

On certain results involving the product of partial theta functions

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Abstract: In this paper, we have established a new identity for the product of two partial theta functions. Certain interesting results have been deduced as the special cases of the main identity.

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1. Introduction

We will use the standard q - notations,

$$[a; q]_{\infty} = \prod_{n=1}^{\infty} (1 - aq^{n-1}),$$

$$[a_1, a_2, \dots, a_r; q]_{\infty} = [a_1; q]_{\infty} [a_2; q]_{\infty} \dots [a_r; q]_{\infty},$$

$$[a; q]_0 = 1,$$

and

$$[a; q]_n = (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1}) \quad \text{for } n \geq 1.$$

Warnaar [6] proved the following generalization of Jacobi's triple product identity,

$$1 + \sum_{n=1}^{\infty} (-)^n q^{n(n-1)/2} (a^n + b^n) = [q, a, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{\left[\frac{aq}{q}; q \right]_{2n} q^n}{[q, a, b, ab; q]_n}. \quad (1.1)$$

The celebrated Jacobi triple product identity [3; Eq. (1.6.1) page 15] is

$$\sum_{n=-\infty}^{\infty} (-)^n a^n q^{n(n-1)/2} = [q, a, q/a; q]_{\infty}. \quad (1.2)$$

Sums of the form

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} \quad (1.3)$$

are called partial theta functions owing to the fact that the sum in (1.2) is usually referred to as a complete theta function or a theta function.

The series

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} \quad (1.4)$$

is called the false theta series (c.f. [Rogers 5]).

In a recent communication Andrews and Warnaar [1] established the following product formula involving partial theta functions,

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} \right) \left(\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} \right) \\ &= [q, a, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{[abq^{n-1}; q]_n q^n}{[q, a, b; q]_n} \\ &= [q, a, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{[ab/q; q]_{2n} q^n}{[q, a, b, ab/q; q]_n}. \end{aligned} \quad (1.5)$$

We shall make use of following results in our analysis.

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} = [q, a; q]_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{[q, a; q]_n}. \quad (1.6)$$

[Andrews, G.E. and Warnaar S.O. 1; (2.1a)]

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} = [a; q]_{\infty} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{[q, a; q]_n}. \quad (1.7)$$

[Andrews, G.E. and Warnaar S.O. 1; (2.1b)]

$$\sum_{n=0}^{\infty} \sum_{r=0}^n A(n, r) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A(n+r, r). \quad (1.8)$$

[Srivastava, H.M. and Manocha, H.L. 4; lemma 1(2) p. 100]

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} = \frac{1}{1+} \frac{aq}{1+} \frac{a(q^2 - q)}{1+} \frac{aq^3}{1+} \frac{a(q^4 - q^2)}{1+} \dots \quad (1.9)$$

[Andrews, G.E. and Berndt B.C. 2; (6.2.29) p.152]

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2; q^2]_{\infty}}{[q; q^2]_{\infty}}. \quad (1.10)$$

[Andrews, G.E. and Berndt B.C. 2; (1.1.7) p.11]

2. Theorem 1.

$$\begin{aligned} & \sum_{n=0}^{\infty} (-)^n a^n q^{n(n-1)/2} \sum_{n=0}^{\infty} (-)^n b^n q^{n(n-1)/2} \\ &= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} - \sum_{s=1}^{\infty} (a^{2s-1} + b^{2s-1}) q^{s(s-1)} \sum_{r=0}^{\infty} (ab)^r q^{(r+s-1)^2} \\ & \quad + \sum_{s=1}^{\infty} (a^{2s} + b^{2s}) q^{s^2} \sum_{r=0}^{\infty} (ab)^r q^{(r+s-1)(r+s)}. \end{aligned} \quad (2.1)$$

Proof of (2.1)

In order to prove (2.1) we shall establish following identity,

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \alpha_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^n \alpha_r \quad . \quad (2.2)$$

We know from Bailey lemma that,

If

$$\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r} \quad (2.3)$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \quad (2.4)$$

Then under suitable conditions of convergence

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \quad (2.5)$$

where u_r, v_r, α_r and δ_r are functions of r alone and all infinite series involved in (2.4) and (2.5) are convergent.Now choosing $u_r = v_r = 1$ in (2.3), (2.4) and (2.5) we get,

If

$$\beta_n = \sum_{r=0}^n \alpha_r \quad (2.6)$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \quad (2.7)$$

Then under suitable convergence conditions we have

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^n \alpha_r, \quad (2.8)$$

which on simplification gives (2.2).

