

SOLUTIONS OF RIEMANN-LIOUVILLE FRACTIONAL MATRIX DIFFERENTIAL EQUATIONS

J. Vasundhara Devi¹, S. Nagamani², K. Jayalakshmi³

¹ GVP - Prof.V.Lakshmikantham, Institute for Advanced Studies,
Department of Mathematics, GVP College of Engineering,
Visakhapatnam - 530 048, AP, India.

² Dr. V. S. Krishna Government Junior College(Boys),
Visakhapatnam - 530 022, AP, India.

³ JNTU(A) College of Engineering,
Ananthapuramu - 515 002, AP, India.

E-mail: jvdevi@gmail.com, nagamanisunkara2013@gmail.com

Dedicated to Prof. M.A. Pathan on his 75th birth anniversary

Abstract: In this paper we have obtained solutions of Riemann - Liouville fractional matrix differential equation (RLFMDE) of the form $D^q X = AX$ and $D^q X = AX + XB$. We have extended these results to 2^{nd} order and 3^{rd} order RLFMDE and then generalized them to n^{th} order RLFMDE.

Keywords: Riemann - Liouville Fractional Matrix Differential Equation, Matrix q - exponential function, fractional cosine matrix function, fractional sine matrix function.

AMS(MOS)Subject Classification: 47G20.

1. Introduction

Matrix is an essential basic element for modeling many physical phenomena where interconnections between variables have to be considered. Such physical phenomena exist in all disciplines such as social, physical and biological sciences, computer sciences and robotics to name a few. As physical systems vary with time, it is quite natural to study the rate of change of these phenomena with respect to time. Thus the study of Matrix differential equations had attracted the attention of many scientists.

Bellman in his book [6] studied Matrix differential equations of the type

$X' = AX, X(t_0) = X_0$ and $X' = AX + XB, X(t_0) = C$ and obtained criteria for the existence of solutions for the fore mentioned matrix differential equations.

Further, observing that networks are represented naturally by graphs and that there exists an isomorphism between graphs and matrices, J. Vasundhara Devi et al [9] introduced the concept of a graph differential equation and studied them through Matrix differential equation.

The theory of fractional calculus had been attracting the attention of many researchers since its inception as a query to Leibnitz by L'Hospital. Over the 300 years of its existence the theory of fractional calculus was fully evolved but its study was restricted to mathematicians only. Further, the theory of Linear fractional differential equations has been developed. With the book of Oldham and Spanier [4] this area caught the attention of applied scientists and the study of fractional differential equations, fractional integral equations and so on took a big leap. There are many interesting books on this topic of research and some of them are [1, 2, 3, 5].

Fractional trigonometry also had its share of development in recent times. See Kilbas [2] for developments in fractional trigonometric functions.

Recently it has been observed that the solutions of second order Caputo scalar fractional differential equations and Riemann - Liouville scalar fractional differential equation of the type ${}^c D^q x + x = 0$ or $D^q x + x = 0, x(t_0) = x^0$ are fractional cosine functions and fractional sine functions [7].

Also the second order Caputo or Riemann - Liouville scalar fractional differential equations of the form ${}^c D^q x - x = 0$ has its solutions as fractional hyperbolic functions [8].

In this paper we study Matrix fractional differential equations of the type $D^q X = AX$ and $D^q X = AX + XB$.

2. Preliminaries

In this section we give the necessary definitions and results required to obtain the main results. As we are studying Fractional Matrix Differential Equations we state below Matrix q - exponential function and other related definitions and results.

Definition 2.1

Let $M_n(\mathbb{R})(n \in \mathbb{N})$ be a set of all matrices $A = [a_{jk}]$ of order $n \times n$ with $a_{jk} \in \mathbb{R}(j, k = 1, 2, \dots, n)$. The matrix q - exponential function e_q^{Az} is defined by

$$e_q^{Az} = z^{q-1} \sum_{k=0}^{\infty} A^k \frac{z^{kq}}{\Gamma(k+1)} \quad \text{where } A \in M_n(\mathbb{R}), z \in \mathbb{C} \setminus \{0\} (R(q) > 0). \quad (2.1)$$

The definition of fractional derivative for $(t - t_0)^w$ by Riemann is given below.

Definition 2.2

Riemann - Liouville fractional derivative for $(t - t_0)^w$ is given by

$$D^q [(t - t_0)^w] = \frac{(t - t_0)^{w-q} \Gamma(1 + w)}{\Gamma(1 + w - q)} . \quad (2.2)$$

Next we present the Riemann - Liouville fractional derivative in-terms of integrals.

