

ON GENERALIZED η -DUALS OF SOME DOUBLE SEQUENCE SPACES

Seraj Ahmed Khan, Ashfaque A. Ansari and Nafis Ahmad*

Department of Mathematics & Statistics,
D.D.U. Gorakhpur University, Gorakhpur-273009, (U.P.) India

*Department of Mathematics,
Shibli National P.G. College, Azamgarh (U.P.) India
E-mail:- 786naksl@gmail.com

Dedicated to Prof. M.A. Pathan on his 75th birth anniversary

Abstract: Ansari and Gupta [1] have generalized the η -dual of some double sequence spaces for $0 < r \leq 1$ and have established their perfectness in relation to η -dual of order r , $0 < r \leq 1$.

We have defined and determined the η -duals of some double sequence spaces of order r , for $r > 0$.

Keywords: η -dual, dual space, perfect spaces, l_r -space, convergent sequence, bounded variation.

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1. Introduction

The notion of double sequences is found Brownich [2]. Hardy [4] introduced the notion of bounded variation double sequences. Later on it was investigated by Moricz [12], Tripathy [12], Basarir [13], Patterson [14], Sonalcan [13] and many others.

Köthe and Toeplitz [8] introduced the idea of dual sequence space, whose main result concerned with η -duals. Later on it was studied by P. Chandra and B.C. Tripathy [15], Cook [4], Kamtham and Gupta [6], Maddox [10] and many other. P. Chandra and B.C. Tripathy [15] have generalized the notation of α -duals on introducing the notation of η -duals of order r , for $r \geq 1$ of sequence spaces and Ansari & Gupta [1] have generalized the notion of α -duals on introducing the notion of η -duals of order r , for $0 < r \leq 1$ of sequence spaces.

In this paper, the space of all, bounded, convergent in Pringsheim's sense, regularity null, absolutely summable, p-absolutely summable, finite, bounded variation regularity convergence, null in Pringsheim's sense, eventually alternating and strongly p-cesaro summable double sequence spaces denoted by ${}^2\omega$, ${}^2l_\infty$, ${}^2C$, ${}^2C^R$, ${}^2C_0$, ${}^2C_0^R$, ${}^2l_1$, ${}^2l_p$, ${}^2\Phi$, ${}^2b_v$, ${}^2\sigma$, ${}^2\omega_p$ respectively and double sequence spaces denoted by $\langle a_{mn} \rangle$ or $\langle X_{mn} \rangle$. According as element of η -duals and element of given spaces respectively.

Throughout the article, the sum without limit means that summation is from $m = 1$ to ∞ and $n = 1$ to ∞ .

List of the some double sequence spaces, whose η -dual will be obtained in this paper are

1. ${}^2C_0 = \{ \langle a_{mn} \rangle \in {}^2\omega : a_{mn} \rightarrow 0 \text{ as } \min(m, n) \rightarrow \infty \}$
2. ${}^2C_0^R = \{ \langle a_{mn} \rangle \in {}^2\omega : a_{mn} \rightarrow 0 \text{ as } \max(m, n) \rightarrow \infty \}$
3.
 - a. ${}^2C^R = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \lim_{n \rightarrow \infty} a_{mn} = L_m \text{ where } L_m \in C \text{ for each } m \in N \right\}$
 - b. ${}^2C^R = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \lim_{n \rightarrow \infty} a_{mn} = J_n \text{ where } J_n \in C \text{ for each } m \in N \right\}$
4. ${}^2C = \{ \langle a_{mn} \rangle \in {}^2\omega : a_{mn} \rightarrow L \text{ as } \min(m, n) \rightarrow \infty \text{ for some } L \in C \}$
5. ${}^2l_\infty = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \sup_{m, n} |a_{mn}| < \infty \right\}$
6. ${}^2l_r = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \sum_m \sum_n |a_{mn}| < \infty \right\}$ where r is real number such that $0 < r < \infty$
7. ${}^2b_v = \{ \langle a_{mn} \rangle \in {}^2\omega : \sum |\Delta^m a_{mn}| < \infty, \sum |\Delta^n a_{mn}| < \infty \text{ and } \sum \sum |\Delta^{mn} a_{mn}| < \infty \}$
8. ${}^2\omega_p = \left\{ \langle a_{ij} \rangle \in {}^2\omega : \lim_{m, n \rightarrow \infty} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n |a_{ij} - L|^P = 0 \text{ for some } L \in C \text{ and } P \text{ is real number} \right\}$
9. The space $({}^2\sigma)$ of all eventually alternating double sequence is define as,

