

SOLUTIONS OF PARAMETRIC QUADRATIC DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$ AND $\mathbb{F}_p$

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Abstract: In this paper, we find the integer solutions to the Diophantine equation $D : x^2 - (a^2b^2c^2 - ab)y^2 - (4ab - 2)x + (4a^2b^2c^2 - 4ab)y + 4(a^2b^2 - a^2b^2c^2) = 0$ over $\mathbb{Z}$ and finite fields $\mathbb{F}_p$ for a, b and c are integers and primes $p \geq 2$, respectively. Moreover, We find positive integer solutions of the equation D in terms of generalized Fibonacci and Lucas sequences.

Keywords and Phrases: Diophantine Equation, Pell Equation, Continued Fractions, Finite fields, Fibonacci and Lucas sequences.

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1. Introduction

We call a Diophantine equation an equation of the form

$$f(x_1, x_2, \dots, x_n) = 0 \tag{1.1}$$

Where f is an n -variable function with $n \geq 2$. If f is a polynomial with integral coefficients, then (1.1) is an algebraic Diophantine equation. An n -tuple $(x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{Z}^n$ satisfying (1.1) is called a solution of the equation. An equation having one or more solutions is called solvable (see[1]). Diophantus's work on equations of type (1.1) was continued by Chinese mathematicians, Arabs and taken to a deeper level by Fermat, Euler, Lagrange, Gauss, and many others. This topic remains of great importance in contemporary mathematics (see[2],[3]). There is no universal method for determining whether a Diophantine equation has a solution, or for finding them all if solutions exists. However, we are quite successful in dealing with polynomials of low degree, or in a small number of variables (see[10]).

In this paper, we consider the Diophantine equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (1.2)$$

where all coefficients and variables are integers. Let $d \neq 1$ be a positive square free integer and N be any fixed integer. Then the equation $x^2 - dy^2 = \pm N$ is known as Pell equation and is named after John Pell, a mathematician who searched for integer solutions to equations of this type in the seventeenth century. For $N = 1$, the Pell equation $x^2 - dy^2 = \pm 1$ is known as classical Pell equation and was studied by Brahmagupta and Bhaskara. The Pell equation $x^2 - dy^2 = \pm 1$ has infinitely many solutions (x_n, y_n) for $n \geq 1$. There are several methods for finding the fundamental solutions of Pell's equation $x^2 - dy^2 = 1$ for a positive non square integer ' d ' e.g. the continued fraction method [12]. Continued fraction plays an important to find solutions of the Pell equations $x^2 - dy^2 = \pm 1$. Whether or not there exist a positive integer solution to the equation $x^2 - dy^2 = 1$ depends on the period length of the continued fraction expansion to $\sqrt{d}$ (see[5], [9]).

Let $a = t, b = 1$ and $c = 1$ then the equation (2.1) reduces to $x^2 - (t^2 - t)y^2 - (4t - 2)x + (4t^2 - 4t)y = 0$ which is established in [16].

If $a = 1$ and $b = 1$ then the Pell equation (2.4) reduces to the particular case of $x^2 - (b^2 - b)y^2 = N$ when $N \in \{\pm 1, \pm 4\}$ which is studied in [11].

2. Preliminaries

The famous Diophantine equation $x^2 - dy^2 = 1$ which is a special case of (1.2) is known as Pell equation (see[2],[10]). This equation has infinitely many integer solutions (x_n, y_n) for $n \geq 1$. The first positive integer solution (x_1, y_1) of this equation is called the fundamental solution, because all other solutions can be derived from it then the n^{th} positive solution is (x_n, y_n) , is defined by the equality $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ for an integer $n \geq 2$ (for further details on Pell and Diophantine equations (see[1]-[16]).

The following lemma and theorems can be found from the references [4], [5], [6], [9], and [13].