Definition 2.3

Riemann - Liouville fractional derivative of $x(t)$ is given by

$$D^q x(t) = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_{t_0}^t (t - s)^{-q} x(s) ds , \quad (t \in \mathbb{R}_+) . \quad (2.3)$$

Next we introduce the definition of Riemann - Liouville fractional derivative of a matrix $A_{n \times n}$

Definition 2.4

The Riemann - Liouville fractional derivative of a matrix $A_{n \times n}$ is defined by

$$D^q(A_{n \times n}) = \left\{ \begin{array}{l} \left[\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t \frac{a_{ij} ds}{(t-s)^q} \right]_{n \times n} \text{ where } A = [a_{ij}]_{n \times n} \in M_n(\mathbb{R}), \\ a_{ij} \in \mathbb{R} (i, j = 1, 2, \dots, n) t_0 \in \mathbb{R}_+ . \end{array} \right\} \quad (2.4)$$

We next state and prove the following Lemma from [2] which we have extended to the matrices in the following section.

Lemma 2.5

Consider the q^{th} order linear Riemann - Liouville fractional differential equation

$$\left. \begin{array}{l} D^q x(t) + x(t) = 0 \quad \text{with initial condition} \\ D^{q-1} x(t_0^+) = b , \quad b \in \mathbb{R} , \quad t_0 \in \mathbb{R}_+ , \quad 0 < q < 1 . \end{array} \right\} \quad (2.5)$$

This problem can be written as

$$\left. \begin{array}{l} D^q x(t) + x(t) = 0 \quad \text{with initial condition} \\ \lim_{t \rightarrow t_0^+} (t - t_0)^{1-q} x(t) = \frac{b}{\Gamma q} . \end{array} \right\} \quad (2.6)$$

Proof : We know that

$$\begin{aligned} D^{q-1} x(t_0^+) &= \lim_{t \rightarrow t_0^+} D^{-(1-q)} x(t) \\ &= \lim_{t \rightarrow t_0^+} I^{1-q} x(t) \\ &= I^{1-q} x(t_0^+) = b . \end{aligned}$$

Then

$$\begin{aligned}
 x(t_0^+) &= I^{q-1}(b) \\
 &= I^{-(1-q)}(b) \\
 &= D^{1-q}(b) \\
 &= \frac{1}{\Gamma q} \frac{d}{dt} \int_{t_0}^t \frac{b}{(t-s)^{1-q-R[1-q]}} ds \\
 &= \frac{b}{\Gamma q} \frac{d}{dt} \int_{t_0}^t \frac{1}{(t-s)^{1-q-0}} ds \\
 &= \frac{b}{\Gamma q} \frac{d}{dt} \int_{t_0}^t (t-s)^{q-1} ds \\
 &= \frac{b}{\Gamma q} \frac{d}{dt} \left[\frac{(t-s)^{q-1+1}}{-(q-1+1)} \right]_{t_0}^t \\
 &= \frac{b}{\Gamma q} \frac{d}{dt} \left[\frac{(t-s)^q}{-q} \right]_{t_0}^t \\
 &= \frac{b}{\Gamma q} \frac{d}{dt} \left[\frac{(t-t_0)^q}{q} \right] \\
 &= \frac{b}{\Gamma q} \frac{q(t-t_0)^{q-1}}{q} \\
 &= \frac{b}{\Gamma q} (t-t_0)^{q-1}
 \end{aligned}$$

$$\text{which implies } (t-t_0)^{1-q} x(t_0^+) = \frac{b}{\Gamma q}$$

$$\text{or } \lim_{t \rightarrow t_0^+} (t-t_0)^{1-q} x(t) = \frac{b}{\Gamma q}$$

3. Main Results

We now state and prove the following theorem which is useful in getting the main results.

Theorem 3.1

Consider the q^{th} order linear Riemann - Liouville fractional matrix differential

equation

$$\left. \begin{aligned} D^q X(t) &= A X(t) \quad \text{with initial condition} \\ D^{q-1} X(t_0^+) &= C, \quad C \in M_{n \times n}(\mathbb{R}), \quad t_0 \in \mathbb{R}_+, \quad 0 < q < 1. \end{aligned} \right\} \quad (3.1)$$

This problem can be written as

$$\left. \begin{aligned} D^q X(t) &= A X(t) \quad \text{with initial condition} \\ \lim_{t \rightarrow t_0^+} (t - t_0)^{1-q} X(t) &= \frac{C}{\Gamma q}. \end{aligned} \right\} \quad (3.2)$$