Again, taking $\alpha_n = (-)^n a^n q^{n(n-1)/2}$ and $\delta_r = (-)^r b^r q^{r(r-1)/2}$ in (2.2) and applying the identity (1.8) on the right hand side we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} (-)^n a^n q^{n(n-1)/2} \sum_{r=0}^{\infty} (-)^r b^r q^{r(r-1)/2} + \sum_{n=0}^{\infty} (ab)^n q^{n(n-1)} \\ &= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} \sum_{n=0}^{\infty} (-)^n q^{n(n-1)/2} q^{nr} (a^n + b^n). \end{aligned} \quad (2.9)$$

On simplification, (2.9) takes the form,

$$\begin{aligned} & \sum_{n=0}^{\infty} (-)^n a^n q^{n(n-1)/2} \sum_{n=0}^{\infty} (-)^n b^n q^{n(n-1)/2} \\ &= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} \left\{ 1 + \sum_{n=1}^{\infty} (-)^n q^{n(n-1)/2} q^{nr} (a^n + b^n) \right\}. \end{aligned} \quad (2.10)$$

if we expand the right hand side of (2.10) we have,

$$\begin{aligned} & \sum_{n=0}^{\infty} (-)^n a^n q^{n(n-1)/2} \sum_{n=0}^{\infty} (-)^n b^n q^{n(n-1)/2} \\ &= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} - (a+b) \sum_{r=0}^{\infty} (ab)^r q^{r^2} + (a^2 + b^2) q \sum_{r=0}^{\infty} (ab)^r q^{r(r+1)} \\ & \quad - (a^3 + b^3) q^2 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)^2} + (a^4 + b^4) q^4 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)(r+2)} \\ & \quad - (a^5 + b^5) q^6 \sum_{r=0}^{\infty} (ab)^r q^{(r+2)^2} + \dots \end{aligned} \quad (2.11)$$

which is precisely (2.1).

If we make use of (1.5) in (2.11) we get,

$$\begin{aligned}
& [q, a, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{[ab/q; q]_{2n} q^n}{[q, a, b, ab/q; q]_n} \\
&= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} - (a+b) \sum_{r=0}^{\infty} (ab)^r q^{r^2} + (a^2 + b^2) q \sum_{r=0}^{\infty} (ab)^r q^{r(r+1)} \\
&\quad - (a^3 + b^3) q^2 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)^2} + (a^4 + b^4) q^4 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)(r+2)} \\
&\quad - (a^5 + b^5) q^6 \sum_{r=0}^{\infty} (ab)^r q^{(r+2)^2} + \dots
\end{aligned} \tag{2.12}$$

Using (1.6) in (2.11) we find,

$$\begin{aligned}
& [q, a; q]_{\infty} [q, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{[q, a; q]_n} \sum_{n=0}^{\infty} \frac{q^n}{[q, b; q]_n} \\
&= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} - (a+b) \sum_{r=0}^{\infty} (ab)^r q^{r^2} + (a^2 + b^2) q \sum_{r=0}^{\infty} (ab)^r q^{r(r+1)} \\
&\quad - (a^3 + b^3) q^2 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)^2} + (a^4 + b^4) q^4 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)(r+2)} \\
&\quad - (a^5 + b^5) q^6 \sum_{r=0}^{\infty} (ab)^r q^{(r+2)^2} + \dots
\end{aligned} \tag{2.13}$$