$${}^2\sigma = \{ \langle a_{mn} \rangle \in {}^2\omega : a_{mn} = -a_{m, n+1} \text{ for all } n \geq n_0 \text{ and } a_{mn} = -a_{m+1, n} \text{ for all } m \geq m_0 \}$$

The spaces ${}^2C^R$, ${}^2C_0^R$, ${}^2C \cap {}^2l_\infty$, ${}^2C_0 \cap {}^2l_\infty$ and ${}^2l_\infty$ are normed linear spaces with the norm given by

$$\|\langle a_{nk} \rangle\| = \sup_{m,n} |a_{mn}|$$

from the above definition it is clear that

$${}^2C^R \subset {}^2C \cap {}^2l_\infty \subset {}^2l_\infty({}^2C) \text{ and } {}^2C_0^R \subset {}^2C_0 \cap {}^2l_\infty \subset {}^2l_\infty({}^2C_0)$$

Definition 1. The α - dual of a subset E of ${}^2\omega$ defined as,

$$E^\alpha = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \sum_m \sum_n |a_{mn} x_{mn}| < \infty \text{ for all } (x_{mn}) \in E \right\}$$

Definition 2. Let E be a non-empty subset of ${}^2\omega$ and $0 < r \leq 1$. Then η - dual of order r of E is denoted by E^η and defined by Ansari and Gupta [1] as,

$$E^\eta = \left\{ \langle a_{mn} \rangle \in {}^2\omega : \sum_m \sum_n |a_{mn} x_{mn}|^r < \infty \text{ for all } (x_{mn}) \in E \right\}$$

Taking $r = 1$ in above definition we also get the α - dual of E . Also a non-empty subset E of ω is said to be perfect or η - reflexive with respect to η - dual, if $E^{\eta\eta} = (E^\eta)^\eta = E$

Definition 3. Let E be a non-empty subset of ${}^2\omega$ and $r > 0$ then we defined the η - dual of order r of E as

$$E^\eta = \left\{ (a_{mn}) \in {}^2\omega : \sum_m \sum_n |a_{mn} x_{mn}|^r < \infty \text{ for all } (x_{mn}) \in E \right\}$$

Taking $r = 1$ in above definition, we also get the α - dual of E . Also, a non-empty subset E of ω is said to be perfect or η - reflexive with respect to η - dual, if $E^{\eta\eta} = E$. In this paper, the sum without limit means that the summation is form $m = 1$ to ∞ and $n = 1$ to ∞ .

Main Results

The proof of the following lemma is obvious in view of the definition of η - dual of double sequences

Lemma

(i) E^η is a linear subspace of ${}^2\omega$ for every $E \subset {}^2\omega$

(ii) $E \subset F \Rightarrow E^\eta \supset F^\eta$

(iii) $E \subset E^{\eta\eta}$ for every $E \subset {}^2\omega$

Theorem 1. $({}^2l_r)^\eta = {}^2l_\infty$ and $({}^2l_\infty)^\eta = {}^2l_r$. The space ${}^2l_r$ and ${}^2l_\infty$ are perfect spaces for $r > 0$.