Lemma 2.1. *Let l be the period length of continued fraction expansion of $\sqrt{d}$. If l is even, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by,*

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no integer solutions. If l is odd, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by $x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}$ and the fundamental solution to the equation $x^2 - dy^2 = -1$ is given by,

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}.$$

Theorem 2.1. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 1$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by,*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n, \text{ with } n \geq 1.$$

Generalized Fibonacci sequence is defined by $F_0(k, s) = 0$, $F_1(k, s) = 1$ and $F_{n+1} = kF_n(k, s) + sF_{n-1}(k, s)$ for $n \geq 1$ and generalized Lucas sequence is defined by $L_0(k, s) = 2$, $L_1(k, s) = k$ and $L_{n+1} = kL_n(k, s) + sL_{n-1}(k, s)$ for $n \geq 1$, respectively. It is well known that $F_n(k, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n(k, s) = \alpha^n + \beta^n$. The above identities are known as Binet's formula. For more information about generalized Fibonacci and Lucas sequences one can refer [7],[8],[11],[13], and [14].

Now we consider the integer solutions of the Diophantine equation

$$D : x^2 - (a^2b^2c^2 - ab)y^2 - (4ab - 2)x + (4a^2b^2c^2 - 4ab)y + 4(a^2b^2 - a^2b^2c^2) = 0 \quad (2.1)$$

over integers. It is very difficult to solve equation (2.1) in this form. That is, we can't determine how many integer solutions (2.1) had and what they are. So we transform (2.1) into an appropriate Diophantine equation which can be easier to solve. To do this let

$$T : \begin{cases} x = u + h, \\ y = v + k \end{cases} \quad (2.2)$$

be a translation for some h and k . In this case the pair h, k is called the base of T and denoted by $T[h; k] = \{h, k\}$. If we apply T to (2.1), then we get

$$\begin{aligned} T(D) = \tilde{D} : (u + h)^2 - (a^2b^2c^2 - ab)(v + k)^2 - (4ab - 2)(u + h) \\ + (4a^2b^2c^2 - 4ab)(v + k) - 4(a^2b^2 - a^2b^2c^2) = 0 \end{aligned} \quad (2.3)$$

Equating the coefficients of u and v of (2.3) on both sides, then we get $2h - 4ab + 2 = 0$ and $-2a^2b^2c^2k + 2abk + 4a^2b^2c^2 - 4ab = 0$ and hence $h = 2ab - 1$ and $k = 2$. So for these values of h and k we get $x = u + 2ab - 1$ and $y = v + 2$. Consequently (2.3) reduces into

$$\tilde{D} : u^2 - (a^2b^2c^2 - ab)v^2 = 1 \quad (2.4)$$

which is a Pell equation.

3. Main Results

3.1. Solutions of (2.4) over $\mathbb{Z}$

Theorem 3.1. *For the Pell equation (2.4), the following are true:*

1. The fundamental solution is $(u_1, v_1) = (2abc^2 - 1, 2c)$

2. (u_n, v_n) is a solution of (2.4), where

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 2abc^2 - 1 & 2a^2b^2c^3 - 2abc \\ 2c & 2abc^2 - 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.1)$$

for $n \geq 1$.

3. The solutions (u_n, v_n) satisfy the recurrence relations $u_n = (2abc^2 - 1)u_{n-1} + (2a^2b^2c^3 - 2abc)v_{n-1}$ and $v_n = (2c)u_{n-1} + (2abc^2 - 1)v_{n-1}$ for $n \geq 2$.

4. The solutions (u_n, v_n) satisfy the equation,

$$(u_n, v_n) = \left(\frac{L_n(4abc^2 - 2, -1)}{2}, 2cF_n(4abc^2 - 2, -1) \right),$$

Where F_n denotes Fibonacci sequence and L_n denotes Lucas sequence.

5. $\sqrt{a^2b^2c^2 - ab} = [abc; \overline{-2c, 2abc}]$

6. The n^{th} solution (u_n, v_n) can be given by

$$\frac{u_n}{v_n} = [abc; \overline{(-2c, 2abc)_{n-1}}, -2c] \quad (3.2)$$

for $n \geq 1$, where $(-2c, 2abc)_{n-1}$ means that there are $n - 1$ successive terms $(-2c, 2abc)$.

