

Relationship between q-Weyl operator and basic analogue of I-function in preview of q-Laplace transform

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Abstract: The q-derivatives and q-integrals are part of so called quantum calculus [1]. In this paper, we investigate how such derivatives and integrals can be possible used in establishing a formula exhibiting relationship between basic analogue of q-Weyl operator and q-Laplace transform, which allows the straight forward derivation of some useful results involving Weyl operator and basic analogue of I-function in terms of q-gamma function [2]. Also some special cases has been discussed.

Keywords: Weyl fractional q-integral operator, q-Laplace transform, basic analogue of I-function.

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Introduction:

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The subject deals with the investigations of q-integrals and q-derivatives of arbitrary order and has gained importance due its various applications in the areas like ordinary fractional calculus, solution of the q-differential and q-integral equations, q-transform analysis [4,5 and 6]. Motivated by these avenues of applications, a number of workers have made use of these operators to evaluate fractional q-calculus, basic analogue of H-function, basic analogue of I-function, general class of q-polynomials etc. Al-Salam [7,8] introduced the q-analogue of Weyl fractional integral operator in the following manner.

$$W_{q,x}^{\alpha} f(x) = \frac{q^{\frac{-\alpha(\alpha-1)}{2}}}{\Gamma_q(\alpha)} \int_x^{\infty} (x - qt)_{\alpha-1} f(tq^{1-\alpha}) d_q(t) \quad ; Re(\alpha) > 0. \quad (1.1)$$

Also, Hahn [3] defined the q-analogues of the well-known classic Laplace transform

$$\Phi(s) = \int_0^\infty e^{-st} f(t) d_q(t). \quad (1.2)$$

By means of the following q-integrals

$${}_qL_s\{f(t)\} = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(s, t) f(t) d_q(t). \quad (1.3)$$

$${}_qL_s\{f(t)\} = \frac{1}{(1-q)} \int_0^\infty e_q(-st) f(t) d_q(t). \quad (1.4)$$

where the q-exponential series is defined as

$$e_q(x) = \sum_{n=0}^\infty \frac{x^n}{(q; q)_n} \quad (1.5)$$

$$E_q(x) = \sum_{n=0}^\infty \frac{(-1)^n (q)^{n(n-1)/2} x^n}{(q; q)_n}. \quad (1.6)$$

R. Jain, et. al. [2] define the basic analogue of I-function in terms of q-gamma function as follows

$$I_{q, A_i, B_i}^{m, n}(z; q) = \frac{1}{2\pi i} \times \int_l \frac{\prod_{j=1}^m \Gamma_q(b_j - \beta_j s) \prod_{j=1}^n \Gamma_q(1 - a_j + \alpha_j s) \pi z^s d_q(s)}{\sum_{i=1}^r \left[\prod_{j=m+1}^{B_i} \Gamma_q(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{A_i} \Gamma_q(a_{ji} - \alpha_{ji} s) \Gamma_q(s) \Gamma_q(1 - s) \sin \pi s \right]}. \quad (1.7)$$

Main Results:

In this section, we will derive the q-Laplace transform of basic analogue of I-function by making make use of equations (1.7) and (1.3)

Proposition q-Laplace transform of basic analogue of I-function

$${}_qL_s\{I_{q, A_i, B_i}^{m, n}(z; q); p\} = \frac{{}_qL_s}{2\pi i} \times \left[\int_l \frac{\prod_{j=1}^m \Gamma_q(b_j - \beta_j s) \prod_{j=1}^n \Gamma_q(1 - a_j + \alpha_j s) \pi c d_q(s)}{\sum_{i=1}^r \left[\prod_{j=m+1}^{B_i} \Gamma_q(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{A_i} \Gamma_q(a_{ji} - \alpha_{ji} s) \Gamma_q(s) \Gamma_q(1 - s) \sin \pi s \right]} \right]$$

$$= \frac{1}{2\pi i} \int_l \Phi(s) {}_qL_s(z^s) d_q(s)$$

where

$$\Phi(s) = \frac{\prod_{j=1}^m \Gamma_q(b_j - \beta_j s) \prod_{j=1}^n \Gamma_q(1 - a_j + \alpha_j s) \pi}{\sum_{i=1}^r \left[\prod_{j=m+1}^{B_i} \Gamma_q(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{A_i} \Gamma_q(a_{ji} - \alpha_{ji} s) \Gamma_q(s) \Gamma_q(1 - s) \sin \pi s \right]}.$$

