

TRANSFORMATION FORMULAE INVOLVING PARTIAL MOCK THETA FUNCTIONS

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Abstract: In this paper certain transformation formulae involving partial mock theta functions have been established.

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1. Introduction Notations and Definitions

Early in 1920, three months before his death, Ramanujan wrote his last letter to Professor G.H. Hardy. In the course of it he said: "I discovered very interesting functions recently which I call 'Mock' theta functions. Unlike the 'False' theta functions they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

The first three pages in which Ramanujan explained what he meant by a "Mock-theta function" are very obscure. They will be made clearer by Hardy's comment that a Mock theta-function is a function defined by a q -series, convergent when $|q| < 1$, for which we can calculate asymptotic formula when q -tends to a rational point $e^{2\pi ir/s}$ of the unit circle, of the same degree of precision as these furnished for the ordinary theta functions by the theory of linear transformation. The last two pages of Ramanujan's notes consist of lists of definitions of four sets of Mock theta-functions with statements of relations connecting members of each of the first three sets; for fairly obvious reasons the functions in the various sets.

(a)

Mock theta functions of order three**Partial mock theta functions
of order three**

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}, \quad f_m(q) = \sum_{n=0}^m \frac{q^{n^2}}{(-q; q)_n^2}. \quad (1.1)$$

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n}, \quad \Phi_m(q) = \sum_{n=0}^m \frac{q^{n^2}}{(-q^2; q^2)_n}. \quad (1.2)$$

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q; q^2)_{n+1}}, \quad \Psi_m(q) = \sum_{n=0}^m \frac{q^{(n+1)^2}}{(q; q^2)_{n+1}}. \quad (1.3)$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\omega q, -\omega^2 q; q)_n}, \quad \chi_m(q) = \sum_{n=0}^m \frac{q^{n^2}}{(-\omega q, -\omega^2 q; q)_n}. \quad (1.4)$$

where $\omega = e^{2\pi i/3}$ (cube root of unity)

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2}, \quad \omega_m(q) = \sum_{n=0}^m \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2}. \quad (1.5)$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}}, \quad \rho_m(q) = \sum_{n=0}^m \frac{q^{2n(n+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}}. \quad (1.6)$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}, \quad \nu_m(q) = \sum_{n=0}^m \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}. \quad (1.7)$$

where $\omega = e^{2\pi i/3}$ (cube root of unity)

(b)

Mock theta functions of order five**Partial mock theta functions
of order five**

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \quad f_{0,m}(q) = \sum_{n=0}^m \frac{q^{n^2}}{(-q; q)_n}. \quad (1.8)$$

$$\Psi_0(q) = \sum_{n=0}^{\infty} (-q; q)_n q^{(n+1)(n+2)/2}, \quad \Psi_{0,m}(q) = \sum_{n=0}^m (-q; q)_n q^{(n+1)(n+2)/2}. \quad (1.9)$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_n}, \quad \chi_{0,m}(q) = \sum_{n=0}^m \frac{q^n}{(q^{n+1}; q)_n}. \quad (1.10)$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} (-q; q^2)_n q^{n^2}, \quad \Phi_{0,m}(q) = \sum_{n=0}^m (-q; q^2)_n q^{n^2}. \quad (1.11)$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}, \quad F_{0,m}(q) = \sum_{n=0}^m \frac{q^{2n^2}}{(q; q^2)_n}. \quad (1.12)$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q)_n}, \quad f_{1,m}(q) = \sum_{n=0}^m \frac{q^{n(n+1)}}{(-q; q)_n}. \quad (1.13)$$

$$\Phi_1(q) = \sum_{n=0}^{\infty} (-q; q^2)_n q^{(n+1)^2}, \quad \Phi_{1,m}(q) = \sum_{n=0}^m (-q; q^2)_n q^{(n+1)^2}. \quad (1.14)$$

$$\Psi_1(q) = \sum_{n=0}^{\infty} (-q; q)_n q^{n(n+1)/2}, \quad \Psi_{1,m}(q) = \sum_{n=0}^m (-q; q)_n q^{n(n+1)/2}. \quad (1.15)$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}}, \quad F_{1,m}(q) = \sum_{n=0}^m \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}}. \quad (1.16)$$

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}}, \quad \chi_{1,m}(q) = \sum_{n=0}^m \frac{q^n}{(q^{n+1}; q)_{n+1}}. \quad (1.17)$$