Proof

1. It is clear that $(u_1, v_1) = (2abc^2 - 1, 2c)$ is the fundamental solution of (2.4). Since $(2abc^2 - 1)^2 - (a^2b^2c^2 - ab)(2c)^2 = 1$.

2. By induction: Let first $n = 1$. Then by equation (3.1) we get $(u_1, v_1) = (2abc^2 - 1, 2c)$ which is the fundamental solution. Let us assume that this equation is satisfied for $n - 1$, that is, $u_{n-1}^2 - (a^2b^2c^2 - ab)v_{n-1}^2 = 1$. We have

to show that this equation is also satisfied for n . Applying (3.1) we find that

$$\begin{aligned}
 \begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} 2abc^2 - 1 & 2a^2b^2c^3 - 2abc \\ 2c & 2abc^2 - 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 2abc^2 - 1 & 2a^2b^2c^3 - 2abc \\ 2c & 2abc^2 - 1 \end{pmatrix} \begin{pmatrix} 2abc^2 - 1 & 2a^2b^2c^3 - 2abc \\ 2c & 2abc^2 - 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 2abc^2 - 1 & 2a^2b^2c^3 - 2abc \\ 2c & 2abc^2 - 1 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} \\
 &= \begin{pmatrix} (2abc^2 - 1)u_{n-1} + (2a^2b^2c^3 - 2abc)v_{n-1} \\ 2cu_{n-1} + (2abc^2 - 1)v_{n-1} \end{pmatrix} \tag{3.3}
 \end{aligned}$$

and hence $u_n^2 - (a^2b^2c^2 - ab)v_n^2 = [(2abc^2 - 1)u_{n-1} + (2a^2b^2c^3 - 2abc)v_{n-1}]^2 - (a^2b^2c^2 - ab)[2cu_{n-1} + (2abc^2 - 1)v_{n-1}]^2 = u_{n-1}^2 - (a^2b^2c^2 - ab)v_{n-1}^2 = 1$. So (u_n, v_n) is also a solution of (2.4).

3. Applying (3.3), We see that $u_n = (2abc^2 - 1)u_{n-1} + (2a^2b^2c^3 - 2abc)v_{n-1}$ and $v_n = (2c)u_{n-1} + (2abc^2 - 1)v_{n-1}$ for $n \geq 2$.
4. Let $d = a^2b^2c^2 - ab$. Then the fundamental solution to the equation (2.4) is $u_1 + v_1\sqrt{d} = 2abc^2 - 1 + 2c\sqrt{d}$.

Let

$$\alpha = 2abc^2 - 1 + 2c\sqrt{d}, \beta = 2abc^2 - 1 - 2c\sqrt{d},$$

$$\alpha + \beta = 4abc^2 - 2, \alpha - \beta = 4c\sqrt{d}, \alpha\beta = 1.$$

Therefore,

$$u_n + v_n\sqrt{d} = (u_1 + v_1\sqrt{d})^n, \quad u_n + v_n\sqrt{d} = \alpha^n, \quad u_n - v_n\sqrt{d} = \beta^n,$$

$$u_n = \frac{1}{2}(L_n(4abc^2 - 2, -1)), \quad v_n = 2cF_n(4abc^2 - 2, -1).$$

Therefore, all positive integer solutions of the equation (2.4) is,

$$(u_n, v_n) = \left(\frac{L_n(4abc^2 - 2, -1)}{2}, 2cF_n(4abc^2 - 2, -1) \right)$$

with $n \geq 1$.