Proof: We know that

$$\begin{aligned} D^{q-1} X(t_0^+) &= \lim_{t \rightarrow t_0^+} D^{-(1-q)} X(t) \\ &= \lim_{t \rightarrow t_0^+} I^{1-q} X(t) \\ &= I^{1-q} X(t_0^+) = C_{n \times n}. \end{aligned}$$

Then

$$\begin{aligned} X(t_0^+) &= I^{q-1}(C_{n \times n}) \\ &= I^{-(1-q)}(C_{n \times n}) \\ &= D^{1-q}(C_{n \times n}) \\ &= \left[\frac{1}{\Gamma q} \frac{d}{dt} \int_{t_0}^t \frac{c_{ij}}{(t-s)^{1-q-R[1-q]}} ds \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{d}{dt} \int_{t_0}^t \frac{1}{(t-s)^{1-q-0}} ds \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{d}{dt} \int_{t_0}^t (t-s)^{q-1} ds \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{d}{dt} \left[\frac{(t-s)^{q-1+1}}{-(q-1+1)} \right]_{t_0}^t \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{d}{dt} \left[\frac{(t-s)^q}{-q} \right]_{t_0}^t \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{d}{dt} \left[\frac{(t-t_0)^q}{q} \right] \right]_{n \times n} \\ &= \left[\frac{c_{ij}}{\Gamma q} \frac{q(t-t_0)^{q-1}}{q} \right]_{n \times n} \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{c_{ij}}{\Gamma q} (t - t_0)^{q-1} \right]_{n \times n} \\
&= \frac{(t - t_0)^{q-1}}{\Gamma q} [c_{ij}]_{n \times n} \\
&= \frac{(t - t_0)^{q-1}}{\Gamma q} C
\end{aligned}$$

$$\text{which implies } (t - t_0)^{1-q} X(t_0^+) = \frac{C}{\Gamma q}$$

$$\text{or } \lim_{t \rightarrow t_0^+} (t - t_0)^{1-q} X(t) = \frac{C}{\Gamma q}$$

In this section we begin with the study of the Reimann - Liouville fractional matrix differential equation (RLFMDE) of the form $D^q X = AX$ and proceed to extend this study to higher order systems.

Theorem 3.2

Consider the system

$$D^q X = AX, D^{q-1} X(t_0) = C \quad (A, C \in M_{n \times m}(R)) , \quad 0 < q < 1. \quad (3.3)$$

The solution of the RLFMDE (3.3) is of the form $X(t) = e_q^{A(t-t_0)} C$ where $e^{A(t-t_0)}$ is the Mittag - Leffler Matrix function.

Proof: We know that from Theorem 3.1, $D^{q-1} X(t_0) = C$ implies $(t - t_0)^{1-q} X(t) \big|_{t=t_0} = \frac{C}{\Gamma q}$. By verification we show that $X(t)$ is the solution of the given system (3.3).

$$\begin{aligned}
\text{Consider } X(t) &= e_q^{A(t-t_0)} C \\
&= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^k (t - t_0)^{kq}}{\Gamma(k+1)q} C \\
&= (t - t_0)^{q-1} \left[\frac{A^0}{\Gamma q} + \frac{A}{\Gamma 2q} (t - t_0)^q + \frac{A^2}{\Gamma 3q} (t - t_0)^{2q} + \dots \right] C
\end{aligned}$$

Differentiating w.r.t. ' t ' we get

$$\begin{aligned}
&= A(t - t_0)^{q-1} \left[\frac{A^0}{\Gamma q} + \frac{A}{\Gamma 2q} (t - t_0)^q + \frac{A^2}{\Gamma 3q} (t - t_0)^{2q} + \dots \right] C \\
&= AX(t)
\end{aligned}$$

Hence $X(t)$ is the solution of the given system.

$$\text{Also } (t - t_0)^{1-q} X(t) \big|_{t=t_0} = (t - t_0)^{1-q} (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^k (t - t_0)^{kq}}{\Gamma(k+1)q} C \big|_{t=t_0} = \frac{C}{\Gamma q}.$$

Hence initial condition is satisfied.

Next we proceed to study the 2^{nd} order system corresponding to (3.3).