(c)

Mock theta functions of order seven

Partial mock theta functions
of order seven

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1}; q)_n}, \quad \mathcal{F}_{0,m}(q) = \sum_{n=0}^m \frac{q^{n^2}}{(q^{n+1}; q)_n} \quad (1.18)$$

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1}; q)_{n+1}}, \quad \mathcal{F}_{1,m}(q) = \sum_{n=0}^m \frac{q^{(n+1)^2}}{(q^{n+1}; q)_{n+1}} \quad (1.19)$$

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1}; q)_{n+1}}, \quad \mathcal{F}_{2,m}(q) = \sum_{n=0}^m \frac{q^{n(n+1)}}{(q^{n+1}; q)_{n+1}} \quad (1.20)$$

2. Identities

In this section we shall establish certain identities by making use of Bailey's transform. Identities established here will be used in next section.

The well known Bailey's transform is stated as,
If

$$\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r}, \quad (2.1)$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \quad (2.2)$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \quad (2.3)$$

subject to the convergence of infinite series and the sequences α_r, u_r, v_r and δ_r are rational functions of r alone.

If in Bailey's transform (2.1)-(2.3) we set $u_r = v_r = 1$ and $\delta_r = z^r$ then we have the following identity; If

$$\beta_n = \sum_{r=0}^n \alpha_r \quad (2.4)$$

then

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \alpha_n z^n = (z; q)_{\infty} \sum_{n=0}^{\infty} \beta_n z^n. \quad (2.5)$$

Again, setting $u_r = v_r = 1$ in Bailey's transform we get

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r = \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n-1} \delta_r. \quad (2.6)$$

Choosing $\delta_r = \frac{(\alpha, \beta; q)_r q^r}{(q, \alpha\beta q; q)_r}$ in (2.6), we have

$$\gamma_n = {}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right] - {}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right]_{n-1}, \quad (2.7)$$

where

$${}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right] = \sum_{r=0}^{\infty} \frac{(\alpha, \beta; q)_r q^r}{(q, \alpha\beta q; q)_r}$$

is basic analogue of Gauss's hypergeometric series and

$${}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right]_n = \sum_{r=0}^n \frac{(\alpha, \beta; q)_r q^r}{(q, \alpha\beta q; q)_r}$$

is a truncated basic hypergeometric series.

Making use of the summation formula [2; App. II (II.8)] and [1; (2.1) p. 389] in (2.7), we have

$$\gamma_n = \frac{(1 - \alpha\beta)}{(1 - \alpha)(1 - \beta)} \left[\frac{(\alpha, \beta; q)_\infty}{(q, \alpha\beta; q)_\infty} - (1 - q^n) \frac{(\alpha, \beta; q)_n}{(q, \alpha\beta; q)_n} \right]. \quad (2.8)$$

Thus, from Bailey's transform we find another identity in the following form,
If

$$\beta_n = \sum_{r=0}^n \alpha_r \quad (2.9)$$

then

$$\frac{(1 - \alpha\beta)}{(1 - \alpha)(1 - \beta)} \sum_{r=0}^{\infty} \alpha_n \left[\frac{(\alpha, \beta; q)_\infty}{(q, \alpha\beta; q)_\infty} - (1 - q^n) \frac{(\alpha, \beta; q)_n}{(q, \alpha\beta; q)_n} \right] = \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} \beta_n. \quad (2.10)$$

3. Transformation Formulae Involving Mock-Theta Functions of order Three

(i) Choosing $\alpha_r = \frac{q^{r^2}}{(-q; q)_r^2}$ in (2.4) and using (1.1), we get

$$\beta_n = f_n(q).$$

Putting these values of α_n and β_n in (2.5), we find

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(-q; q)_n^2} = (z; q)_\infty \sum_{n=0}^{\infty} f_n(q) z^n, \quad (3.1)$$

provided $|z| < 1$.

