}$, where (4.15) for $p = 1$ is Kober fractional integral of the second kind of order α and parameter γ . Hence this author called (4.15) the Kober fractional integral of the second kind of order α and parameter γ in the real matrix-variate case. This author has extended various fractional integrals in the scalar variable case to the corresponding matrix-variate cases in the real and complex domains.

Conclusion

Starting from a Gauss hypergeometric function, both in the real scalar variable case and in the matrix-variate case, various limiting forms are considered. These create various paths of different types. The path of ${}_1F_0$ or binomial function going to ${}_0F_0$ or exponential function is shown to be connected to various problems in different areas. If the limiting forms are the stable or ideal cases in physical situations then the paths leading to these models give the q -versions of these models or such q -versions can describe the unstable or chaotic neighborhoods of stable situations and path leading to stable situations. All other paths can be similarly extended, which will produce rich sources of interesting and useful results, especially the path of Bessel function ${}_0F_1$ going to exponential function ${}_0F_0$. These remain as open problems.

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