Using (1.7) in (2.11) we obtain

$$\begin{aligned}
& [a, b; q]_{\infty} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{[q, a; q]_n} \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{[q, b; q]_n} \\
&= \sum_{r=0}^{\infty} (ab)^r q^{r(r-1)} - (a+b) \sum_{r=0}^{\infty} (ab)^r q^{r^2} + (a^2 + b^2) q \sum_{r=0}^{\infty} (ab)^r q^{r(r+1)} \\
&\quad - (a^3 + b^3) q^2 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)^2} + (a^4 + b^4) q^4 \sum_{r=0}^{\infty} (ab)^r q^{(r+1)(r+2)}
\end{aligned}$$

$$-(a^5 + b^5)q^6 \sum_{r=0}^{\infty} (ab)^r q^{(r+2)^2} + \dots \quad (2.14)$$

3. Special Cases

In this section we shall discuss certain special cases of the results established in previous section.

(i) Taking $b=a$ in (2.11) we get,

$$\begin{aligned} & \left[\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} \right]^2 \\ &= \sum_{r=0}^{\infty} a^{2r} q^{r(r-1)} - 2a \sum_{r=0}^{\infty} a^{2r} q^{r^2} + 2a^2 q \sum_{r=0}^{\infty} a^{2r} q^{r(r+1)} \\ & - 2a^3 q^2 \sum_{r=0}^{\infty} a^{2r} q^{(r+1)^2} + 2a^4 q^4 \sum_{r=0}^{\infty} a^{2r} q^{(r+1)(r+2)} - 2a^5 q^6 \sum_{r=0}^{\infty} a^{2r} q^{(r+1)^2} + \dots \end{aligned} \quad (3.1)$$

Taking $a = -q$ in (3.1) and then using (1.10) we get,

$$\begin{aligned} & \sum_{r=0}^{\infty} q^{r(r+1)} + 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} \\ & + 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} \dots = \frac{[q^2; q^2]_{\infty}^2}{[q; q^2]_{\infty}^2}. \end{aligned} \quad (3.2)$$

Taking aq for a in (3.1) and using (1.9) we have

$$\begin{aligned} & \sum_{r=0}^{\infty} a^{2r} q^{r(r+1)} - 2aq \sum_{r=0}^{\infty} a^{2r} q^{r(r+2)} + 2a^2 q^3 \sum_{r=0}^{\infty} a^{2r} q^{r(r+3)} \\ & - 2a^3 q^6 \sum_{r=0}^{\infty} a^{2r} q^{r(r+4)} + 2a^4 q^{10} \sum_{r=0}^{\infty} a^{2r} q^{r(r+5)} \dots \\ &= \left\{ \frac{1}{1+} \frac{aq}{1+} \frac{a(q^2 - q)}{1+} \frac{aq^3}{1+} \frac{a(q^4 - q^2)}{1+} \dots \right\}^2. \end{aligned} \quad (3.3)$$

Taking $a=q$ in (3.1) we get,

$$\left[\sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2} \right]^2$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} q^{r(r+1)} - 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} \\
&- 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} - 2q^{15} \sum_{r=0}^{\infty} q^{r(r+6)} + \dots
\end{aligned} \tag{3.4}$$

(3.4) gives series expansion of the square of false theta function.

Taking $b=-a$ in (2.11) we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} (-a)^n q^{n(n-1)/2} \sum_{n=0}^{\infty} a^n q^{n(n-1)/2} \\
&= \sum_{r=0}^{\infty} (-)^r a^{2r} q^{r(r-1)} + 2a^2 q \sum_{r=0}^{\infty} (-)^r a^{2r} q^{r(r+1)} \\
&\quad + 2a^4 q^4 \sum_{r=0}^{\infty} (-)^r a^{2r} q^{(r+1)(r+2)} + \dots \quad .
\end{aligned} \tag{3.5}$$

For $a=q$, (3.5) yields

$$\begin{aligned}
&\frac{[q^2; q^2]_{\infty}}{[q; q^2]_{\infty}} \sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2} \\
&= \sum_{r=0}^{\infty} (-)^r q^{r(r+1)} + 2q^3 \sum_{r=0}^{\infty} (-)^r q^{r(r+3)} + 2q^{10} \sum_{r=0}^{\infty} (-)^r q^{r(r+5)} + \dots \quad , \tag{3.6}
\end{aligned}$$

which is another form of false theta function.