(ii) Putting the values of α_n, β_n given in (3.1) in (2.10) we obtain,

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha\beta q; q)_\infty} f(q) &= \frac{(1 - \alpha\beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2}}{(q, \alpha\beta; q)_n (-q; q)_n^2} \\ &\quad - \frac{(1 - \alpha\beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+n}}{(q, \alpha\beta; q)_n (-q; q)_n^2} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} f_n(q). \end{aligned} \quad (3.2)$$

(iii) Putting $\alpha_r = \frac{q^{r^2}}{(-q^2; q^2)_r}$ in (2.4) and using (1.2), we find

$$\beta_n = \Phi_n(q).$$

Substituting these values of α_n and β_n in (2.5) and (2.10), we have following transformation respectively;

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(-q^2; q^2)_n} = (z; q)_\infty \sum_{n=0}^{\infty} \Phi_n(q) z^n, \quad |z| < 1. \quad (3.4)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \Phi(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2}}{(q, \alpha \beta; q)_n (-q^2; q^2)_n} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+n}}{(q, \alpha \beta; q)_n (-q^2; q^2)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \Phi_n(q). \end{aligned} \quad (3.5)$$

(iv) Taking $\alpha_r = \frac{q^{(r+1)^2}}{(q; q^2)_{r+1}}$ in (2.4) and using (1.3), we get

$$\beta_n = \Psi_n(q). \quad (3.6)$$

Putting the values of α_n, β_n from (3.6) in (2.5) and (2.10), we obtain following two transformation respectively;

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} z^n}{(q; q^2)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \Psi_n(q) z^n, \quad |z| < 1. \quad (3.7)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \Psi(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{(n+1)^2}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+3n+1}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \Psi_n(q). \end{aligned} \quad (3.8)$$

(v) Taking $\alpha_r = \frac{q^{r^2}}{(-\omega q, -\omega^2 q; q)_r}$ in (2.4) and making use of (1.4), we find

$$\beta_n = \chi_n(q). \quad (3.9)$$

Putting these values of α_n, β_n in (2.5) and (2.10), we have

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(-\omega q, -\omega^2 q; q)_n} = (z; q)_\infty \sum_{n=0}^{\infty} \chi_n(q) z^n, \quad |z| < 1. \quad (3.10)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \chi(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2}}{(q, \alpha \beta; q)_n (-\omega q, -\omega^2 q; q)_n} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+n}}{(q, \alpha \beta; q)_n (-\omega q, -\omega^2 q; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \chi_n(q). \end{aligned} \quad (3.11)$$

(vi) Taking $\alpha_r = \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2}$ in (2.4) and using (1.5), we get

$$\beta_n = \omega_n(q).$$

If we put these values of α_n and β_n in (2.5) and (2.10), we have

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(q; q^2)_{n+1}^2} = (z; q)_\infty \sum_{n=0}^{\infty} \omega_n(q) z^n, \quad |z| < 1. \quad (3.12)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \omega(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n(n+1)}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}^2} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n^2+3n}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}^2} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \omega_n(q). \end{aligned} \quad (3.13)$$

(vii) Taking $\alpha_r = \frac{q^{2r(r+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}}$ in (2.4) and using (1.6), we find

$$\beta_n = \rho_n(q).$$

Putting these values in (2.5) and (2.10), we have

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(\omega q, \omega^2 q; q^2)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \rho_n(q) z^n, \quad |z| < 1. \quad (3.14)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \rho(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n(n+1)}}{(q, \alpha \beta; q)_n (\omega q, \omega^2 q; q^2)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n^2+3n}}{(q, \alpha \beta; q)_n (\omega q, \omega^2 q; q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \rho_n(q). \end{aligned} \quad (3.15)$$

(viii) Taking $\alpha_r = \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}$ in (2.4) and using (1.7), we find

$$\beta_n = \nu_n(q).$$

For these values of α_n and β_n (2.5) and (2.10), yields

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n}{(-q; q^2)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \nu_n(q) z^n, \quad |z| < 1. \quad (3.16)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \nu(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n(n+1)}}{(q, \alpha \beta; q)_n (-q; q^2)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+2n}}{(q, \alpha \beta; q)_n (-q; q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \nu_n(q). \end{aligned} \quad (3.17)$$

4. Transformation Formulae for Fifth order Mock-Theta Functions

(i) Taking $\alpha_r = \frac{q^{r^2}}{(-q; q)_r}$ in (2.4) and using (1.8), we get $\beta_n = f_{0,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we get,

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(-q; q)_n} = (z; q)_\infty \sum_{n=0}^{\infty} f_{0,n}(q) z^n, \quad |z| < 1, \quad (4.1)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} f_0(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2}}{(q, \alpha \beta; q)_n (-q; q)_n} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+n}}{(q, \alpha \beta; q)_n (-q; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} f_{0,n}(q), \end{aligned} \quad (4.2)$$

respectively.