Proof: Let $\langle a_{mn} \rangle \in {}^2l_\infty$ we have

$$\sum_m \sum_n |a_{mn} x_{mn}|^r < \infty \text{ for all } (x_{mn}) \in {}^2l_r$$

Hence

$${}^2l_\infty \subseteq ({}^2l_r)^\eta \quad (1)$$

Conversely, let $\langle a_{mn} \rangle \notin {}^2l_\infty$, then there exist a single sequence $\langle a_{i,n_i} \rangle$ such that $a_{i,n_i} \geq i^s$ for some $s > 0$ consider the double sequence $\langle x_{mn} \rangle$ defined by

$$x_{mn} = \begin{cases} i^{-s} & \text{if } m = i, n = n_i, i \in N \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle x_{mn} \rangle \in {}^2l_r$ but $\langle a_{mn} x_{mn} \rangle \notin {}^2l_r$. Hence

$$({}^2l_r)^\eta \subseteq {}^2l_\infty \quad (2)$$

From (1) and (2) we have

$$({}^2l_r)^\eta = {}^2l_\infty$$

Similarly we can prove that,

$$({}^2l_\infty)^\eta = {}^2l_r$$

Since,

$${}^2l_\infty^{\eta\eta} = ({}^2l_\infty^\eta)^\eta = ({}^2l_r)^\eta = {}^2l_\infty$$

and

$${}^2l_r^{\eta\eta} = ({}^2l_r^\eta)^\eta = ({}^2l_\infty)^\eta = {}^2l_r$$

Hence the spaces ${}^2l_\infty$ and ${}^2l_r$ are perfect.

Theorem 2. $({}^2\omega_p \cap {}^2l_\infty)^\eta = {}^2l_r$ and the space ${}^2\omega_p \cap {}^2l_\infty$ is not perfect.

Proof: Since ${}^2\omega_p \cap {}^2l_\infty \subset {}^2l_\infty$. By lemma (ii)

$${}^2l_\infty \subset ({}^2\omega_p \cap {}^2l_\infty)^\eta \text{ and } {}^2l_r \subset ({}^2\omega_p \cap {}^2l_\infty)^\eta \quad (\because {}^2l_\infty = {}^2l_r \text{ from theorem 1}) \quad (3)$$

Let $(a_{mn}) \notin l_r$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}|^r = \infty$$

Consider the sequence (x_{mn}) defined as, $x_{mn} = 1$ for all $m, n \in N$, then

$$(x_{mn}) \in {}^2\omega_p \cap {}^2l_{\infty}$$

but

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}x_{mn}|^r &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}^r| |x_{mn}|^r \\ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}x_{mn}| &= \infty \end{aligned}$$

Hence

$$(a_{mn}) \notin ({}^2\omega_p \cap {}^2l_{\infty})^{\eta}$$

therefore,

$$({}^2\omega_p \cap {}^2l_{\infty}) \subset {}^2l_r \tag{4}$$

From (3) and (4), we have

$$({}^2\omega_p \cap {}^2l_{\infty})^{\eta} = {}^2l_r$$

therefore

$$({}^2\omega_p \cap {}^2l_{\infty})^{\eta\eta} = (({}^2\omega_p \cap {}^2l_{\infty})^{\eta})^{\eta} = {}^2l_r^{\eta} = {}^2l_{\infty} \neq {}^2\omega_p \cap {}^2l_{\infty}$$

Hence the space ${}^2\omega_p \cap {}^2l_{\infty}$ is not perfect.

Theorem 3. ${}^2C_0 = {}^2C^{\eta} = {}^2l_r$ and the sequences spaces ${}^2C_0, {}^2C$ are not perfect, for $r > 0$

Proof: Since ${}^2C_0 \subset {}^2l_{\infty}$. By lemma (ii) ${}^2l_{\infty}^{\eta} \subset {}^2C_0^{\eta}$ and by theorem (1)

$$\begin{aligned} {}^2l_r &= {}^2l_{\infty}^{\eta} \\ \Rightarrow {}^2l_r &\subset {}^2C_0^{\eta} \end{aligned} \tag{5}$$