Theorem 3.3

Consider the system

$$\left. \begin{aligned} D^\alpha X(t) &= A^2 X(t) \quad \text{with initial data} \\ D^{\alpha-1} X(t_0) &= I_{n \times n} D^{\alpha-2} X(t_0) = O_{n \times n}. \end{aligned} \right\} \quad (3.4)$$

Then the general solution is given by $c_1 X(t) + c_2 Y(t)$ where c_1 and c_2 are arbitrary constants and $X(t)$ and $Y(t)$ are infinite series solutions of the form

$$\begin{aligned} X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{2k} (t - t_0)^{2kq}}{\Gamma(2k+1)q} \quad \text{and} \\ Y(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{2k+1} (t - t_0)^{(2k+1)q}}{\Gamma(2k+2)q} \quad \text{where} \quad \alpha = 2q, \quad \frac{1}{2} < q < 1, t \in \mathbb{R}_+. \end{aligned}$$

Proof: Given $\alpha = 2q$ where $\frac{1}{2} < q < 1$. We transform the given system into two first order system of equations

$$D^q X(t) = AY(t), \quad D^q Y(t) = AX(t) \quad (3.5)$$

with initial conditions

$$\lim_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n}, \quad \lim_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n}. \quad (3.6)$$

We know that from Theorem (3.1)

$$\begin{aligned} \lim_{t \rightarrow t_0} D^{q-1} X(t) &= I_{n \times n} \quad \text{implies} \quad (t - t_0)^{1-q} X(t) \Big|_{t=t_0} = \frac{I_{n \times n}}{\Gamma q} \\ \text{and} \quad \lim_{t \rightarrow t_0} D^{q-1} Y(t) &= O_{n \times n} \quad \text{implies} \quad (t - t_0)^{1-q} Y(t) \Big|_{t=t_0} = O_{n \times n}. \end{aligned}$$

By verification we show that $X(t)$ and $Y(t)$ are the solutions of the given system.

$$\begin{aligned} \text{Consider} \quad X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{2k} (t - t_0)^{2kq}}{\Gamma(2k+1)q} \\ &= (t - t_0)^{q-1} \left[\frac{A^0}{\Gamma q} + \frac{A^2}{\Gamma 3q} (t - t_0)^{2q} + \frac{A^4}{\Gamma 5q} (t - t_0)^{4q} + \dots \right] \end{aligned}$$

Differentiating w.r.t. 't' we get

$$D^q(X(t)) = \frac{A^2}{\Gamma 2q} (t - t_0)^{2q-1} + \frac{A^4}{\Gamma 4q} (t - t_0)^{4q-1} + \dots$$

Again differentiating w.r.t. ' t ' we get

$$\begin{aligned} D^{2q}X(t) &= \frac{A^2}{\Gamma q}(t-t_0)^{q-1} + \frac{A^4}{\Gamma 3q}(t-t_0)^{3q-1} + \dots \\ &= A^2X(t) \end{aligned}$$

Hence $X(t)$ is the solution of (3.4) .

Similarly $Y(t)$ is the solution of (3.4) .

$$\text{Also } (t-t_0)^{1-q}X(t) \Big|_{t=t_0} = (t-t_0)^{1-q}(t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{2k}(t-t_0)^{2kq}}{\Gamma(2k+1)q} \Big|_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}$$

and similarly $(t-t_0)^{1-q}Y(t) \Big|_{t=t_0} = O_{n \times n}$.

Hence initial conditions are satisfied.

We next state and prove theorem as 3^{rd} order system corresponding to (3.3).

Theorem 3.4

Consider the system

$$D^\alpha X(t) = A^3X, \quad 2 < \alpha < 3, \quad t \in \mathbb{R}_+ \quad (3.7)$$

with initial data

$$D^{\alpha-1}X(t_0) = I_{n \times n}, \quad D^{\alpha-2}X(t_0) = O_{n \times n}, \quad D^{\alpha-3}X(t_0) = O_{n \times n}. \quad (3.8)$$

Then the general solution of (3.7) and (3.8) is given by $c_1X(t) + c_2Y(t) + c_3Z(t)$ (c_1, c_2 and c_3 being arbitrary constants) where $X(t), Y(t)$ and $Z(t)$ are infinite series solutions of the form

$$\begin{aligned} X(t) &= (t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{3k}(t-t_0)^{3kq}}{\Gamma(3k+1)q}, \\ Y(t) &= (t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{3k+1}(t-t_0)^{(3k+1)q}}{\Gamma(3k+2)q}, \\ Z(t) &= (t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{3k+2}(t-t_0)^{(3k+2)q}}{\Gamma(3k+3)q}, \quad \text{where } \alpha = 3q, \quad \frac{2}{3} < q < 1, \quad t \in \mathbb{R}_+. \end{aligned}$$