(ii) Taking $\alpha_r = (-q; q)_r q^{(r+1)(r+2)/2}$ in (2.4) and using (1.9), we find $\beta_n = \Psi_{0,n}(q)$. For these values of α_n and β_n in (2.5) and (2.10) yield following results respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} (-q; q)_n q^{(n+1)(n+2)/2} z^n = (z; q)_\infty \sum_{n=0}^{\infty} \Psi_{0,n}(q) z^n, \quad |z| < 1, \quad (4.3)$$

and

$$\frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \Psi_0(q) = \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta, -q; q)_n}{(q, \alpha \beta; q)_n} q^{(n+1)(n+2)/2}$$

$$-\frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta, -q; q)_n q^{n+(n+1)(n+2)/2}}{(q, \alpha\beta; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} \Psi_{0,n}(q), \quad (4.4)$$

(iii) Taking $\alpha_r = \frac{q^r}{(q^{r+1}; q)_r}$ in (2.4) and by an appeal of (1.10), we get $\beta_n = \chi_{0,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we get the following transformations respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n z^n}{(q^{n+1}; q)_n} = (z; q)_{\infty} \sum_{n=0}^{\infty} \chi_{0,n}(q) z^n, \quad |z| < 1, \quad (4.5)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha\beta q; q)_{\infty}} \chi_0(q) &= \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta; q)_n (q^{n+1}; q)_n} \\ &- \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n}}{(q, \alpha\beta; q)_n (q^{n+1}; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} \chi_{0,n}(q). \end{aligned} \quad (4.6)$$

(iv) Taking $\alpha_r = (-q; q^2)_r q^{r^2}$ in (2.4) and using (1.11), we find $\beta_n = \Phi_{0,n}(q)$. For these values of α_n and β_n in (2.5) and (2.10) yields following results respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} (-q; q^2)_n q^{n^2} z^n = (z; q)_{\infty} \sum_{n=0}^{\infty} \Phi_{0,n}(q) z^n, \quad |z| < 1, \quad (4.7)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha\beta q; q)_{\infty}} \Phi_0(q) &= \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q^2)_n q^{n^2}}{(q, \alpha\beta; q)_n} \\ &- \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q^2)_n q^{n^2+n}}{(q, \alpha\beta; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} \Phi_{0,n}(q). \end{aligned} \quad (4.8)$$

(v) Choosing $\alpha_r = \frac{q^{2r^2}}{(q; q^2)_r}$ in (2.4) and applying (1.12), we have $\beta_n = F_{0,n}(q)$. For these values of α_n and β_n , (2.5) and (2.10) yield following results respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2} z^n}{(q; q^2)_n} = (z; q)_{\infty} \sum_{n=0}^{\infty} F_{0,n}(q) z^n, \quad |z| < 1, \quad (4.9)$$

and

$$\frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha\beta q; q)_{\infty}} F_0(q) = \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n^2}}{(q, \alpha\beta; q)_n (q; q^2)_n}$$

$$-\frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n^2+n}}{(q, \alpha\beta; q)_n (q; q^2)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} F_{0,n}(q). \quad (4.10)$$

(vi) Taking $\alpha_r = \frac{q^{r(r+1)}}{(-q; q)_r}$ in (2.4) and using (1.13), we get $\beta_n = f_{1,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we have following transformations respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n}{(-q; q)_n} = (z; q)_{\infty} \sum_{n=0}^{\infty} f_{1,n}(q) z^n, \quad |z| < 1, \quad (4.11)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha\beta q; q)_{\infty}} f_1(q) &= \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n(n+1)}}{(q, \alpha\beta; q)_n (-q; q)_n} \\ &- \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+2n}}{(q, \alpha\beta; q)_n (-q; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} f_{1,n}(q). \end{aligned} \quad (4.12)$$

(vii) Choosing $\alpha_r = (-q; q^2)_r q^{(r+1)^2}$ in (2.4) and making use of (1.14), we have $\beta_n = \Phi_{1,n}(q)$. For these values of α_n and β_n in (2.5) and (2.10) yields following results respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{(n+1)^2} z^n}{(-q; q^2)_n} = (z; q)_{\infty} \sum_{n=0}^{\infty} \Phi_{1,n}(q) z^n, \quad |z| < 1, \quad (4.13)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha\beta q; q)_{\infty}} \Phi_1(q) &= \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q^2)_n q^{(n+1)^2}}{(q, \alpha\beta; q)_n} \\ &- \frac{(1-\alpha\beta)}{(1-\alpha)(1-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q^2)_n q^{n^2+3n+1}}{(q, \alpha\beta; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha\beta q; q)_n} \Phi_{1,n}(q). \end{aligned} \quad (4.14)$$