Again, let $(a_{mn}) \in {}^2C_0^{\eta}$

$$\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}x_{mn}| < \infty \text{ for every } (x_{mn}) \in {}^2C_0$$

$$\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}^r z_{mn}| < \infty, \text{ for every } z_{mn} = x_{mn}^r \in {}^2C_0$$

$$\Rightarrow (a_{mn})^r \in {}^2C_0^\alpha = {}^2l_1$$

$$\Rightarrow (a_{mn}) \in {}^2l_r$$

Thus

$${}^2C_0^\eta \subset {}^2l_r \quad (6)$$

From (5) and (6), we have

$${}^2C_0^\eta = {}^2l_r \quad (7)$$

Since, ${}^2C_0 \subset {}^2C \subset {}^2l_\infty$ by lemma (ii)

$${}^2l_\infty \subset {}^2C^\eta \subset {}^2C_0^\eta$$

From the theorem (1) and result (7) we get

$${}^2l_r = {}^2l_\infty^\eta = {}^2C^\eta = {}^2C_0^\eta = {}^2l_r$$

$$\Rightarrow {}^2C_0^\eta = {}^2C^\eta = {}^2l_r$$

therefore

$${}^2C_0^{\eta\eta} = ({}^2C_0^\eta)^\eta = {}^2l_r^\eta = {}^2l_\infty \neq {}^2C_0$$

and

$${}^2C^{\eta\eta} = ({}^2C^\eta)^\eta = {}^2l_r^\eta = {}^2l_\infty \neq {}^2C$$

Hence the sequence spaces ${}^2C_0$ and ${}^2C$ are not perfect with respect to η - dual for $r > 0$.

Theorem 4. $({}^2C_0 \cap {}^2l_\infty)^\eta = ({}^2C \cap {}^2l_\infty)^\eta = {}^2l_r$ and the spaces ${}^2C_0 \cap {}^2l_\infty$ and ${}^2C \cap {}^2l_\infty$ are not perfect, for $r > 0$.

Proof: Since ${}^2C_0 \subset {}^2l_\infty$, therefore ${}^2C_0 \cap {}^2l_\infty = {}^2C_0$

$$\Rightarrow ({}^2C_0 \cap {}^2l_\infty)^\eta = {}^2C_0^\eta = {}^2l_r \quad (\text{by theorem 3}) \quad (8)$$

Also, ${}^2C \subset {}^2l_\infty$, therefore ${}^2C \cap {}^2l_\infty = {}^2C$

$$\Rightarrow ({}^2C \cap {}^2l_\infty)^\eta = {}^2C^\eta = {}^2l_r \quad (\text{by theorem 3}) \quad (9)$$

From (8) and (9), we find

$$({}^2C_0 \cap {}^2l_\infty)^\eta = ({}^2C \cap {}^2l_\infty)^\eta = {}^2l_r$$

$$({}^2C_0 \cap {}^2l_\infty)^{\eta\eta} = (({}^2C_0 \cap {}^2l_\infty)^\eta)^\eta = {}^2l_r^\eta = {}^2l_\infty \neq {}^2C_0 \cap {}^2l_\infty$$

and

$$({}^2C \cap {}^2l_\infty)^{\eta\eta} = (({}^2C \cap {}^2l_\infty)^\eta)^\eta = {}^2l_r^\eta = {}^2l_\infty \neq {}^2C \cap {}^2l_\infty$$

Hence, the spaces ${}^2C_0 \cap {}^2l_\infty$ and ${}^2C \cap {}^2l_\infty$ are not perfect.

Theorem 5. $({}^2\sigma)^\eta = {}^2l_r$ the space ${}^2\sigma$ is not perfect, for $r > 0$.

Proof: We have

$${}^2\sigma \subseteq {}^2l_\infty \Rightarrow {}^2l_r \subset ({}^2\sigma)^\eta \quad (10)$$

let $(x_{mn}) \in ({}^2\sigma)^\eta$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}x_{mn}|^r < \infty \text{ for all } \langle a_{mn} \rangle \in {}^2\sigma$$

consider the sequence $\langle a_{mn} \rangle$ defined as,

$$a_{mn} = 1 = -a_{m+1,n} = -a_{m,n+1}, \text{ for all } m, n \in N$$

then $\langle a_{mn} \rangle \in {}^2\sigma$

$$\begin{aligned} &\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^r < \infty \\ &\Rightarrow (x_{mn}) \in {}^2l_r \\ &\Rightarrow ({}^2\sigma)^\eta \subseteq {}^2l_r \end{aligned} \quad (11)$$

From (10) and (11), we get

$$({}^2\sigma)^\eta = {}^2l_r$$

The rest of the proof follows from theorem (1).

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