Proof: We write $\alpha = 3q$ and transform the given IVP to a system of equations of the q^{th} order, $\frac{2}{3} < q < 1$ as

$$D^qX(t) = AZ(t), \quad D^qY(t) = AX(t) \quad \text{and} \quad D^qZ(t) = AY(t) \quad (3.9)$$

and corresponding initial conditions

$${}^{Lt}_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n}, {}^{Lt}_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n}, {}^{Lt}_{t \rightarrow t_0} D^{q-1} Z(t) = O_{n \times n}. \quad (3.10)$$

From theorem (3.1)

$$\begin{aligned} {}^{Lt}_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n} & \text{ implies } (t - t_0)^{1-q} X(t) \big/_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}, \\ {}^{Lt}_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n} & \text{ implies } (t - t_0)^{1-q} Y(t) \big/_{t=t_0} = O_{n \times n} \\ \text{and } {}^{Lt}_{t \rightarrow t_0} D^{q-1} Z(t) = O_{n \times n} & \text{ implies } (t - t_0)^{1-q} Z(t) \big/_{t=t_0} = O_{n \times n}. \end{aligned}$$

By verification we show that $X(t), Y(t)$ and $Z(t)$ are solutions of the given system.

$$\begin{aligned} \text{Consider } X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{3k}(t - t_0)^{3kq}}{\Gamma(3k + 1)q} \\ &= (t - t_0)^{q-1} \left[\frac{A^0}{\Gamma q} + \frac{A^3}{\Gamma 4q} (t - t_0)^{3q} + \frac{A^6}{\Gamma 7q} (t - t_0)^{6q} + \dots \right] \end{aligned}$$

Differentiating w.r.t. ' t ' we get

$$D^q X(t) = \frac{A^3}{\Gamma 3q} (t - t_0)^{3q-1} + \frac{A^6}{\Gamma 6q} (t - t_0)^{6q-1} + \dots$$

Again differentiating w.r.t. ' t ' we get

$$D^{2q} X(t) = \frac{A^3}{\Gamma 2q} (t - t_0)^{2q-1} + \frac{A^6}{\Gamma 5q} (t - t_0)^{5q-1} + \dots$$

Again differentiating w.r.t. ' t ' we get

$$\begin{aligned} D^{3q} X(t) &= A^3 \frac{(t - t_0)^{q-1}}{\Gamma 2q} + A^6 \frac{(t - t_0)^{4q-1}}{\Gamma 4q} + \dots \\ &= A^3 X(t) \end{aligned}$$

Hence $X(t)$ is the solution of the given system .

Similarly $Y(t)$ and $Z(t)$ are also the solutions of the given system.

$$\text{Also } (t - t_0)^{1-q} X(t) \big/_{t=t_0} = (t - t_0)^{1-q} (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{A^{3k}(t - t_0)^{3kq}}{\Gamma(3k + 1)q} \big/_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}$$

and similarly $(t - t_0)^{1-q} Y(t) \big/_{t=t_0} = O_{n \times n}$, $(t - t_0)^{1-q} Z(t) \big/_{t=t_0} = O_{n \times n}$.

Hence initial conditions are satisfied.

4. Riemann - Liouville Fractional Matrix Differential Equation

This section deals with the study of the Riemann - Liouville fractional matrix differential equation (RLFMDE) of the form $D^q X = AX + XB$. We further proceed to extend this study to higher order systems.

Theorem 4.1

Consider the system

$$\left. \begin{aligned} D^q X &= AX + XB, \quad 0 < q < 1 \quad \text{with initial data} \\ D^{q-1} X(t_0) &= C, \quad (A, B, C \in M_{n \times n}(R)), \quad t_0 \in \mathbb{R}_+ \end{aligned} \right\} \quad (4.1)$$

Suppose that AB and BC commute with each other.

Then the solution of (4.1) is of the form $X(t) = e^{(A+B)(t-t_0)} C$.

Proof: We know that from theorem (3.1), $D^{q-1} X(t_0) = C$ implies $(t - t_0)^{1-q} X(t) \big|_{t=t_0} = \frac{C}{\Gamma_q}$.

By verification we show that $X(t)$ is the solution of the given system.

Consider $X(t) = e_q^{(A+B)(t-t_0)} C$

$$\begin{aligned} D^q X(t) &= (A + B) e^{(A+B)(t-t_0)} C \\ &= AX + XB \quad (\text{Since } A, B \text{ and } B, C \text{ commute with each other}). \end{aligned}$$

Hence $X(t)$ is the solution of the given system.

Also $(t - t_0)^{1-q} X(t) \big|_{t=t_0} = (t - t_0)^{1-q} e_q^{(A+B)(t-t_0)} \big|_{t=t_0} = \frac{C}{\Gamma_q}$. Hence the initial condition is satisfied.