5.

$$\begin{aligned}
\sqrt{d} &= \sqrt{a^2b^2c^2 - ab} \\
&= abc + \sqrt{a^2b^2c^2 - ab} - abc \\
&= abc + \frac{1}{\frac{1}{\sqrt{a^2b^2c^2 - ab} - abc}} \\
&= abc + \frac{1}{\frac{\sqrt{a^2b^2c^2 - ab} + abc}{a^2b^2c^2 - ab - a^2b^2c^2}} \\
&= abc + \frac{1}{\frac{\sqrt{a^2b^2c^2 - ab} + 2abc - abc}{-ab}} \\
&= abc + \frac{1}{\frac{2abc}{-ab} + \frac{\sqrt{a^2b^2c^2 - ab} - abc}{-ab}} \\
&= abc + \frac{1}{-2c + \frac{1}{\frac{1}{-ab}}} \\
&= abc + \frac{1}{-2c + \frac{1}{\frac{-ab(\sqrt{a^2b^2c^2 - ab} + abc)}{a^2b^2c^2 - ab - a^2b^2c^2}}} \\
&= abc + \frac{1}{-2c + \frac{1}{\sqrt{a^2b^2c^2 - ab} + abc}} \\
&= abc + \frac{1}{-2c + \frac{1}{\frac{1}{2abc + \frac{1}{-2c + \frac{2abc}{\sqrt{a^2b^2c^2 - ab} - abc}}}}}
\end{aligned}$$

Hence the continued fraction expansion of $\sqrt{d}$ is $\sqrt{d} = [abc; \overline{-2c, 2abc}]$.

6. Note that $\frac{u_1}{v_1} = \frac{2abc^2 - 1}{2c}$, which means that (u_1, v_1) is the fundamental solution. Let us assume that (u_n, v_n) is a solution of (2.4). Then by (3.2), we

derive

$$\begin{aligned} \frac{u_{n+1}}{v_{n+1}} &= abc + \frac{1}{-2c + \frac{1}{2abc + \frac{1}{-2c + \frac{1}{2abc + \frac{1}{\dots + 2abc + \frac{1}{2}}}}}} \\ &= \frac{(2abc^2 - 1)u_n + (2a^2b^2c^3 - 2abc)v_n}{2cu_n + (2abc^2 - 1)v_n}. \end{aligned}$$

So (u_{n+1}, v_{n+1}) is also a solution of (2.4). Since $u_{n+1}^2 - (a^2b^2c^2 - ab)v_{n+1}^2 = u_n^2 - (a^2b^2c^2 - ab)v_n^2 = 1$.

Using the above theorem we can give the following corollary.

Corollary 3.1. *The base of the transformation (2.2) is the fundamental solution of (2.4), that is, $T[h; k] = \{h, k\} = \{u_1, v_1\}$.*

Proof. We proved that $(u_1, v_1) = (2abc^2 - 1, 2c)$ is the fundamental solution of (2.4). Also we showed in (2.3) that $h = 2ab - 1$ and $k = 2$. Hence the result is clear.

We see that (2.1) could be transformed into (2.4) using the transformation (2.2). Also we showed that $x = u + 2ab - 1$ and $y = v + 2$. So we can retransfer all results from (2.4) to (2.1) by using the inverse of (2.2). Thus we can give the following main theorem.

Theorem 3.2. *Let (2.1) be the Diophantine equation. Then we give the following:*

1. *The fundamental solution of (2.1) is $(x_1, y_1) = (2abc^2 + 2ab - 2, 2c + 2)$.*
2. *For the sequence $\{(x_n, y_n)\}_{n \geq 1} = \{(u_n + 2ab - 1, v_n + 2)\}$, where $\{(u_n, v_n)\}$ defined in (3.1), (x_n, y_n) is a solution of (2.1). So (2.1) has infinitely many integer solutions $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.*
3. *The solutions (x_n, y_n) satisfy the recurrence relations $X_n = (2abc^2 - 1)X_{n-1} + (2a^2b^2c^3 - 2abc)Y_{n-1} - 4a^2b^2c^3 - 4a^2b^2c^2 + 2abc^2 + 4abc + 4ab - 2$ for $n \geq 2$ and $Y_n = 2cX_{n-1} + (2abc^2 - 1)Y_{n-1} - 4abc^2 - 4abc + 2c + 4$ for $n \geq 2$.*
4. *The n^{th} solution (x_n, y_n) can be given by $\frac{X_n - 2ab + 1}{Y_n - 2} = [abc; \overline{(-2c, 2abc)}_{n-1}, 2]$, and $\frac{X_n - 2ab + 1}{Y_n - 2} = \left(\frac{L_n(4abc^2 - 2, -1)}{2}, 2cF_n(4abc^2 - 2, -1) \right)$ for $n \geq 1$.*