(viii) Substituting $\alpha_r = (-q; q)_r q^{r(r+1)/2}$ in (2.4) and using (1.15), we get $\beta_n = \Psi_{1,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10), we obtain following results respectively.

$$(zq; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2} z^n}{(-q; q)_n} = (z; q)_{\infty} \sum_{n=0}^{\infty} \Psi_{1,n}(q) z^n, \quad |z| < 1, \quad (4.15)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \Psi_1(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q)_n}{(q, \alpha \beta; q)_n} q^{n(n+1)/2} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (-q; q)_n q^{(n^2+3n)/2}}{(q, \alpha \beta; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \Psi_{1,n}(q). \end{aligned} \quad (4.16)$$

(ix) Taking $\alpha_r = \frac{q^{2r(r+1)}}{(q; q^2)_{r+1}}$ in (2.4) and making use of (1.16), we find $\beta_n = F_{1,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we get following transformations respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(q; q^2)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} F_{1,n}(q) z^n, \quad |z| < 1, \quad (4.17)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} F_1(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n(n+1)}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n^2+3n}}{(q, \alpha \beta; q)_n (q; q^2)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} F_{1,n}(q). \end{aligned} \quad (4.18)$$

(x) Putting $\alpha_r = \frac{q^r}{(q^{r+1}; q)_{r+1}}$ in (2.4) and appealing of (1.17), we get $\beta_n = \chi_{1,n}(q)$. For these values of α_n and β_n in (2.5) and (2.10) yield following results respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^n z^n}{(q^{n+1}; q)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \chi_{1,n}(q) z^n, \quad |z| < 1, \quad (4.19)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \chi_1(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{2n}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \chi_{1,n}(q). \end{aligned} \quad (4.20)$$

5. Transformation Formulae for Mock-Theta Functions of order Seven

(i) Taking $\alpha_r = \frac{q^{r^2}}{(q^{r+1}; q)_r}$ in (2.4) and using (1.18), we get $\beta_n = \mathcal{F}_{0,n}(q)$. For these

values of α_n and β_n in (2.5) and (2.10) yield following results respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q^{n+1}; q)_n} = (z; q)_\infty \sum_{n=0}^{\infty} \mathcal{F}_{0,n}(q) z^n, \quad |z| < 1, \quad (5.1)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \mathcal{F}_0(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_n} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+n}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_n} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \mathcal{F}_{0,n}(q). \end{aligned} \quad (5.2)$$

(ii) Taking $\alpha_r = \frac{q^{(r+1)^2}}{(q^{r+1}; q)_{r+1}}$ in (2.4) and using (1.19), we get $\beta_n = \mathcal{F}_{1,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we obtain following transformations respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} z^n}{(q^{n+1}; q)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \mathcal{F}_{1,n}(q) z^n, \quad |z| < 1, \quad (5.3)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \mathcal{F}_1(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{(n+1)^2}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+3n+1}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \mathcal{F}_{1,n}(q). \end{aligned} \quad (5.4)$$

(iii) Substituting $\alpha_r = \frac{q^{r(r+1)}}{(q^{r+1}; q)_{r+1}}$ in (2.4) and using (1.20), we get $\beta_n = \mathcal{F}_{2,n}(q)$. Putting these values of α_n and β_n in (2.5) and (2.10) we obtain following transformations respectively.

$$(zq; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n}{(q^{n+1}; q)_{n+1}} = (z; q)_\infty \sum_{n=0}^{\infty} \mathcal{F}_{2,n}(q) z^n, \quad |z| < 1, \quad (5.5)$$

and

$$\begin{aligned} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} \mathcal{F}_2(q) &= \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n(n+1)}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} \\ &- \frac{(1 - \alpha \beta)}{(1 - \alpha)(1 - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^{n^2+2n}}{(q, \alpha \beta; q)_n (q^{n+1}; q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} \mathcal{F}_{2,n}(q). \end{aligned} \quad (5.6)$$