Next we proceed to study the 2^{nd} order differential system corresponding to (4.1).

Theorem 4.2

Consider the system

$$D^\alpha X = A^2 X + 2AXB + B^2, \quad 1 < \alpha < 2, \quad \alpha = 2q, \quad \frac{1}{2} < q < 1, \quad t \in \mathbb{R}_+ \quad (4.2)$$

$$\text{with initial conditions } D^{\alpha-1} X(t_0) = I_{n \times n}, \quad D^{\alpha-2} X(t_0) = O_{n \times n}. \quad (4.3)$$

The general solution is given by $c_1 X(t) + c_2 Y(t)$ where c_1 and c_2 are arbitrary constants and $X(t)$ and $Y(t)$ are infinite series solutions of the form

$$\begin{aligned} X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{2kq}}{\Gamma(2k + 1)q}, \\ Y(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{2k+1} (t - t_0)^{(2k+1)q}}{\Gamma(2k + 2)q}. \end{aligned}$$

Proof: Given $\alpha = 2q$ where $\frac{1}{2}, < q < 1$. We transform the given system into two first order system equations.

$$D^q X(t) = (A + B)Y(t) \text{ and } D^q Y(t) = (A + B)X(t) \quad (4.4)$$

$$\text{with initial conditions } \lim_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n}, \quad \lim_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n}. \quad (4.5)$$

We know that from theorem (3.1),

$$\lim_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n} \text{ implies } (t - t_0)^{1-q} X(t) \big|_{t=t_0} = \frac{I_{n \times n}}{\Gamma q} \text{ and}$$

$$\lim_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n} \text{ implies } (t - t_0)^{1-q} Y(t) \big|_{t=t_0} = O_{n \times n}.$$

By verification we show that $X(t)$ and $Y(t)$ are the solutions of the given system .

$$\begin{aligned} \text{Consider } X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{2kq}}{\Gamma(2k + 1)q} \\ &= \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k+1)q-1}}{\Gamma(2k + 1)q} \end{aligned}$$

Differentiating w.r.t., 't' we get

$$D^q X(t) = \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{2kq-1} \Gamma(2k + 1)q}{\Gamma(2k + 1)q \Gamma 2kq}$$

Differentiating w.r.t., 't' we get

$$\begin{aligned} D^{2q} X(t) &= \sum_{k=1}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k-1)q-1} \Gamma 2kq}{\Gamma(2k - 1)q \Gamma 2kq} \\ &= \sum_{k=1}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k-1)q-1}}{\Gamma(2k - 1)q} \end{aligned}$$

Consider

$$\begin{aligned} &A^2 X + 2AXB + XB^2 \\ &= A^2 \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k+1)q-1}}{\Gamma(2k + 1)q} + 2A \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k+1)q-1}}{\Gamma(2k + 1)q} B \\ &+ \sum_{k=0}^{\infty} \frac{(A + B)^{2k} (t - t_0)^{(2k+1)q-1}}{\Gamma(2k + 1)q} B^2 \\ &= A^2 \left[\frac{(t - t_0)^{q-1}}{\Gamma q} + \frac{(A + B)^2 (t - t_0)^{3q-1}}{\Gamma 3q} + \frac{(A + B)^4 (t - t_0)^{5q-1}}{\Gamma 5q} + \dots \right] \end{aligned}$$

$$\begin{aligned}
& + 2A \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^2(t-t_0)^{3q-1}}{\Gamma 3q} + \frac{(A+B)^4(t-t_0)^{5q-1}}{\Gamma 5q} + \dots \right] B \\
& + \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^2(t-t_0)^{3q-1}}{\Gamma 3q} + \frac{(A+B)^4(t-t_0)^{5q-1}}{\Gamma 5q} + \dots \right] B^2 \\
& = (A^2 + 2AB + B^2) \frac{(t-t_0)^{q-1}}{\Gamma q} + (A^2 + 2AB + B^2)(A+B)^2 \frac{(t-t_0)^{3q-1}}{\Gamma 3q} \\
& + (A^2 + 2AB + B^2)(A+B)^4 \frac{(t-t_0)^{5q-1}}{\Gamma 5q} + \dots \\
& = (A+B)^2 \frac{(t-t_0)^{q-1}}{\Gamma q} + (A+B)^4 \frac{(t-t_0)^{3q-1}}{\Gamma 3q} + (A+B)^6 \frac{(t-t_0)^{5q-1}}{\Gamma 5q} + \dots \\
& = \sum_{k=1}^{\infty} \frac{(A+B)^{2k}(t-t_0)^{(2k-1)q-1}}{\Gamma(2k-1)q} \\
& = D^{2q}X(t).
\end{aligned}$$

Hence

$$D^{2q}X = A^2X + 2AXB + XB^2.$$

Thus $X(t)$ is the solution of the given system.