Now, ${}_qL_s(z^s) = \frac{(q; q)_s}{z^{s+1}}$, we have

$${}_qL_s \{I_{q, A_i, B_i}^{m, n}(z; q); p\} = \frac{{}_qL_s}{2\pi i} \int_l \Phi(s) \frac{(q; q)_s}{z^{s+1}} d_q(s) = \hat{I}_q(z; p). \quad (1.8)$$

In this section, we establish a theorem with the help of above preposition involving q -Laplace transform of basic analogue of I -function and q -Weyl operator.

Theorem 2.1: Let $\mu > 0, \beta > 0, \lambda > 0$ and $a \in R$, let $W_{q, x}^\mu$ be the q -Weyl integral operator, then

$$\begin{aligned} W_{q, x}^\mu \{p^{-\lambda} \hat{I}_q(z; p)\} &= p^{\mu-\lambda-1} q^{-\mu(\mu-1)/2} \frac{1}{2\pi i} \int_l \Phi(s) \frac{\Gamma_q(1 - \mu - \lambda + s)}{\Gamma_q(1 + \lambda + s) p^s} d_q(s) \\ &= p^{\mu-\lambda-1} q^{-\mu(\mu-1)/2} \times \\ &\quad \left\{ I_{q, A_{i+1}, B_{i+1}; r}^{m, n+1} \left(\begin{matrix} (a_j, \alpha_j)_{1, n}; (\mu - \lambda, 1); (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1, m}; (-\lambda, 1); (b_{ji}, \beta_{ji})_{m+1, B_i} \end{matrix} \middle| p^{-1}; q \right) \right\} \end{aligned}$$

Proof: To prove theorem we apply equation (1.8) to the left side of (2.1) we get,

$$W_{q, x}^\mu \{p^{-\lambda} \hat{I}_q(z; p)\} = W_{q, x}^\mu \left\{ p^{-\lambda} \frac{1}{2\pi i} \int_l \Phi(s) \frac{(q; q)_s}{z^{s+1}} d_q(s) \right\}$$

where

$$\Phi(s) = \frac{\prod_{j=1}^m \Gamma_q(b_j - \beta_j s) \prod_{j=1}^n \Gamma_q(1 - a_j + \alpha_j s) \pi}{\sum_{i=1}^r \left[\prod_{j=m+1}^{B_i} \Gamma_q(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{A_i} \Gamma_q(a_{ji} - \alpha_{ji} s) \Gamma_q(s) \Gamma_q(1 - s) \sin \pi s \right]}.$$

Thus making use of equation (1.1) and changing the order of integration which is justified under the existence conditions of basic analogue of I-function, we have

$$\begin{aligned} & W_{q,x}^\mu \left\{ p^{-\lambda} \frac{1}{2\pi i} \int_l \Phi(s) \frac{(q;q)_s}{z^{s+1}} d_q(s) \right\} \\ &= \frac{1}{2\pi i} \int_l \Phi(s) (q;q)_s \Gamma_q(-\mu + \lambda + 1 + s) p^{-\lambda-1-s-\mu} q^{\mu(\mu-1)/2} d_q(s) \end{aligned}$$

Hence we get

$$\begin{aligned} W_{q,x}^\mu \{ p^{-\lambda} \hat{I}_q(z; p) \} &= p^{\mu-\lambda-1} q^{-\mu(\mu-1)/2} \frac{1}{2\pi i} \int_l \Phi(s) \frac{\Gamma_q(1-\mu-\lambda+s)}{\Gamma_q(1+\lambda+s) p^s} d_q(s) \\ &= p^{\mu-\lambda-1} q^{-\mu(\mu-1)/2} \times \\ &\quad \left\{ I_{q, A_{i+1}, B_{i+1}; r}^{m, n+1} \left(\begin{matrix} (a_j, \alpha_j)_{1,n}; (\mu-\lambda, 1); (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1,m}; (-\lambda, 1); (b_{ji}, \beta_{ji})_{m+1, B_i} \end{matrix} \middle| p^{-1}; q \right) \right\} \end{aligned}$$