6. Special Cases

(i) (3.2) can be written as,

$$\frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha \beta q; q)_\infty} f(q) = \frac{1}{(1+q)^2} \sum_{n=0}^{\infty} \frac{(\alpha q, \beta q; q)_n q^{(n+1)^2}}{(q, \alpha \beta q; q)_n (-q^2; q)_n^2} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} f_n(q). \quad (6.1)$$

(ii) Taking $\alpha = \beta = -1$ in (6.1), we get

$$\frac{(-q; q)_\infty^2}{(q; q)_\infty^2} f(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q; q)_n^2 (1+q^{n+1})^2} + \sum_{n=0}^{\infty} \frac{(-1; q)_n^2 q^n}{(q; q)_n^2} f_n(q). \quad (6.2)$$

(iii) For $\alpha = \beta = -q$, in (6.1) yields

$$\frac{(-q^2; q)_\infty^2}{(q, q^3; q)_\infty^2} f(q) = \frac{1}{(1+q)^2} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q, q^3; q)_n} + \sum_{n=0}^{\infty} \frac{(-q; q)_n^2 q^n}{(q, q^3; q)_n} f_n(q). \quad (6.3)$$

(iv) Taking $\alpha = 0$ in (6.1) we have,

$$\frac{(\beta q; q)_\infty}{(q; q)_\infty} f(q) = \frac{1}{(1+q)^2} \sum_{n=0}^{\infty} \frac{(\beta q; q)_n q^{(n+1)^2}}{(q; q)_n (-q^2; q)_n^2} + \sum_{n=0}^{\infty} \frac{(\beta; q)_n q^n}{(q; q)_n} f_n(q). \quad (6.4)$$

(v) For $\beta = q$, (6.4) yields

$$f(q) = \sum_{n=0}^{\infty} \frac{(1-q^{n+1})q^{(n+1)^2}}{(-q; q)_{n+1}^2} + \sum_{n=0}^{\infty} f_n(q) q^n. \quad (6.5)$$

(vi) Taking $\beta = -q$ in (6.4) we have

$$\frac{(-q^2; q)_\infty}{(q; q)_\infty} f(q) = \frac{1}{(1+q)^2} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q, -q^2; q)_n} + \sum_{n=0}^{\infty} \frac{(-q; q)_n}{(q; q)_n} q^n f_n(q). \quad (6.6)$$

(vii) Taking $z = q$ in (3.1), we find,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q)_n^2} = (1-q) \sum_{n=0}^{\infty} f_n(q) q^n. \quad (6.7)$$

(viii) Taking $z = -q$ (3.1) yields,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(-q; q)_n^2} = (1+q) \sum_{n=0}^{\infty} f_n(q) (-1)^n q^n. \quad (6.8)$$

(ix) Also, for $z = -1$, (3.1) gives

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1)^n}{(-q; q)_n^2} = 2 \sum_{n=0}^{\infty} (-1)^n f_n(q). \quad (6.9)$$

(x) (4.2) can be written as,

$$\frac{(\alpha q, \beta q; q)_{\infty}}{(q, \alpha \beta q; q)_{\infty}} f_0(q) = \sum_{n=0}^{\infty} \frac{(\alpha q, \beta q; q)_n q^{(n+1)^2}}{(q, \alpha \beta q; q)_n (-q; q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n q^n}{(q, \alpha \beta q; q)_n} f_{0,n}(q). \quad (6.10)$$

(xi) For $\alpha = -1$, (6.10) yields

$$\frac{(-q, \beta q; q)_{\infty}}{(q, -\beta q; q)_{\infty}} f_0(q) = \sum_{n=0}^{\infty} \frac{(\beta q; q)_n q^{(n+1)^2}}{(q, -\beta q; q)_n (1 + q^{n+1})} + \sum_{n=0}^{\infty} \frac{(-1, \beta; q)_n q^n}{(q, -\beta q; q)_n} f_{0,n}(q). \quad (6.11)$$

(xii) Taking $\beta = 0$ in (6.11) we have,

$$\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} f_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q; q)_n (1 + q^{n+1})} + \sum_{n=0}^{\infty} \frac{(-1; q)_n q^n}{(q; q)_n} f_{0,n}(q). \quad (6.12)$$

(xiii) Taking $\alpha = \beta = 0$ in (6.10) we get,

$$f_0(q) = (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^2; q^2)_n (1 + q^{n+1})} + (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} f_{0,n}(q). \quad (6.13)$$

Similar other results can also be scored. One interesting paper mock-theta functions is [3].

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