Example 3.1. Consider $d = a^2b^2c^2 - ab$. Let $a = 1, b = 2$ and $c = 3$ then $(u_1, v_1) = (35, 6)$ is the fundamental solution of the Pell equation $u^2 - 34v^2 = 1$ and hence $(u_2, v_2) = (2449, 420)$ and $(u_3, v_3) = (171395, 29394)$ and etc. After substituting the values of a, b, u, v, h and k in the transformation (2.2) we get, $(x_1, y_1) = (38, 8)$, $(x_2, y_2) = (2452, 422)$, $(x_3, y_3) = (171398, 29396)$ and etc.

Further we can check the integer solutions of the equation $u^2 - 34v^2 = 1$ using Fibonacci and Lucas sequences. Substitute $a = 1, b = 2, c = 3$ and $n = 2$ in 4th result of Theorem 3.1 we find $(u_2, v_2) = (2449, 420)$, $(x_2, y_2) = (2452, 422)$ and etc.

3.2. Solutions of (2.4) over finite fields $\mathbb{F}_p$

In this section, we will find the integer solutions of (2.4) over finite fields $\mathbb{F}_p$ for primes $p \geq 2$, where $a, b, c \in \mathbb{F}_p^*$. For $p = 2, b = 1$ and $c = 1$ then the equation $u^2 - (a^2 - a)v^2 \equiv 1 \pmod{2}$ has two solutions $(1, 0), (1, 1)$ and for $p = 3$, the equation $u^2 - (a^2 - a)v^2 \equiv 1 \pmod{3}$ has six solutions $(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)$ for $a = 1$ and has four solutions $(0, 1), (0, 2), (1, 0), (2, 0)$ for $a = 2$.

Now we consider the general case for primes $p \geq 5$. Let $a^2 - a \equiv d \pmod{p}$. Then equation (2.4) transforms to

$$\tilde{D}_p^d : u^2 - dv^2 \equiv 1 \pmod{p} \quad (3.4)$$

Let $\tilde{D}_p^d(\mathbb{F}_p) = \{(u, v) \in \mathbb{F}_p \times \mathbb{F}_p : u^2 - dv^2 \equiv 1 \pmod{p}\}$. Then we can give the following theorem.

Theorem 3.3. Let $\tilde{D}_p^d$ be the Diophantine equation in (3.4). Then the number of solutions in

$$\tilde{D}_p^d(\mathbb{F}_p) = \begin{cases} p - 1, & \text{for } d \in \mathbb{Q}_p, \\ p + 1, & \text{for } d \notin \mathbb{Q}_p, \end{cases}$$

where $\mathbb{Q}_p$ denotes the set of quadratic residues modulo p .

Proof. Refer [16].

Note that for $p = 2$, (2.1) has two solutions $(1, 0), (1, 1)$ and for $p = 3$ it has six solutions $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)$ for $a = 1, b = 1, c = 1$ and has four solutions $(0, 0), (0, 2), (1, 1), (2, 1)$ for $a = 2, b = 1, c = 1$. So we can give the following result.

Theorem 3.4. Let (2.1) be the Diophantine equation. Then

$$D(\mathbb{F}_p) = \begin{cases} p - 1, & \text{for } a^2b^2c^2 - ab \in \mathbb{Q}_p, \\ p + 1, & \text{for } a^2b^2c^2 - ab \notin \mathbb{Q}_p, \end{cases}$$

where $D(\mathbb{F}_p)$ denotes the set of integer solutions of (2.1) over $\mathbb{F}_p$ for primes $p \geq 5$.