Special cases

Corollary 1: By setting $r = 1$ and $q = 1$ in equation (1.7) and above theorem, then it takes the form of Fox's H-function [9]

$$\begin{aligned} H_{A,B}^{m,n}(z) &= \frac{1}{2\pi i} \times \\ &\int_l \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s) \pi z^s d_q(s)}{\prod_{j=m+1}^{B_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{A_i} \Gamma(a_{ji} - \alpha_{ji} s) \Gamma_q(s) \Gamma(1-s) \sin \pi s} \end{aligned}$$

and in this case result becomes

$$\begin{aligned} W_x^\mu \{ p^\lambda \hat{H}(z) \} &= p^{\mu-\lambda-1} \frac{1}{2\pi i} \int_l \Phi(s) \frac{\Gamma(1-\mu+\lambda+s)}{\Gamma(1+\lambda+s) p^s} d(s) \\ &= p^{\mu-\lambda-1} \times \\ &\quad \left\{ H_{A+1, B+1; r}^{m, n+1} \left(\begin{matrix} (a_j, \alpha_j)_{1,n}; (\mu-\lambda, 1); (a_j, \alpha_j)_{n+1, A} \\ (b_j, \beta_j)_{1,m}; (-\lambda, 1); (b_j, \beta_j)_{m+1, B} \end{matrix} \middle| p^{-1} \right) \right\} \end{aligned}$$

Corollary 2: By setting $q=1$ in the above preposition, we get well-known results established by D. Singh [10]

$$L_s \{ I_{q, A_i, B_i}^{m,n}(z); p \} = \frac{1}{2\pi i} \int_l \Phi(s) \frac{\Gamma(s+1)}{z^{s+1}} d(s) = I(z; p).$$

Corollary 3: By setting $q=1$ in the above preposition, we get well-known results established by D. Singh [10]

$$\begin{aligned} W_x^\mu \{p^{-\lambda} \hat{I}(z; p)\} &= p^{\mu-\lambda-1} \frac{1}{2\pi i} \int_l \Phi(s) \frac{\Gamma(1-\mu+\lambda+s)}{\Gamma(1+\lambda+s)p^s} d(s) \\ &= p^{\mu-\lambda-1} \times \\ &\quad \left\{ I_{q, A_{i+1}, B_{i+1}; r}^{m, n+1} \left(\begin{matrix} (a_j, \alpha_j)_{1, n}; (\mu-\lambda, 1); (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1, m}; (-\lambda, 1); (b_{ji}, \beta_{ji})_{m+1, B_i} \end{matrix} \middle| p^{-1} \right) \right\}. \end{aligned}$$

References

- [1] V. Kac and P. Cheung, Quantum Calculus, University, Springer-Verlag, New York, 2002
- [2] D. K. Jain and Farooq Ahmad, Some transformation formula for basic analogue of I -function, Asian Journal of Mathematics and Statistics, 5(4); 158-162, 2012.
- [3] W. Hahn, Beitrage Zur theorie dar Heinesch Reihen, die 24 intigrade dr hypergeometrischen q -differenzengleichung, das q -analogan dr laplace transformation. Math Nachr, 2(1949), 340-379.
- [4] Podlubny I, Fractional Differential Equations, Academic Press, 1999.
- [5] Samko G, Kilbas A.A., Marichev, Fractional integrals and Derivatives, theory and applications, Gordan and Breach, Yverdon, 1993.
- [6] Kilbas A, Srivastava M.H. and Trujillo J.J., Theory and Applications of fractional differential, North Holland Mathematics studies 204, 2006.
- [7] W.A. Al. Salam, q -analogues of Cauchy's formula, Proc. Am. Math. Soc. 17 (1952-1953) 182-184.
- [8] W.A. Al. Salam, Some fractional q -integrals and q -derivatives, Proc. Edinb Math. Soc. 15 (1969), 135-140.
- [9] Fox. C, A formal solution of certain dual integral equations, Amer.Math. Soc. 119, 1965.

- [10] Dinesh Singh, A study of fractional calculus operators with special reference to integro-differential equations, A Ph.D. thesis, Jiwaji University, Gwalior, (2011)