Similarly $Y(t)$ is also the solution of the given system.

Also

$$(t-t_0)^{1-q}X(t) \big/_{t=t_0} = (t-t_0)^{1-q}(t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A+B)^{2k}(t-t_0)^{2kq}}{\Gamma(2k+1)q} \big/_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}$$

and similarly

$$(t-t_0)^{1-q}Y(t) \big/_{t=t_0} = O_{n \times n}.$$

Hence initial conditions are satisfied.

Next we proceed to study the 3^{rd} order RLMFDE corresponding to (3.1).

Theorem 4.3

Consider the system

$$\begin{aligned}
D^\alpha X &= A^3X + 3A^2XB + 3AXB^2 + XB^3, \\
2 &< \alpha < 3, \alpha = 3q, \frac{2}{3} < q < 1, t \in \mathbb{R}_+ \quad (4.6)
\end{aligned}$$

with initial conditions

$$D^{\alpha-1}X(t_0) = I_{n \times n}, \quad D^{\alpha-2}X(t_0) = O_{n \times n}, \quad D^{\alpha-3}X(t_0) = O_{n \times n}. \quad (4.7)$$

Then the general solution is given by $c_1X(t) + c_2Y(t) + c_3Z(t)$ where c_1, c_2 and c_3 are arbitrary constants and $X(t), Y(t)$ and $Z(t)$ are infinite series solutions of the form

$$\begin{aligned} X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{3k}(t - t_0)^{3kq}}{\Gamma(3k + 1)q}, \\ Y(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{3k+1}(t - t_0)^{(3k+1)q}}{\Gamma(3k + 2)q}, \\ Z(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{3k+2}(t - t_0)^{(3k+2)q}}{\Gamma(3k + 3)q} \end{aligned}$$

Proof: Given $\alpha = 3q$ where $\frac{2}{3} < q < 1$. We transform the given system into three first order system of equations

$$D^q X(t) = (A + B)Z(t), D^q Y(t) = (A + B)X(t), D^q Z(t) = (A + B)Y(t) \quad (4.8)$$

with initial conditions

$${}^{Lt}_{t \rightarrow t_0} D^{q-1} X(t) = I_{n \times n}, {}^{Lt}_{t \rightarrow t_0} D^{q-1} Y(t) = O_{n \times n}, {}^{Lt}_{t \rightarrow t_0} D^{q-1} Z(t) = O_{n \times n}. \quad (4.9)$$

We know that from Theorem (3.1) ,

$$\begin{aligned} {}^{Lt}_{t \rightarrow t_0} D^{q-1} X(t) &= I_{n \times n} \text{ implies } (t - t_0)^{1-q} X(t) \big/_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}, \\ {}^{Lt}_{t \rightarrow t_0} D^{q-1} Y(t) &= O_{n \times n} \text{ implies } (t - t_0)^{1-q} Y(t) \big/_{t=t_0} = O_{n \times n} \quad \text{and} \\ {}^{Lt}_{t \rightarrow t_0} D^{q-1} Z(t) &= O_{n \times n} \text{ implies } (t - t_0)^{1-q} Z(t) \big/_{t=t_0} = O_{n \times n}. \end{aligned}$$

By verification we show that $X(t), Y(t)$ and $Z(t)$ are solutions of the given system.

Consider

$$\begin{aligned} X(t) &= (t - t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A + B)^{3k}(t - t_0)^{3kq}}{\Gamma(3k + 1)q} \\ &= \sum_{k=0}^{\infty} \frac{(A + B)^{3k}(t - t_0)^{(3k+1)q-1}}{\Gamma(3k + 1)q} \end{aligned}$$

Differentiating w.r.t., 't' we get

$$D^q X(t) = \sum_{k=0}^{\infty} \frac{(A + B)^{3k}(t - t_0)^{3kq-1} \Gamma(3k + 1)q}{\Gamma(3k + 1)q \Gamma 3kq}$$

Again differentiating w.r.t., 't' we get

$$D^{2q}X(t) = \sum_{k=1}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k-1)q-1} \Gamma 3kq}{\Gamma(3k-1)q \Gamma 3kq}$$

Again differentiating w.r.t., 't' we get

$$\begin{aligned} D^{3q}X(t) &= \sum_{k=1}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k-2)q-1} \Gamma(3k-1)q}{\Gamma(3k-1)q \Gamma(3k-2)q} \\ &= \sum_{k=1}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k-2)q-1}}{\Gamma(3k-2)q} \end{aligned}$$