Comparing (3.4) and (3.6) we have,

$$\begin{aligned}
&\frac{[q; q^2]_{\infty}^2}{[q^2; q^2]_{\infty}^2} \left[\sum_{r=0}^{\infty} (-)^r q^{r(r+1)} + 2q^3 \sum_{r=0}^{\infty} (-)^r q^{r(r+3)} + 2q^{10} \sum_{r=0}^{\infty} (-)^r q^{r(r+5)} + \dots \right]^2 \\
&= \sum_{r=0}^{\infty} q^{r(r+1)} - 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} - 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} \dots \quad .
\end{aligned} \tag{3.7}$$

Taking $b=q$ in (2.12) we get,

$$[q; q]_{\infty}^2 [a; q]_{\infty} \sum_{n=0}^{\infty} \frac{[a; q]_{2n} q^n}{[q, a; q]_n^2}$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} a^r q^{r^2} - (a+q) \sum_{r=0}^{\infty} a^r q^{r(r-1)} + q(a^2+q^2) \sum_{r=0}^{\infty} a^r q^{r(r+2)} \\
&\quad - (a^3+a^3)q^3 \sum_{r=0}^{\infty} a^r q^{r(r+3)} + (a^4+q^4)q^6 \sum_{r=0}^{\infty} a^r q^{r(r+4)} \\
&\quad - (a^5+q^5)q^{10} \sum_{r=0}^{\infty} a^r q^{r(r+5)} + \dots \quad .
\end{aligned} \tag{3.8}$$

For $a=q$, (3.8) yields

$$\begin{aligned}
&[q; q]_{\infty}^3 \sum_{n=0}^{\infty} \frac{[q; q]_{2n} q^n}{[q; q]_n^4} = [q; q]_{\infty}^3 \sum_{n=0}^{\infty} \frac{[q; q^2]_n [-q; q]_n q^n}{[q; q]_n^3} \\
&= \sum_{r=0}^{\infty} q^{r(r+1)} - 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} - 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} \\
&\quad + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} - 2q^{15} \sum_{r=0}^{\infty} q^{r(r+6)} + \dots \quad .
\end{aligned} \tag{3.9}$$

For $a=0$, (3.8) gives

$$\begin{aligned}
&[q; q]_{\infty}^2 \sum_{n=0}^{\infty} \frac{q^n}{[q; q]_n^2} \\
&= 1 - q + q^3 - q^6 + q^{10} - q^{15} + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}
\end{aligned} \tag{3.10}$$

Right hand side of (3.10) is false theta function.

Taking $a=b=q$ in (2.13) we get,

$$\begin{aligned}
&[q; q]_{\infty}^4 \left\{ \sum_{n=0}^{\infty} \frac{q^n}{[q; q]_n} \right\}^2 \\
&= \sum_{r=0}^{\infty} q^{r(r+1)} - 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} \\
&\quad - 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} \dots \quad .
\end{aligned} \tag{3.11}$$

Comparing (3.10) and (3.11) we get,

$$\begin{aligned}
& \left\{ \sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2} \right\}^2 \\
&= \sum_{r=0}^{\infty} q^{r(r+1)} - 2q \sum_{r=0}^{\infty} q^{r(r+2)} + 2q^3 \sum_{r=0}^{\infty} q^{r(r+3)} \\
&\quad - 2q^6 \sum_{r=0}^{\infty} q^{r(r+4)} + 2q^{10} \sum_{r=0}^{\infty} q^{r(r+5)} \dots \quad .
\end{aligned} \tag{3.12}$$

which is (3.4)

A number of similar results can also be deduced.

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