Example 3.2. Let $a = 2, b = 2$ and $c = 1$. Then $12 \in \mathbb{Q}_{13}$ and $12 \notin \mathbb{Q}_{29}$. So

$$D_{13}^{12}(\mathbb{F}_{13}) = \left\{ \begin{array}{l} (0, 1), (0, 12), (1, 0), (2, 6), (2, 7), (6, 2), \\ (6, 11), (7, 9), (7, 11), (11, 6), (11, 7), (12, 0) \end{array} \right\}$$

$$D_{29}^{12}(\mathbb{F}_{29}) = \left\{ \begin{array}{l} (1, 0), (2, 14), (2, 15), (3, 7), (3, 22), (4, 9), (4, 20), (7, 2), (7, 27), (10, 1), \\ (10, 28), (12, 13), (12, 16), (14, 3), (14, 26), (17, 13), (17, 16), (19, 1), (19, 28), (22, 2), \\ (22, 27), (25, 9), (25, 20), (26, 7), (26, 22), (27, 14), (27, 15), (15, 3), (15, 26), (28, 0) \end{array} \right\}$$

References

- [1] Andreescu T, Andrica D and Cucurezeanu I, An Introduction to Diophantine Equations, Birkhauser, 2010.
- [2] Barbeau E J, Pell's Equation, Springer - Verlag Series, New York, Berlin, Heidelberg, 2003.
- [3] Jacobson M J and Williams H C, Solving the Pell Equations, Springer, 2006.
- [4] Jones J P, Representation of Solutions of Pell Equations Using Lucas Sequences, Acto Academia Paed. Agr., Section Mathematica, 30 (2003), 75-86.
- [5] Keskin R, Solutions of Some Quadratic Diophantine equations, Computers and Mathematics with Applications, 60 (2010), 2225-2230.
- [6] Keskin R and Siar Z, Positive integer solutions of the diophantine equation $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$, Proc. Indian Acad. Sci. (Math. Sci.) 124(3) (2014) 301-313.
- [7] Keskin R and Demirturk B, Integer Solutions of Some Diophantine equations Via Fibonacci and Lucas numbers, J. Integer seq. 12 (2009).
- [8] Koshy T, Fibonacci and Lucas numbers with Applications, John Wiley and Sons, New York, 2001.
- [9] Mollin R A, Simple continued fraction solutions for Diophantine equations, Expositions Math. 19(1) (2001) 55-73.
- [10] Niven I, Zuckerman H S and Montgomery H L, An Introduction to the theory of Numbers, John Wiley and Sons, Inc., New York, 1991.

- [11] Peker B and Senay H, Solutions of the Parametric Pell Equation $x^2 - (b^2 - b)y^2 = N$ via Generalized Fibonacci and Lucas Numbers, *Int. J. Contemp. Math. Sciences*, 8(13) (2013) 637-642.
- [12] Sadhasivam V, Kalaimani T and Ambika S, Solutions of the Pell Equation $x^2 - (a^2b^2c^2 + 2ab)y^2 = N$ when $N \in \{\pm 1, \pm 4\}$, *Journal of Ramanujan Society of Mathematics and Mathematical Sciences* 3(2) (2014) 79-88.
- [13] Smarandache F, Method to solve the Diophantine Equation $ax^2 - by^2 + c = 0$, In collected papers, Vol.1. lupton, AZ: Erhus University Press, 1996.
- [14] Tekcan A, The Pell equation $x^2 - Dy^2 = \pm 4$, *Appl. Math. Sci. (Ruse)* 1 (2007), 5-8, 363-369.
- [15] Tekcan A, Continued Fractions Expansion of $\sqrt{D}$ and Pell Equation $x^2 - Dy^2 = 1$, *Mathematica Moravica*, 15(2) (2015) 19-27.
- [16] Tekcan A and Ozkoc A, Quadratic Diophantine Equation $x^2 - (t^2 - t)y^2 - (4t - 2)x + (4t^2 - 4t)y = 0$, *Bull. Malays. Math. Sci. Soc. (2)* 33(2) (2010), 273-280.