Consider $A^3X + 3A^2XB + 3AXB^2 + XB^3$

$$\begin{aligned} &= A^3 \left[\sum_{k=0}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k+1)q-1}}{\Gamma(3k+1)q} \right] + 3A^2 \left[\sum_{k=0}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k+1)q-1}}{\Gamma(3k+1)q} \right] B \\ &+ 3A \left[\sum_{k=0}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k+1)q-1}}{\Gamma(3k+1)q} \right] B^2 + \left[\sum_{k=0}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k+1)q-1}}{\Gamma(3k+1)q} \right] B^3 \\ &= A^3 \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^3(t-t_0)^{4q-1}}{\Gamma 4q} + \frac{(A+B)^6(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \right] \\ &+ 3A^2 \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^3(t-t_0)^{4q-1}}{\Gamma 4q} + \frac{(A+B)^6(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \right] B \\ &+ 3A \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^3(t-t_0)^{4q-1}}{\Gamma 4q} + \frac{(A+B)^6(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \right] B^2 \\ &+ \left[\frac{(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^3(t-t_0)^{4q-1}}{\Gamma 4q} + \frac{(A+B)^6(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \right] B^3 \\ &= (A^3 + 3A^2B + 3AB^2 + B^3) \frac{(t-t_0)^{q-1}}{\Gamma q} \\ &\quad + (A^3 + 3A^2B + 3AB^2 + B^2)(A+B)^3 \frac{(t-t_0)^{4q-1}}{\Gamma 4q} \\ &\quad + (A^3 + 3A^2B + 3AB^2 + B^3)(A+B)^6 \frac{(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \\ &= \frac{(A+B)^3(t-t_0)^{q-1}}{\Gamma q} + \frac{(A+B)^6(t-t_0)^{4q-1}}{\Gamma 4q} + \frac{(A+B)^9(t-t_0)^{7q-1}}{\Gamma 7q} + \dots \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{(3k-2)q-1}}{\Gamma(3k-2)q} \\
 &= D^{3q}X(t)
 \end{aligned}$$

which implies that $X(t)$ is the solution of the given system.

Similarly $Y(t)$ and $Z(t)$ are also the solutions of the given system.

Also

$$(t-t_0)^{1-q}X(t) \Big|_{t=t_0} = (t-t_0)^{1-q}(t-t_0)^{q-1} \sum_{k=0}^{\infty} \frac{(A+B)^{3k}(t-t_0)^{3kq}}{\Gamma(3k+1)q} \Big|_{t=t_0} = \frac{I_{n \times n}}{\Gamma q}$$

$$\text{and similarly } (t-t_0)^{1-q}Y(t) \Big|_{t=t_0} = O_{n \times n}, \quad (t-t_0)^{1-q}Z(t) \Big|_{t=t_0} = O_{n \times n}.$$

Hence initial conditions are satisfied .

Conclusion: In this paper we have studied matrix differential equations of the type $D^q X = AX$ and $D^q X = AX + XB$ and introduced fractional Cosine Matrix function and fractional Sine Matrix function as solutions of second order fractional matrix differential equations.

References

- [1] Herrmann, R. Fractional Calculus An Introduction For physicists., published by World Scientific Publishing Co. Pte. Ltd., Singapore , (2011).
- [2] Kilbas, Anatoly. A., Srivastava, Hari M. and Trujillo, Juan J., Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006).
- [3] Lakshmikantham, V., Leela. S. and Vasundhara Devi, J., Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cottenham, UK (2009).
- [4] Oldham. K. B. and Spanier. J., The Fractional Calculus, Academic Press, New York, 1974.
- [5] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.
- [6] Richard Bellman, Introduction to Matrix Analysis, Mc Graw - Hill Book Company, New York, 1960.

- [7] Vasundhara Devi, J., Deo, S.G., and Nagamani, S., On Fractional Trigonometric Functions and Their Generalizations, Dynamic Systems and Applications 22 (2013) 337 - 354.
- [8] Vasundhara Devi, J. and Nagamani, S., On Fractional Hyperbolic Functions and Their Generalizations, Nonlinear Studies, CSP - Cambridge, UK; I&S - Florida, USA, (2013) Vol. 20, No.3, pp. 331 - 348.
- [9] Vasundhara Devi, J., Ravikumar R. V. G. and Giri Babu. N., On Graph Differential Equations and Its associated Matrix Differential Equations, Malaya Journal of Matematic , Vol.1, No.1, 2013, pp.1-9.